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Different types of primary ideals of near-rings and their graphs

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Abstract

From the last few decades the study of the graph theory based on algebraic structures have been taken into account. Many researchers have studied the graphs via groups, rings, near-rings and seminearrings and vice versa. In our study, first we introduce different type of v-primary ideals $(v = 0,1,2,3,c)$ which are the generalization of v-prime ideals $(v = 0,1,2,3,c)$ in near-rings. Then, we provide some important characterizations of these newly initiated ideals. In addition, we explore some relationships among these ideals. Throughout, we furnish our results by providing suitable examples. Finally, as an application, we provide characterizations of different graphs associated with these v-primary ideals $(v = 0,1,2,3,c)$ in near-rings.

Keywords: near-rings, prime ideal, almost prime ideal, primary ideal, graph.

1. Introduction

A (right) near-ring is an algebraic structure N with operation "$+$" and "$\cdot$" where N is a group under "$+$", semigroup under "$\cdot$" and N satisfies (right) distributive law i.e., for any a, b, $c \in N$; $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$. Similarly, one can define a left near-ring by introducing the left distributive law. We refer (Pilz, 2011) for the fundamental concepts and notions of near-rings. In this note, we deal with (right) near-ring. A subset I of a near-ring N is said to be a right ideal of N if: (i) $(I,+) \trianglelefteq (N,+)$, (ii) For every $IN \subseteq I$. Similarly, I is said to be left ideal if (i) $(I,+) \trianglelefteq (N,+)$, (ii) $a_1(a_2+i)-a_1a_2 \in I$, for all $a_1,\ a_2 \in N$ and $i \in I$. If I is a left as well as right ideal of a near-ring, then I is called simply an ideal of near-ring. An ideal I is said to be prime ideal of N if $I_1I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ where I_1 and I_2 are all ideals of N. Various types of prime ideals in literature have been discussed in ((Birkenmeier, Heatherly & Lee 1993), (Fröhlich, 1958), (Holcombe, 1970) & (Ramakotaiah & Rao, 1979)). Almost prime ideal has been endorsed by B. Elavarasan (Elavarasan, 2011) in near-rings. I is called an almost prime ideal if for $I_1I_2 \subseteq I$ and $I_1I_2 \nsubseteq I^2 \Longrightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ where I_1 and I_2 are ideals of N. Further author recognized some relationships involving almost prime and prime ideals as well in (Elavarasan, 2011). Recently, almost prime ideal have been introduced in gamma near-ring by (Khan, Muhammad, Taouti & Maki, 2018). Notions of 0-(1-2)-prime ideals have been defined in ((Birkenmeier, Heatherly & Lee 1993), (Holcombe, 1970) & (Ramakotaiah & Rao, 1979)). Subsequently, (Ramakotaiah & Rao, 1979) introduced the notions of 0-prime, 1-prime and 2-prime ideals of a near-rings. Following (Birkenmeier, Heatherly & Lee, 1993), I is said to be a type-zero or simply a prime ideal of N if $I_1I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$. Further, I is called 3-prime ideal of N if $aNb \subseteq I$ then $a \in I$ or $b \in I$ (Groenewald, 1991). Likewise, I is called 2-prime ideal if for two subgroup K_1, K_2 of $(N,+)$ such that $K_1K_2 \subseteq I \Rightarrow K_1 \subseteq I$ or $K_2 \subseteq I$. It is well-known that 3-prime implies 2-prime implies 1-prime implies 0-prime, however the converse doesn't exist in any of the implication. Recently, P-ideals and their P-properties in near-rings have been introduced in (Atagun, KamacI, Tastekin & Sezgin, 2019). On the other hand, few concepts of near-rings have been shifted towards seminearrings in (Koppula, Kedukodi & Kuncham, 2020). A. Ali introduced commutativity of a 3-prime near-ring on Jordan ideals in (Ali, 2020). G. Wendt discussed the primeness and primitivity of near-rings in

(Wendt, 2021). Further, T. Gaketem introduced some application of ideals of nLA-rings in (Gaketem, 2022). Recently, completely 2-absorbing ideals of N-groups have been discussed in (Sahoo, Shetty, Groenewald, Harikrishnan & Kuncham, 2021).

Graph theory is an important subject which converts algebraic structures in to the graphs for batter understanding. We refer ((Godsil & Royle, 2001), (Bondy & Murty, 1976)) for basic concepts of graph theory. In (Beck, 1988) author introduced the graph of commutative ring by considering the elements of ring as vertices of graph and there exist an edge between vertices x, y if $x.y = 0$. Further, (Anderson & Livingston, 1999) associate graph to commutative ring by using the concepts of zero divisors graph. In (Lipkovski, 2012) author associates the digraph with commutative ring and also discussed some of the properties related to degree and loop of digraph. In (Hausken & Skinner, 2013) author introduced digraph of commutative rings and also discussed some properties of digraph of commutative ring which gives the information about ring. Moreover, (Bhavanari, Kunham & Kedukodi, 2010) define the graph of an ideal of near-ring and also introduce the terminologies strong vertex cut and ideal symmetric graph. Prohelika Das (Das, 2016) define the diameter, girth and coloring of the strong zero-divisor graph of near-rings. Recently, graph of prime intersection of ideals in ring have been introduced in (Rajkhowa & Saikia, 2020).

In this study, we define the notions of ν-primary ideals ($\nu = 0,1,2,3, c$) which are the generalizations of ν-prime ideals ($\nu = 0,1,2,3, c$) in near-rings. Further, we investigate that 0-prime ideal is always 0-primary but converse is not true. We also establish the relations among different ν-primary ideals ($\nu = 0,1,2,3, c$) as well as with ν-prime ideals and verify these relations by suitable examples. Furthermore, several characterizations are obtained and supported by suitable examples. Finally, we define the graphs of ν-primary ideals ($\nu = 0,1,2,3, c$) in near-ring and verify these definitions of ν-primary ideals ($\nu = 0,1,2,3, c$) by using the concepts of subgraphs.

2. Primary ideals in near-rings

In this section, we introduce and discuss different types of v-primary ideals ($v = 0,1,2,3,c$) of near-rings. We also investigate some relationships among them.

Definition 2.1 A proper ideal P of near-ring N is called 0-primary if for all I_1, I_2 are ideals of N so that $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or ${I_2}^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Example 2.2 Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a right near-ring with two operations defined in the tables set 1.

Here, $\{0\}$, $\{0,\ 1\}$, $\{0,\ 2\}$, $\{0,\ 1,\ 2,\ 3\}$ and N are all ideals of N. If we choose, $P = \{0,$ $2\}$ then for all $I_1,\ I_2$ of N such that $I_1 I_2 \subseteq P$ must implies $I_1 \subseteq P$ or $I_2^n \subseteq P$. For clarification we can take $I_1 = \{0,\ 1,\ 2,\ 3\}$ and $I_2 = \{0,\ 1\}$ are ideals of N such that $I_1 I_2 = \{0\} \subseteq P$ implies $I_2^2 \subseteq P$. This implies that P is 0-primary ideal of N. Similarly, one can justify for the remaining ideals.

Proposition 2.3 Let I be an ideal of a zero-symmetric near-ring N. Then, I is a 0-primary ideal if and only if every zero-divisor in N/I is nilpotent.

Proof. Let I be a 0-primary ideal and consider N/I is a non-trivial. Let $n + I \in N/I$ be a zero-divisor and $n_1 \in N/I$. Let $n_1 n\ + I = (n_1 + I)(n + I) = 0 + I \Rightarrow\ n_1 n \in I,\ n_1 \notin I \Rightarrow$ $n^k \in I$ for some $k \in \mathbb{Z}^+$. Hence $(n + I)^k = n^k + I = 0 + I \Rightarrow n\ + I$ is nilpotent. Conversely, let N/I be a non-trivial and every nonzero zero-divisor in N/I is nilpotent. Since $I \neq N$, let n_1, $n_2 \in N$ such that $n_1 \cdot n_2 \in I$, then each $n_1 \in I$ or $n_1 \notin I$, suppose $n_1 \notin I$ then consider $(n_2 + I)(n_1 + I) = n_2 \cdot n_1 + I = 0 + I \Rightarrow n_2 \cdot n_1 = 0$, so $n_2 + I$ is a zero-divisor and there exist $k >$ 0 such that $n_2^k + I = (n_2 + I)^k = I \Rightarrow n_2^k \in I$, hence I is a 0-primary ideal.

Example 2.4 Suppose $N = \{0, 1, 2, 3\}$ be a zero-symmetric near-ring under the addition and multiplication defined in the tables set 2.

Clearly, $P = \{0,\ 1\}$ is 0-primary ideal and the quotient $N/P = \{0 + P, 2 + P\}$ along with operations given in tables set 3.

Here the zero divisors of N/P are $0 + P$ and $2 + P$, which are nilpotents.

Intersection of any two 0-primary ideals of a near-ring is a 0-primary ideal, we provide an example for this fact below.

Example 2.5 Suppose $N = \{0, 1, 2, 3\}$ be a commutative near-ring with $" + "$ and $" \cdot "$ defined in the tables set 2 in Example 2.4. Let us consider 0-primary ideals are $P_1 = \{0,\ 1\}$ and $P_2 = \{0,\ 2\}$ of a near-ring N. And $P_1 \cap P_2 = \{0\}$ is also a 0-primary ideal of N.

Proposition 2.6 Each 0-prime is a 0-primary ideal of N.

Proof. Let P be a 0-prime ideal of a near-ring N. Then, $ab \in P$ implies $a \in P$ or $b \in P$, for all $a, b \in N$ while considering $n = 1$, the result follows.

Remark 2.7 Each maximal ideal of N is 0-prime and hence a 0-primary ideal. It implies that a maximal ideal is a 0-primary.

Definition 2.8 A proper ideal I of a near-ring N is said to be a semi-primary ideal, if for ideal J of N, $J^2 \subseteq I \Rightarrow J \subseteq I$.

It is well known that the intersection of prime ideals in near-ring is also a semi-prime ideal. We also know that the intersection of minimal prime ideals of N is a semi-prime ideal of N such that the ideal I can be written as the intersection of all prime ideals containing I.

However, the intersection of two primary ideals need not be a semi-primary ideal for instance see in Example 2.5 i.e., $I = \{0\}$ is the intersection of 0-primary ideals $\{0, 1\}$ and $\{0, 2\}$, but I is not a semi-primary ideal i.e., $P_2^2 \subseteq \{0\} = I$, but $P_2 \nsubseteq I$.

Remark 2.9 Every 0-primary ideal is not a semi-primary ideal.

To verify above remarks we refer Example 2.5, in which $\{0, 2\}$ is 0-primary ideal but not a semi-primary ideal.

Definition 2.10 A proper ideal P of N is called 1-primary ideal, if for all I_1, I_2 are right ideals of N, such that $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Example 2.11 Consider $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a right near-ring with addition and multiplication defined in the tables set 4.

Here $\{0\}, \{0, 2, 4, 6\}$, $\{0, 2\}$ and N are right ideals of N. Suppose $P = \{0, 4\}$, then for all right ideals I_1, I_2 of N, such that $I_1 I_2 \subseteq P$ must implies $I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in \mathbb{Z}^+$. To verify this let us consider $I_1 = \{0, 2, 4, 6\}$ and $I_2 = \{0, 2\}$ are non-travail right ideals of N, such that $I_1 I_2 = \{0, 4\} \subseteq P$ implies $I_1 \nsubseteq P$ but $I_2^2 = \{0\} \subseteq P$. As a result, P is 1-primary ideal of N.

Definition 2.12 A proper ideal P of N is called 2-primary ideal if for all N-subgroups I_1 and I_2, such that $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Proposition 2.13 Let N be a near-ring. Then the given statements are equivalent.

(i) P is a 2-primary ideal of N.

(ii) If A be an N-subgroup and B is an ideal of N, then $AB \subseteq P$ implies $A \subseteq P$ or $B^k \subseteq P$ where $k \in \mathbb{Z}^+$.

Proof. $(i) \Rightarrow (ii)$ If P is 2-primary ideal and B is an N-subgroup then (ii) is straightaway.

$(ii) \Rightarrow (i)$ Let A and B are two N-subgroups of N such that $AB \subseteq P$. Let A is not a subset of P and assume that $B^k \subseteq (P: A) = \{n \in N: An \subseteq P\} = S$. Since, S is an ideal of N, so if $r \in S$ and $n, n_1 \in N$, then for all $a \in A$, $a(-n + r + n) = -an + ar + an \in P$, as P is an ideal. Thus, $a[(n + r)\ n_1 - nn_1] = (an + ar)n_1\ - ann_1 \in P$ which implies $Anr \subseteq Ar \subseteq P$. Hence $AS \subseteq P$ but we have assumed that $A \nsubseteq P$ implies $S \subseteq P$ so $B^k \subseteq S \subseteq P$.

Proposition 2.14 Let P be a 2-primary ideal and $A_1, . . . , A_k$ are N-subgroups. Then $A_1A_2\ . . . A_k \subseteq P$ implies $A_i^n \subseteq P$ for some $i \in \{1, . . . , k\}$ and $n \in \mathbb{Z}^+$.

Proof. Let $A_1A_2\ . . . A_k \subseteq P$ and $A_1 \nsubseteq P$ such that $(A_2, . . ., A_k)^n \subseteq (P: A_1)$. Thus $A_1 \cdot (P: A_1) \subseteq P$ implies $(P: A_1) \subseteq P$ given that P is 2-primary ideal. By using Proposition 2.13 (ii), we get $(A_2, . . ., A_k)^n \subseteq P$. Similarly, we can repeat procedure for $A_2 \nsubseteq P$ and eventually $A_i^n \subseteq P$ for some $i \in \{1, . . ., k\}$ where $i \neq 2$.

Definition 2.15 An ideal P of N is called 3-primary ideal if for all $x, y \in N$, $xNy \subseteq P$ implies $x \in P$ or $y^n \in P$ for some $n \in \mathbb{Z}^+$.

Example 2.16 Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a (right) near-ring under the addition and multiplication defined in tables set 5.

Here, $P = \{0, 6\}$ is a 3-primary ideal of N. One can check for all $x, y \in N$, $xNy \subseteq P$ must implies $x \in P$ or $y^n \in P$ for some $n \in \mathbb{Z}^+$.

Definition 2.17 A proper ideal P of N is called (completely) c-primary ideal if for all $x, y \in N$, $xy \in P \Rightarrow x \in P$ or $y^n \in P$ for some $n \in \mathbb{Z}^+$.

Example 2.18 Suppose $N = \{0, 1, 2, 3, 4, 5\}$, where " $+$" and " $\cdot$" are defined in the tables set 6.

Clearly, right ideal $P = \{0, 2, 4\}$ satisfies the conditions of c-primary ideal of N as $1 \cdot 3 = 0 \in P$ implies $3^2 = 0 \in P$ and $5 \cdot 3 = 0 \Rightarrow 3^2 = 0 \in P$.

Refer to Example 2.11, it is easy to verify that $P = \{0, 4\}$ is a 0-primary ideal. Thus, every 1-primary ideal is a 0-primary ideal. From Example 2.11, we have observed that an ideal $P = \{0, 4\}$ is 1-primary ideal but it is not 1-prime ideal. Similarly, from Example 2.16, we can see that $P = \{0, 6\}$ is 3-primary ideal but it is not 3-prime ideal as $3N7 = \{0\} \subseteq P$ but 3 or 7 doesn't belong to P. Similarly, in Example 2.18, $P = \{0, 2, 4\}$ is a c-primary ideal which is not a c-prime ideal. However, it is easy to verify that an ideal P is simultaneously c-primary, 3-primary, 2-primary, 1-primary and 0-primary ideal. Hence, we concluded that:

$$c - \text{primary} \Rightarrow 3 - \text{primary} \Rightarrow 2 - \text{primary} \Rightarrow 1 - \text{primary} \Rightarrow 0 - \text{primary}.$$

Refer to Example 2.11, $\{0,\ 4\}$ is the v-primary ideal $(v = 0,1,2,3, c)$ which is the justification of above implications. But the converse doesn't hold true in the above implication. Because, in Example 2.2, $\{0\}$ is 0-primary ideal but not 1-primary ideal. Similarly, $\{0\}$ is not 3-primary ideal i.e, $3N4 = \{0\} \subseteq \{0\}$ but $3,\ 4 \notin \{0\}$ or $3^n,\ 4^n \notin \{0\}$ for some $n \in Z^+$. Further, One can check that $\{0\}$ is not a c-primary ideal. In Example 2.11, $\{0\}$ is a 1-primary ideal but not 2-primary. Simalarly, $\{0\}$ is not a 3-primary ideal i.e, $3N7 = \{0\} \subseteq \{0\}$ but $3,\ 7 \notin \{0\}$ or 3^n, $7^n \notin \{0\}$.

After discussing different types of primary ideals in a near-ring now we introduce v-primary near-rings $(v = 0,1,2,3, c)$.

Definition 2.19 A near-ring N is said to be a ν-primary near-ring ($\nu = 0,1,2,3,c$) if an ideal $\{0\}$ is ν-primary ideal of N.

We can state that N is a 0-primary near-ring, if for all ideals A and B of N, such that $AB \subseteq \{0\}$ implies $A \subseteq \{0\}$ or $B^n \subseteq \{0\}$. In a similar manner, one can define remaining ν-primary near-rings ($\nu = 1,2,3,c$).

Example 2.20 In Example 2.2, $\{0\}$ is a 0-primary ideal of near-ring N. Consequently, N is a 0-primary near-ring.

Proposition 2.21 Each 0-prime near-ring is a 0-primary near-ring.

Example 2.22 Every integral near-rings are prime near-rings and hence primary (0-primary) near-rings.

Following (Pilz, 2011), if I is an ideal of N, then prime radical of N is the intersection of all prime ideals containing I and is denoted by $\wp(I)$ i.e., $\wp(I) = \bigcap_{P \supseteq I} P$, where P is prime. Hence, if $n \in \wp(I) \Rightarrow \exists\ k \in \mathbb{N}: n^k \in I$. In other words, I is a semiprime ideal of N iff $\wp(I) = I$. Likewise rings, we will see that if I is the ν-primary ideal ($\nu = 0,1,2,3,c$) of a near-ring, then its prime radical is the corresponding ν-prime ideal.

Example 2.23 Refer to Example 2.2, $P = \{0, 2\}$ is 0-primary ideal and $\sqrt{\{0,2\}} = \{0, 1, 2, 3\}$ is a 0-prime ideal of N.

It is easy to verify that if an ideal I is a ν-primary, then its prime radical is a ν-prime which we have already seen in Example 2.23. But the converse doesn't hold true i.e., if the prime radical of an ideal I is ν-prime then it is not necessary that I is a ν-primary ideal.

Proposition 2.24 Let I be the both primary and semi-prime ideal of N. Then I is a prime ideal.

It is well known that an ideal I is a c-prime ideal (or completely prime), if $a, b \in N$, $ab \in I$ implies $a \in I$ or $b \in I$.

Definition 2.25 Let Q be a c-primary (completely primary) ideal of N such that $\sqrt{Q} = P$, where P is a c-prime ideal of N. Then we call Q a cP-primary ideal.

Definition 2.26 Let Q be a cP-primary ideal of N. Then, for $n \in N - Q$, so that $(Q: n) = \{a \in N: an \in Q\}$.

Proposition 2.27 Let Q be a cP-primary ideal of N and let $n \in N$. Then below relations hold:

(i) If $n \in Q$, then $(Q: n) = N$;

(ii) If $n \notin Q$, then $(Q: n)$ is cP-primary ideal and $\sqrt{(Q: n)} = P$;

(iii) If $n \notin P$, then $(Q: n) = Q$.

Remark 2.28 Let Q be a cP-primary ideal of N such that $\sqrt{(Q: n)}$ is c-prime and $\sqrt{Q_i} = P_i$, then it must be contained in the set $\sqrt{(Q: n)}$ where $n \in N$.

We illustrate Proposition 2.27 and Remark 2.28 in the below example.

Example 2.29 Refer to Example 2.2, we have $Q = \{0, 2\}$ is c-primary ideal. Then, only the possible c-prime ideal of N containing Q is the ideal $P = \{0, 1, 2, 3\}$ and hence, it is a prime radical of Q. Which implies Q is a cP-primary. On the other hand, let $3 \in N$ and consider $(Q: 3) = \{n \in N: 3n \in Q\} = \{0, 1, 2, 3\}$, which is clearly a c-prime ideal. Hence $(Q: 3)$ is an associated c-prime ideal of a c-primary ideal Q. Thus, every associated c-prime ideal must be contained in $\sqrt{(Q: x)}$.

3. Applications of v-primary ideals $(v = 0, 1, 2, 3, c)$ towards graph theory

In this section, we provide applications of v-primary ideals $(\boldsymbol{v = 0, 1, 2, 3, c})$ towards graph theory. In this regard, we provide characterizations of different graphs through v-primary ideals $(\boldsymbol{v = 0, 1, 2, 3, c})$ of near-rings.

Definition 3.1 Let P be the v-primary ideals $(v = 0,1,2,3, c)$ of near-ring N. Then, the graph of v-primary ideal is denoted by $G_{v-P}(N)$ consists of all the members of N as vertices and $x, y \in N$ are connected by directional edge from x to y (resp., y to x), if $xy \in P$ (resp., $yx \in P$). Similarly, if for any $x, y \in N$ such that $xy = yx \in P$, then there exists an undirected edge between x and y.

Example 3.2 Refer to Example 2.2, in which $P = \{0,2\}$ is 0-primary ideal of a near-ring N. Then graph associated with 0-primary ideal is denoted by $G_{0-P}(N)$ and is shown in Fig. 1.

According to the definition of 0-primary ideal $G_{I_1 I_2}(N) \sqsubseteq G_{0-P}(N)$ implies $G_{I_2^2}(N) \sqsubseteq G_{0-P}(N)$ but $G_{I_1}(N)$ is not a subgraph of $G_{0-P}(N)$. This shows the definition of 0-primary ideal by their graphs which can be check easily.

Example 3.3 In Example 2.11, $P = \{0, 4\}$ is the 1-primary ideal of N. The graph of 1-primary ideal is denoted by $G_{1-P}(N)$ and is shown in Fig. 2.

In Example 2.11, the graph $G_{I_1 I_2}(N) \sqsubseteq G_{1-P}(N)$ implies that $G_{I_2^2}(N) \sqsubseteq G_{1-P}(N)$ and hence satisfies the definition of 1-primary ideal.

Example 3.4 In Example 2.16, $P = \{0, 6\}$ is a 3-primary ideal. Then the graph $G_{3-P}(N)$ is presented in Fig. 3.

The graph of $G_{3-P}(N)$ is different from the graph of 3-prime ideal because every prime ideal is primary but converse does not hold in general.

Example 3.5 In Example 2.18, $P = \{0, 2, 4\}$ is c-primary ideal. The graph of $G_{c-P}(N)$ is shown in figure 4.

Theorem 3.6 Let P be an ideal of near-ring. If P be 3-primary ideal, then P is a strong vertex cut of $G_{3-P}(N)$. If P be 3-semiprimary ideal and P is strong vertex cut of $G_{3-P}(N)$, then P is 3-primary.

Proof. Let P be a 3-primary ideal of N. If $P = N$, then the prove is straightforward. Let us suppose that P is not equal to N and $a, b \in N - P$ i.e., $a \neq b$. Let us suppose that there exists an edge between a and b of $G_{3-P}(N) \Rightarrow aNb \subseteq P$ or $bNa \subseteq P$. Since P is 3-primary ideal of N implies $a \in P$ or $b^n \in P$ which contradicts to our assumption that $a, b \in N - P$. So, as a result $G_{3-P}(N)$ has a strong vertex cut P. On the contrary, assume that P is 3-semiprimary ideal and a strong vertex cut of $G_{3-P}(N)$. Then, we have to prove that P is 3-primary ideal of N. For this, let $a, b \in N$ such that $aNb \subseteq P$. As we know that P is 3-semiprimary ideal so $a = b \Rightarrow a \in P$. Let $a \neq b$ choose $a, b \in N - P$ for possible condition. As we know that P is strong vertex cut of $G_{3-P}(N)$, so edge does not exist between a and b of $G_{3-P}(N) \Rightarrow aNb \nsubseteq P$ and $bNa \nsubseteq P$. A contradiction arises, so $aNb \subseteq P$ implies $a \in P$ or $b^n \in P$. This prove that P is 3-primary ideal.

Lemma 3.7 (i) Let P be a 3-primary ideal of N and a be the vertex of $G_{3-P}(N)$. If $\deg(\mathrm{a}) = \deg(0)$, then $a \in P$.

(ii) Suppose N is a zero-symmetric near-ring and vertex of $G_{3-P}(N)$ is a. If $a \in P$, then $\deg(a) = \deg(0)$.

(iii) Suppose N is a zero-symmetric near-ring and P be its 3-primary ideal. Then $a \in P$ if and only if $\deg(a) = \deg(0)$ in $G_{3-P}(N)$.

Proof. $(i) \Rightarrow$ Suppose $\deg(a) = \deg(0)$, then $aNb \subseteq P$ or $bNa \subseteq P$, for all $b \in N$ i.e., $a \neq b$. Assume that $aNb \subseteq P$, for all $b \in N$. If $P = N$, then $x \in P$. Let $P \neq N$ and choose $b \in N - P$. It is clear that P is 3-primary ideal of N and $aNb \subseteq P$ implies $a \in P$ or $b^n \in P$ for $n \in Z^+$, which proves condition (i). Now to prove (ii), let $a \in P$ and if $a = 0$, then the proof is straightforward. On the other hand, let $a \neq 0$ and $\deg(a) \leq \deg(0)$, then there is vertex b such that b is not adjacent to a in $G_{3-P}(N)$. Hence we can conclude that $aNb \nsubseteq P$ and $bNa \nsubseteq P$. Now according to our supposition $a \in P$ and P is an ideal of N implies $aN \subseteq P$. Thus $aNb \subseteq P$. As we know that N is zero-symmetric, then $Pb \subseteq P \Rightarrow aNb \subseteq P$ which contradicts our supposition. Hence $\deg(a) = \deg(0)$. Condition (iii) is follows form (i) and (ii).

Theorem 3.8 Suppose P is an ideal of N;

(i) *If* N is zero-symmetric and P is 3-primary ideal, then $G_{3-P}(N)$ is ideal symmetric.

(ii) If $G_{3-P}(N)$ is ideal symmetric with P is 3-semiprimary and for every $a \in N$, $\deg(a) = \deg(0)$ in $G_{3-P}(N) \Rightarrow a \in P$, then P is 3-primary and P is a strong vertex cut of $G_{3-P}(N)$.

Proof. To show condition (i), suppose a, b are any two vertices of $G_{3-P}(n)$ having an edge between a and b. Then, $aNb \subseteq P$ or $bNa \subseteq P$. Let $aNb \subseteq P$ and P is 3-primary ideal $\Rightarrow$ $x \in P$ or $b^n \in P$, where n is any positive integer. Since N is a zero-symmetric, by using Lemma 3.7 (ii), $\deg(a) = \deg(0)$ or $\deg(b) = \deg(0)$. It implies $G_{3-P}(N)$ is an ideal symmetric. To verify (ii), suppose $a, b \in N$, $aNb \subseteq P$. According to given condition P is 3-semiprimary implies $a = b \Rightarrow a \in P$. Suppose $a \neq b$ then there exists an edge between a and b in $G_{3-P}(N)$. Since $G_{3-P}(N)$ is ideal symmetric, $\deg(a) = \deg(0)$ or $\deg(y) = \deg(0)$ implies $a \in P$ or $b^n \in P$. As P is 3-primary ideal of N. So by Theorem 3.6, P is a strong vertex cut of $G_{3-P}(N)$.

Conclusion

In this paper, we have introduced the notions of v-primary ideals $(v = 0,1,2,3,c)$ in near-ring, which are the generalization of v-prime ideals $(v = 0,1,2,3,c)$. We verify these defined ideals by examples and counter examples. We have investigated the relations of these ideals among each other as well. During this we have established that $c-\text{primary} \Rightarrow 3-\text{primary} \Rightarrow 2-\text{primary} \Rightarrow 1-\text{primary} \Rightarrow 0-\text{primary}$, but converse of this implication is not true. Furthermore, we have proved some logical results and verified them through examples. Finally, we have studied different types of graphs associated with v-primary ideals $(v = 0,1,2,3,c)$ and also proved some algebraic results by using the concepts of graph theory. In future, v-primary ideals $(v = 0,1,2,3,c)$ and their defined relations with other ideals in near-ring will help to differentiate and introduce more algebraic structure, which are not yet initiated in near-rings. One can also study the sequential machine via using these newly established ideals and the graphs associated with them.

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Tables set 1

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	1
2	0	0	2	2	0	0	2	2
3	0	0	2	2	0	1	2	3
4	0	1	0	1	4	4	4	4
5	0	1	0	1	4	5	4	5
6	0	1	2	3	4	4	6	6
7	0	1	2	3	4	5	6	7

Tables set 2

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	0	0
3	0	1	0	1

Tables set 3

$+$	$0+P$	$2+P$
$0+P$	$0+P$	$2+P$
$2+P$	$2+P$	$0+P$

$\cdot$	$0+P$	$2+P$
$0+P$	$0+P$	$0+P$
$2+P$	$0+P$	$0+P$

Tables set 4

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0
2	0	2	0	2	0	2	2	0
3	0	3	0	3	0	3	3	0
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	5	4
6	4	6	4	6	4	6	6	4
7	4	7	4	7	4	7	7	4

***Tables set* 5**

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	3	0	1	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

Tables set 6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	1	0	5	1
2	0	4	2	0	4	2
3	0	3	3	0	3	3
4	0	2	4	0	2	4
5	0	1	5	0	1	5

Figure

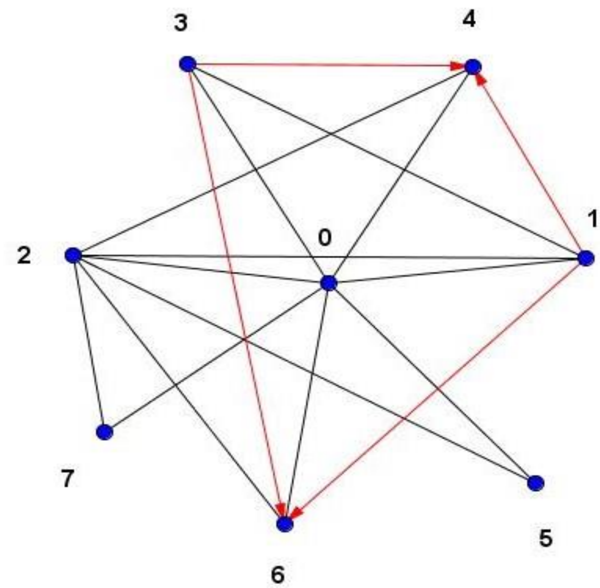

Figure 1. $\mathbf{G_{0-P}(N)}$

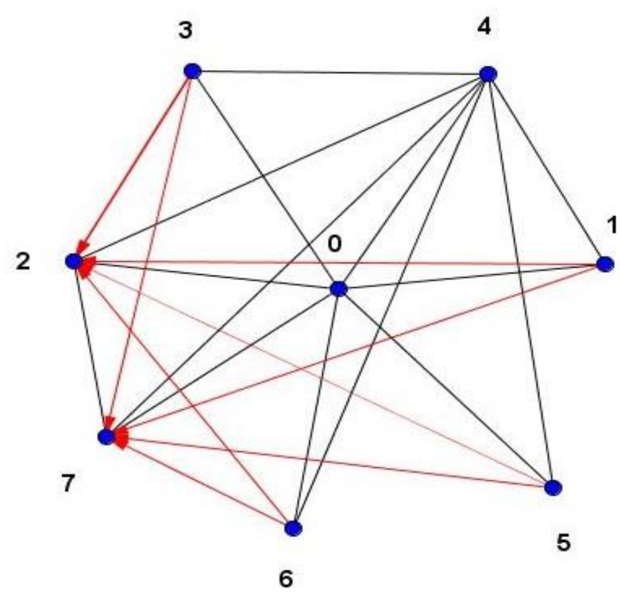

Figure 2. $\mathbf{G_{1-P}(N)}$

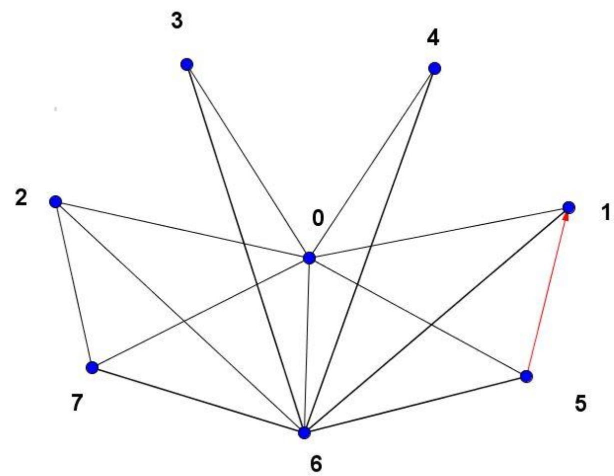

Figure 3. $\mathbf{G_{3-P}(N)}$

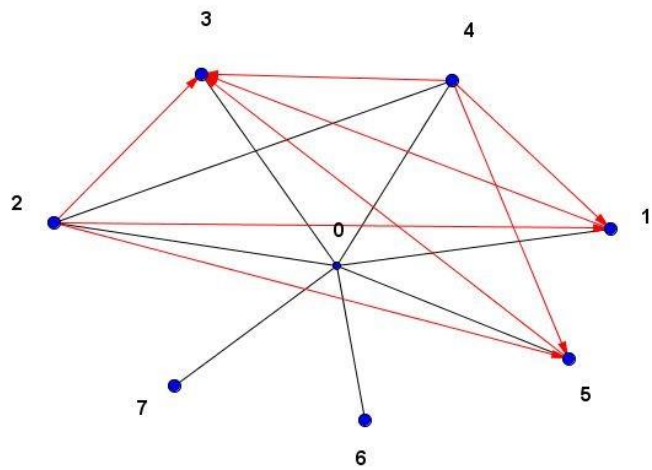

Figure 4. $\mathbf{G_{c-P}(N)}$