Similarity Measure between Pythagorean Fuzzy Sets based on Lower, Upper and Middle Fuzzy Sets with Applications to Pattern Recognition and Multicriteria Decision Making with PF-TODIM

Zahid Hussain, Farida Khanum, Shams Ur Rahman, Rashid Hussain*

Department of Mathematical Sciences, Karakoram International University, Gilgit-Baltistan, 15100, Pakistan.

*Corresponding author's Email address: rashid.ali@kiu.edu.pk

Abstract

Similarity measure (SM) plays an important role to differentiate between objects. Similarity measure of a Pythagorean fuzzy set (PFS) is very useful and effective to show the discrimination between two Pythagorean fuzzy sets (PFSs). Therefore, in this paper, we suggest new similarity measure between PFSs based on the information taken by converting the PFSs into their lower, upper and middle fuzzy sets (FSs) to calculate the degree of similarity between PFSs. We construct axiomatic definition for new SM of PFSs. Furthermore, we put forward a new way to express the similarity measure of PFSs to show the competency, reliability and applicability. To exhibit the reasonability and usefulness of proposed methods, we present several practical examples related to pattern recognition and multicriteria decision making problems. Finally, we construct an algorithm for Portuguese of interactive and multiple attributes decision making (TODIM) method based proposed similarity measures to handle complex multicriteria decision making problems related to day to day life. Our final results show that the suggested method is reasonable, reliable and useful in managing different life setting complex decision making problems in the context of Pythagorean fuzzy sets domain.

Keywords: Fuzzy Set, Pythagorean Fuzzy Sets, Similarity Measures, Pattern Recognition, TODIM, Multicriteria Decision Making

1. Introduction

The concept of fuzzy sets was first proposed by (Zadeh, 1965). With a widely spread use in various fields, fuzzy sets not only provide us a broad opportunity to measure uncertainties in more powerful and logical way, but also are a meaningful way to represent vague concepts in natural language. It is known that most systems based on crisp set theory or two valued logics are somehow difficult for handling imprecise and vague information. In this sense, fuzzy sets can be used and provide solutions to more real world problems. Furthermore, to treat more imprecise and vague information in daily life, various applications and extensions of fuzzy sets are made by researchers, such as multi attribute decision making (MADM) is a significant branch and it plays an important role in human activities is given by (Garg, 2017, Zeng et al., 2018). Recently, many tools have introduced for representing and communicating uncertainty. For example, (Zadeh, 1965), introduced the fuzzy set (FS). The theory of fuzzy set has obtained a lot of attention for many years, but the weakness of FS, just has a membership degree (MD) and ignores unsure data in real DM problems. To overcome this disadvantage of FS, (Atanassov, 1986) gave the amazing concept of intuitionistic fuzzy sets (IFSs), with the characterization of membership degree (MD) and non-membership degree (NMD) and the degree of hesitancy. Thus, IFSs are more powerful and capable than fuzzy sets. The condition of IFSs state that the sum of MD and NMD is always less or equal to 1. However, there exist circumstances where the sum of MD and NMD is more than one. To overcome, this disadvantage, the new generalization of IFSs called Pythagorean fuzzy sets (PFSs) was introduced by (Yager, 2013; Yager & Abbasov, 2013), and Pythagorean membership grades in multicriteria decision making by (Yager, 2013). The range of PFSs is much wider than IFS, in which the sum of the square of MD and NMD is restricted to less than or equal to 1. For example, (0.6, 0.7), is that situation in which both IFSs cannot evaluate the attribute value of (0.6, 0.7) because $0.6 + 0.7 > 1$ and $(0.6)^2 + (0.7)^2 < 1$, respectively. Therefore, we can say that PFSs are more powerful and rigorous than IFSs to handle incomplete information related to daily life. Fuzzy sets have many applications in almost all fields such as clustering, image processing, mathematical programming, fuzzy control, pattern recognition, water quality, engineering, medical diagnosis, business management, decision making, data mining, etc. Application of fuzzy sets in soil science, fuzzy logic, fuzzy measurements and fuzzy decisions is given by (Mcbratney & Odeh, 1997), an application of fuzzy set theory to inventory control models is suggested by (Gen et al., 1997). On possible and necessary inclusion of intuitionistic fuzzy sets is given by (Grzegorzewski, 2011). Image enhancement using fuzzy set is proposed by (Pal & King, 1980). The similarity measures based on lower, upper and middle fuzzy sets corresponding to PFSs, provides us a new way to construct similarity measures between PFSs. On the basis of numerical analysis results, we find that our new construction can handle amicably different problems related to daily life especially problems involving in pattern recognition and multicriteria decision making process. As a whole, our proposed construction method for similarity measures between PFSs can provide a more effective way for measuring similarity degrees between PFSs. Similarity measure is a significant instrument to determine the degree of SMs between two objects. (Kaufman & Rousseeuw, 1990), introduced few models to present traditional similarity measures with applications in various leveled group investigation. Different similarity measures between FSs has been introduced. (Dengfeng & Chuntian, 2002), introduced a few SMs between IFSs utilized in design recognition. (Liang & Shi, 2003) proposed similarity measures between IFSs and furthermore present the connections between these measures with applications to design recognition is given by (Mitchell, 2003). Deciphered IFSs as ensembles of ordered FSs from the statistical perspective to change techniques is suggested by (Dengfeng & Chuntian, 2002). (Liang & Shi, 2019) utilized numerical comparisons to express that their suggested SMs are more reliable than those of (Dengfeng & Chuntian, 2002). (Hung & Yang, 2004) expressed a few SMs between IFSs dependent on Hausdorff distance which are very much utilized with linguistic variables. (Xu & Chen, 2008) gave the comparisons of distance and SMs between IFSs. Pythagorean fuzzy sets are being used in variety of applications in almost all fields including pattern recognition, multicriteria decision making, engineering, medical diagnosis, business management, clustering etc. The similarity measure of constrained Pythagorean fuzzy sets (CPFSs) is presented by (Pan et al., 2021) suggested the complex distance measure of PFS and apply it to pattern recognition. (Yang & Hussain, 2019) suggested distance and SM between hesitant fuzzy sets and apply it to clustering. (Li & Lu, 2019) suggested many new distance and SM between PFSs with its applications and give the concept of normalized Hamming and

Hausdorff distance. Fuzzy entropy for Pythagorean fuzzy sets with application to multicriterion decision making is coined by (Yang & Hussain, 2018). Many distance and similarity measure of PFSs with its applications are discussed in this literature (Hussain & Yang, 2019), (Li & Lu, 2019), (Peng & Garg, 2019), (Verma & Merijo, 2019) and (Zhao & Chen, 2019). Similarity Measures for New Hybrid Models: mF Sets and mF Soft Sets s suggested by (Akram & Waseem 2019) .A hybrid method for complex Pythagorean fuzzy decision making, Mathematical Problems in Engineering is put forwarded by (Akram et al., 2021). Minimum spanning tree hierarchical clustering algorithm: A new Pythagorean fuzzy similarity measure for the analysis of functional brain networks is given by (Habib et al., 2022). A new outranking method for multicriteria decision making with complex Pythagorean fuzzy information (Akram et. al., 2022). Belief and plausibility measures on Pythagorean fuzzy sets and its applications with BPI-VIKOR is proposed by (Hussain et al., 2022) An integrated ELECTRE-I approach for risk evaluation with hesitant Pythagorean fuzzy information is proposed by (Akram et. al., 2022). The similarity measure between two PFSs is very useful to indicate the degree of resemblance between two objects. Generally, information systems in PFSs are carried out by its lower, upper and middle fuzzy sets. In this manuscript, we put forward new construction for similarity measures between PFSs based on lower, upper and middle fuzzy sets with the objective to develop useful and reasonable similarity measures between PFSs. We have utilized TODIM methods because the decision making outcome is determined by computing the degree of gain or loss of an alternative relative to the rest, to better reflect the behavioral preference of the decision makers.

The rest of the manuscript is arranged as follows. Section 2 consists of some basic preliminaries about PFSs. In Section 3, we introduced new similarity measure between PFSs based on the lower, upper and center FSs. In Section 4, we show the performance of our proposed similarity measures using different examples. In section 5, we use Pythagorean fuzzy TODIM method to manage problem involving multicriteria decision making. At the end, conclusion is conveyed in Section 6.

2. Preliminaries

The basic concepts of IFS and PFS are respectively given in following section.

Definition 1. An intuitionistic fuzzy set (IFS) S in X is defined by (Atanassov, 1999), as an object of the following form as,

$$S=\left\{\left(x_i,\mu_{\hat{S}}\left(x_i\right),\nu_{\hat{S}}\left(x_i\right)\right):x_i\in X\right\},$$

where $\mu_{\hat{S}}\left(x_i\right):X\rightarrow\left[0,1\right],$ denotes the degree of membership of $x_i\in S$ and $\nu_{\hat{S}}\left(x_i\right):X\rightarrow\left[0,1\right],$ denotes the degree of non-membership of $x_i\in\hat{S}$ and $0\le\mu_{\hat{S}}\left(x_i\right)+\nu_{\hat{S}}\left(x_i\right)\le 1.$ The degree of non - determinacy of IFS $\hat{S}$ is symbolized by the following relation $\pi_{\hat{S}}\left(x_i\right)=1-\left(\mu_{\hat{S}}\left(x_i\right)+\nu_{\hat{S}}\left(x_i\right)\right).$

Definition 2. A Pythagorean fuzzy set PFS G in X is given by (Yager & Abbasov, 2013) as

$$G=\left\{\left\langle x_i,\mu_G\left(x_i\right),\nu_G\left(x_i\right)\right\rangle:x_i\in X\right\},$$

where $\mu_G\left(x_i\right):X\rightarrow\left[0,1\right],$ represents the degree of membership and $\nu_G\left(x_i\right):X\rightarrow\left[0,1\right],$ represents the degree of membership. For every $x_i\in B$ with the condition that

$$0\le\mu_G^2\left(x_i\right)+\nu_G^2\left(x_i\right)\le 1.$$

Definition 3. (Atanassov, 1999). For all $P\in PFSs\left(X\right),$ the following expression is termed as Pythagorean index of the element $x_i\in G$

$\pi_G\left(x_i\right)=\sqrt{1-\left\{\mu_G^2\left(x_i\right)+\nu_G^2\left(x_i\right)\right\}},$ It is obvious that $0\le\pi_G^2\left(x_i\right)\le 1,\ \forall\ x_i\in X.$

Definition 4. Let P_1, P_2 and P_3 are three PFSs on X . A similarity SM $S(G,H)$ is mapped as, $S: PFSs(X) \times PFSs(X) \rightarrow [0,1]$ have the following operations

(S_1) $0 \leq S(G,H) \leq 1$;

(S_2) $S(G,H) = 1 \; if \; G = H$;

(S_3) $S(G,H) = S(H,G)$;

(S_4) $S(G,H) \geq S(G,I) \; and \; S(H,I) \geq S(G,I) \; if \; G \subseteq H \subseteq I$;

(S_5) $S(G,H) = 0 \; if \; G = X \; and \; H = \varnothing \; or \; G = \varnothing \; and \; H = X$.

Definition 5. (Peng et al., 2017) If G and H be two PFSs on X the following operations can be define as follows:

(1) $G^c = \{\langle x_i, \nu_G(x_i), \mu_G(x_i) \rangle : x_i \in X\}$;

(2) $G \subseteq H \; iff \; \forall x_i \in X, \; \mu_G(x_i) \leq \mu_H(x_i) \; and \; \nu_G(x_i) \geq \nu_H(x_i)$;

(3) $G = H \; iff \; \forall x_i \in X, \; \mu_G(x_i) = \mu_H(x_i) \; and \; \nu_G(x_i) = \nu_H(x_i)$;

(4) $G \cap H = \{x_i, min(\mu_G(x_i), \mu_H(x_i)), max(\nu_G(x_i), \nu_H(x_i))\}$;

(5) $G \cup H = \{x_i, max(\mu_G(x_i), \mu_H(x_i)), min(\nu_G(x_i), \nu_H(x_i))\}$.

3. Construction of new similarity measures

In this section, we construct some new and useful similarity measures between two PFSs. We use the similar idea of (Hwang & Yang, 2013) and define some similarity measures on PFSs based on lower, upper and middle Pythagorean fuzzy set. Let us take a PFS

$G=\left\{\left\langle x_i,\mu_G\left(x_i\right),\nu_G\left(x_i\right)\right\rangle : x_i\in X\right\}$, we first define the lower, upper and middle Pythagorean fuzzy sets with reference to (Hwang & Yang, 2013; Gregorszewsk, 2011). Assume that the lower, upper and middle Pythagorean fuzzy sets are denoted by G^L, G^U *and* G^M respectively as follows:

$$G^L=\left\{\left\langle x_i,\mu_G^L\left(x_i\right),\nu_G^L\left(x_i\right)\right\rangle : x_i\in X\right\},\ \mu_G^L\left(x_i\right)=\mu_G^2\left(x_i\right);$$

$$G^U=\left\{\left\langle x_i,\mu_G^U\left(x_i\right),\nu_G^U\left(x_i\right)\right\rangle : x_i\in X\right\},\ \mu_G^U\left(x_i\right)=\mu_G^2\left(x_i\right)+\pi_G^2\left(x_i\right)=1-\nu_G^2\left(x_i\right);$$

$$G^M=\left\{\left\langle x_i,\mu_G^M\left(x_i\right)\right\rangle : x_i\in X\right\},\ \mu_G^M\left(x_i\right)=\frac{\mu_G^2\left(x_i\right)+1-\nu_G^2\left(x_i\right)}{2}.$$

First, we extend the similarity measures between IFSs (Hwang & Yang, 2013) to the similarity measures between Pythagorean fuzzy sets as follows:

$$\tilde{S}_C\left(G,H\right)=1-\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\nu_G^2\left(x_i\right)\right|-\left|\mu_H^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\} \tag{1}$$

$$\tilde{S}_H\left(G,H\right)=1-\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|-\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\} \tag{2}$$

$$\begin{aligned}\tilde{S}_L\left(G,H\right)=1&-\frac{1}{4n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\nu_G^2\left(x_i\right)\right|-\left|\mu_H^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\}\\&-\frac{1}{4n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|-\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\}\end{aligned} \tag{3}$$

$$\tilde{S}_O\left(G,H\right)=1-\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|^2\right\}} \tag{4}$$

$$\tilde{S}_{DC}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{m}_G(i)-\tilde{m}_H(i)\right|^p} \tag{5}$$

where $\tilde{m}_G(i)=\left(\mu_G^2\left(x_i\right)+1-\nu_G^2\left(x_i\right)\right)/2$ and $\tilde{m}_H(i)=\left(\mu_H^2\left(x_i\right)+1-\nu_H^2\left(x_i\right)\right)/2$ $1\le P<\infty$.

$$\tilde{S}_{HF}\left(G,H\right)=\frac{1}{2}\left(\rho\tilde{\mu}^{2}\left(G,H\right)+\rho\tilde{\nu}^{2}\left(G,H\right)\right) \tag{6}$$

where

$$\rho\tilde{\mu}^{2}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\mu_{G}^{2}\left(x_{i}\right)-\mu_{H}^{2}\left(x_{i}\right)\right|^{p}} \text{ and } \rho\tilde{\nu}^{2}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\nu_{G}^{2}\left(x_{i}\right)-\nu_{H}^{2}\left(x_{i}\right)\right|^{p}}$$

$$\tilde{S}_{e}^{p}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\tilde{\phi}_{tGH}(i)+\tilde{\phi}_{fGH}(i)\right)^{p}} \tag{7}$$

where

$$\tilde{\phi}_{tGH}(i)=\left|\mu_{G}^{2}\left(x_{i}\right)-\mu_{H}^{2}\left(x_{i}\right)\right|/2 \text{ and } \tilde{\phi}_{fGH}(i)=\left|\left(1-\nu_{G}^{2}\left(x_{i}\right)\right)-\left(1-\nu_{H}^{2}\left(x_{i}\right)\right)\right|/2 \quad 1\leq P<\infty$$

$$\tilde{S}_{s}^{p}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\tilde{Q}_{s1}\left(x_{i}\right)+\tilde{Q}_{s2}\left(x_{i}\right)\right)} \tag{8}$$

$$\tilde{Q}_{s1}\left(x_{i}\right)=\left|\tilde{m}_{G1}\left(x_{i}\right)-\tilde{m}_{H1}\left(x_{i}\right)\right|/2,\ \tilde{Q}_{s2}\left(x_{i}\right)=\left|\tilde{m}_{G2}\left(x_{i}\right)-\tilde{m}_{H2}\left(x_{i}\right)\right|/2,\ \tilde{m}_{G1}(i)=\left(\tilde{\mu}_{G}^{2}\left(x_{i}\right)+\tilde{m}_{G}(i)\right)/2,$$

$$\tilde{m}_{G2}(i)=\left(\tilde{m}_{G}(i)+1-\tilde{\nu}_{G}(i)\right)/2,\ \tilde{m}_{H1}(i)=\left(\tilde{\mu}_{\mathbf{B}}^{2}(i)+\tilde{m}_{H}\left(x_{i}\right)\right)/2,\ \tilde{m}_{H2}(i)=\left(\tilde{m}_{B}(i)+1-\tilde{\nu}_{G}^{2}(i)\right)/2.$$

$$\tilde{S}_{HB}\left(G,H\right)=\frac{1}{2}\left(\tilde{\rho}_{\mu}^{2}\left(G,H\right)+\tilde{\rho}_{\nu}^{2}\left(G,H\right)\right) \tag{9}$$

where $\tilde{\rho}_{\mu}^{2}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{\mu}_{G}^{2}\left(x_{i}\right)-\tilde{\mu}_{H}^{2}\left(x_{i}\right)^{p}\right|}$ and $\tilde{\rho}_{\nu}^{2}\left(G,H\right)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{\nu}_{G}^{2}\left(x_{i}\right)-\tilde{\nu}_{H}^{2}\left(x_{i}\right)\right|^{p}}$

Next, we extend the above defined similarity measures Eqs. (1) - (9) between two PFSs *G* and *H* to the similarity measures between two PFSs *G* and *H* based on below, above and center (bac) fuzzy sets respectively as follows:

$$\tilde{S}_{lumc}\left(G,H\right)=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|+\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \\ +\frac{1}{2}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \tag{10}$$

$$\tilde{S}_{lumH}\left(G,H\right)=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|+\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \\ +\frac{1}{2}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \tag{11}$$

$$\tilde{S}_{lumL}\left(G,H\right)=1-\frac{1}{3n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|+\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \\ \frac{1}{2}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right| \tag{12}$$

$$\tilde{S}_{lumO}\left(G,H\right)=1-\frac{1}{3}\left(\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|^2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^2} \right. \\ \left. +\sqrt{\frac{1}{4n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^2}\right) \tag{13}$$

$$\tilde{S}_{lumDC}\left(G,H\right)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^p} \right. \\ \left. +\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^p}\right) \tag{14}$$

$$\tilde{S}_{lumHB}\left(G,H\right)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^p} \right. \\ \left. +\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)+\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|^p}\right) \tag{15}$$

$$\tilde{S}_{leum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right.$$
$$\left.+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \tag{16}$$

$$\tilde{S}_{lsum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right.$$
$$\left.+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \tag{17}$$

$$\tilde{S}_{lhum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{2^{p}}{3^{p}n}\sum_{i=1}^{n}\left|\mu_{G}^{2}(x_{i})-\mu_{H}^{2}(x_{i})\right|^{p}}+\sqrt[p]{\frac{2^{p}}{3^{p}n}\sum_{i=1}^{n}\left|\nu_{G}^{2}(x_{i})-\nu_{H}^{2}(x_{i})\right|^{p}}\right.$$
$$\left.+\sqrt[p]{\frac{1}{3^{p}n}\sum_{i=1}^{n}\left|\mu_{G}^{2}(x_{i})-\mu_{H}^{2}(x_{i})+\nu_{H}^{2}(x_{i})-\nu_{G}^{2}(x_{i})\right|^{p}}\right) \tag{18}$$

To show the reasonability and usefulness of our proposed similarity measures of Eqs. (10) – (18), we put forward the following examples.

4. Demonstration of Results and Application

In the following section, we present some practical examples related to pattern recognition and multicriteria decision making to show reasonability and practicability of our proposed similarity measures of Eqs. (10) – (18) as follows:

Example 1. Let *G* and *H* be two PFSs which are express in the following Table 1,

Table 1, reflects the calculation of suggested similarity measures using Eqs. (1) - (9). The numerical result shows the reasonability of our proposed methods of calculating similarities between PFSs. Next, we utilized our newly constructed similarity measures Eqs. (10) - (18) based on lower upper and middle fuzzy sets to calculate similarity measures between PFSs.

Examples 2. Let *G* and *H* be two PFSs which are express in the Table 2. From the numerical simulations of Table 2, it is clear that the newly constructed similarity measures of Eqs. (10) - (18) based on lower, upper and middle fuzzy sets are reasonable and appropriate. In the following subsection, we apply our proposed similarity measures of Eqs. (10) – (18) in handling problems related pattern recognition.

4.1 Application to Pattern Recognition

In this subsection, we particularly use our similarity measures of Eqs. (10) – (18) in application in pattern recognition. We utilize the law of maximum similarity between two PFSs to measure the similarity between two PFSs.

Example 3. Let G_1 and G_2 are two given patterns in PFSs in finite universe of discourse X.

$$G_1 = \{(x,\ 0.70,\ 0.70)\}\ and\ G_2 = \{(x,\ 0.80,\ 0.30)\}.$$

We have a sample Q which is expressed by the PFSs

$$Q = \{(x,\ 0.70,\ 0.50)\}$$

The objective is to recognize the sample Q with one of the given patterns G_1 and G_2, using the principle of maximum degree of similarity between two PFSs the procedure of transmission Q to G_m is

$$S_{PFS,m} = arg \max_{1\le i\le 2}\left(S_{PFS}\left(G_i, Q\right)\right)$$

Utilizing the proposed similarity measures of Eqs. (10) – (18) between PFSs, we have the following $G_i,\ i = 1,\ 2\ and\ Q$. The calculation of similarity measures of Eqs. (10) – (18) are given as follows:

$G_i,\ i = 1,\ 2\ and\ Q$. The calculation of similarity measures Eqs. (1) - (18) are given as follows

$$\tilde{S}_C\left(G_1,Q\right) = 0.88,\quad \tilde{S}_C\left(G_2,Q\right) = 0.896,\quad \tilde{S}_{lumC}\left(G_1,Q\right) = 0.64,\quad \tilde{S}_{lumC}\left(G_2,Q\right) = 0.86,$$

$$\tilde{S}_H\left(G_1,Q\right) = 0.88,\quad \tilde{S}_H\left(G_2,Q\right) = 0.86,\quad \tilde{S}_{lumH}\left(G_1,Q\right) = 0.64,\quad \tilde{S}_{lumH}\left(G_2,Q\right) = 0.86,$$

$$\tilde{S}_L\left(G_1,Q\right) = 0.88,\quad \tilde{S}_L\left(G_2,Q\right) = 0.86,\quad \tilde{S}_{lumL}\left(G_1,Q\right) = 0.88,\quad \tilde{S}_{lumL}\left(G_2,Q\right) = 0.86,$$

$$\tilde{S}_0\left(G_1,Q\right)=0.65,\quad \tilde{S}_O\left(G_2,Q\right)=0.86,\quad \tilde{S}_{lumO}\left(G_1,Q\right)=0.99,\quad \tilde{S}_{lumO}\left(G_2,Q\right)=0.86,$$

$$\tilde{S}_{Dc}\left(G_1,Q\right)=0.88,\quad \tilde{S}_{DC}\left(G_2,Q\right)=0.86,\quad \tilde{S}_{lumDC}\left(G_1,Q\right)=0.88,\quad \tilde{S}_{lumDC}\left(G_2,Q\right)=0.86,$$

$$\tilde{S}_{HB}\left(G_1,Q\right)=0.88,\quad \tilde{S}_{HB}\left(G_2,Q\right)=0.86,\quad \tilde{S}_{lumHB}\left(G_1,Q\right)=0.88,\quad \tilde{S}_{lumHB}\left(G_2,Q\right)=0.86,$$

$$\tilde{S}_e^p\left(G_1,Q\right)=0.88,\quad \tilde{S}_e^P\left(G_2,Q\right)=0.86,\quad \tilde{S}_{lume}^p\left(G_1,Q\right)=0.88,\quad \tilde{S}_{lume}^P\left(G_2,Q\right)=0.86,$$

$$\tilde{S}_s^p\left(G_1,Q\right)=0.79,\quad \tilde{S}_S^P\left(G_2,Q\right)=0.86,\quad \tilde{S}_{lumS}^p\left(G_1,Q\right)=0.88,\quad \tilde{S}_{lumS}^P\left(G_2,Q\right)=0.86,$$

$$\tilde{S}_h^p\left(G_1,Q\right)=0.93,\quad \tilde{S}_h^P\left(G_2,Q\right)=0.99,\quad \tilde{S}_{lumh}^p\left(G_1,Q\right)=0.95,\quad \tilde{S}_{lumh}^P\left(G_2,Q\right)=0.92.$$

Intuitively, we expect that Q should belong to the pattern G_1. The above numerically computed results shows that the sample Q belongs to the pattern G_1 according to the principle of maximum degree of similarity between PFSs. Most of the similarity measures are agreed expect few conflicts with negligible error. This might the result of poor approximation. So we can conclude that the sample Q belong to the pattern G_1 according to the principle of maximum degree of similarity between PFSs.

Next, we propose Pythagorean fuzzy TODIM to apply our propose similarity measure in an application to daily life matters containing complex multicriteria decision making process.

5. Pythagorean Fuzzy TODIM

Step 1: The Pythagorean fuzzy decision matrix with respect to alternatives $\tilde{H}_i, i=1,2,3,\ldots,m$ to the criteria $\tilde{C}_j, j=1,2,3,\ldots,n$ is given as follows:

$$\tilde{R}=\left(r_{ij}\right)_{m\times n}=\begin{array}{c|cccccc} \tilde{H}_i/\tilde{C}_j & \tilde{C}_1 & \tilde{C}_2 & . & . & . & \tilde{C}_n \\ \hline \tilde{H}_1 & r_{11} & r_{12} & . & . & . & r_{1j} \\ \tilde{H}_2 & r_{21} & r_{22} & . & . & . & r_{2j} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ \tilde{H}_m & r_{i1} & r_{i2} & . & . & . & r_{ij} \end{array}$$

Step 2: The transform of the decision matrix to the normalized Pythagorean decision matrix as follows:

$$\tilde{L}=\left(l_{ij}\right)_{m\times n}=\begin{cases} r_{ij} & \text{for beneficial attribute} \\ \left(r_{ij}\right)^{c} & \text{for cost attribute} \end{cases}$$

Step 3: Calculation of the relative weights of criteria by following formula

$$\tilde{w}_{jr}=\tilde{w}_j/\tilde{w}_r,\ 0\le w_{jr}\le 1$$

$$\tilde{w}_r=max\left[\tilde{w}_j : j=1,2,3,...,n\right] and\ 0\le\tilde{w}_{jr}\le 1$$

Step 4: Calculate the dominance degree of each alternative $\tilde{H}_i$ over each alternative $\tilde{H}_t$ with respect to the criterion by $\tilde{C}_j$ using,

$$\tilde{\phi}_j\left(\tilde{H}_i,\tilde{H}_t\right)=\begin{cases}\sqrt{\dfrac{\tilde{w}_{rj}\ d\left(\tilde{I}_{ij},\tilde{I}_{tj}\right)}{\sum_{j=1}^{n}\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}>\tilde{I}_{tj}\\ 0 & if\ \ \tilde{I}_{ij}=\tilde{I}_{tj}\\ -\dfrac{1}{\tilde{\theta}}\sqrt{\dfrac{\sum_{j=1}^{n}\tilde{w}_{rj}d\left(\tilde{I}_{ij},\tilde{I}_{tj}\right)}{\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}<\tilde{I}_{tj}\end{cases}$$

Step 5: Calculate the overall dominance degree of $\tilde{H}_i$ over each alternative $\tilde{H}_t$ using

$$\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)=\sum_{j=1}^{n}\tilde{\phi}_j\left(\tilde{H}_i,\tilde{H}_t\right)$$

$\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)$ denotes the measurement of dominance of alternative $\tilde{H}_i$ over alternative $\tilde{H}_t$

Step 6: Derive the overall value of each alternative $\tilde{H}_i$ by using

$$\varepsilon_i=\frac{\sum_{i=1}^{n}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)-\underset{i}{min}\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right)}{\underset{i}{max}\left\{\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right\}-\underset{i}{min}\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right)}$$

Clearly, $0\leq\varepsilon_i\leq 1$, and we select the greater value of ε_i and consider this as better alternative $\tilde{H}_i$. Thus, one can choose the appropriate alternative, in accordance with a descending order of the overall values of all the alternatives.

Step: 7 Find the best alternatives according to the given values.

Example 4. (Application in vehicle selection)

Consider a customer wants to purchase a new model car. Assume that four types of cars as alternative A_j (j= 1,2,3,4) are available in the market. The customer considers the following four criterias before buying a new model car:

C_1: *Economical to operate*, C_2 : *Price*, C_3 : *Reliable*, C_4 : *Capacious*.

We noticed that C_2 is cost attribute while other three are benefit attributes. The evaluated values of alternatives A_i over criteria are given in the decision making Table 3.

As D_2 is cost attribute so we have to find the compliment of C_2 for normalize the decision matrix and is given in Table 4.

Assume that the weights are known, so we denote these four criterias as C_j where j=1,2,3,4 and their respective weighted vector is represented by $w=(0.40,\ 0.30,\ 0.20,\ 0.10)$.

Step 3: Since w_1 is maximum of all other weights than C_1 is reference criteria and the weight $w_r = 0.4$. So, the relative weight of all the given criteria C_j (j=1,2,3,4) are

$$w_r = max\ \{w_j,\ j=1,\ 2,\ 3,\ 4\}$$

$$w_{1r} = 0.4/0.4 = 1,\quad w_{2r} = 0.2/0.4 = 0.5,\quad w_{3r} = 0.1/0.4 = 0.25,\quad w_{4r} 0.3/0.4 = 0.75$$

$$\sum_{i=1}^{n} w_{jr} = 1 + 0.75 + 0.5 + 0.25 = 2.5 = \theta$$

Step 4: Calculation of the degree of dominance by using

$$d(G,H) = \frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i) - \mu_H^2(x_i)\,|\,\nu\,|\,\nu_G^2(x_i) - \nu_H^2(x_i)\right|$$

$$\phi_1\left(A_x, A_t\right)\ \left(\mathrm{x}, \mathrm{y} = 1, 2, 3, 4\right)$$

The calculated values are given in Table 5, 6, 7, and 8 respectively.

The overall dominance degree of A_i is mentioned in Table 9.

The overall value of dominance degree of A_i over each alternative A_t can be find as

$$\varepsilon_i = \frac{\sum_{i=1}^{n} \delta\left(A_x, A_t\right) - min_i\left(\sum_{i=1}^{m} \delta\left(A_x, A_t\right)\right)}{max_i\left\{\sum_{i=1}^{m} \delta\left(A_x, A_t\right)\right\} - min_i \sum_{i=1}^{m} \delta\left(A_x, A_t\right))}$$

$$\varepsilon_1 = \frac{3.1506 - \left(-6.0942\right)}{7.528 - \left(-6.0942\right)} = 0.6787,\ \varepsilon_2 = \frac{-6.0942 - \left(-6.0942\right)}{7.528 - \left(-6.0942\right)} = 0$$

$$\varepsilon_3 = \frac{7.528 - \left(-6.0942\right)}{7.528 - \left(-6.0942\right)} = 1,\ \varepsilon_4 = \frac{5.9042 - \left(-6.0942\right)}{7.528 - \left(-6.0942\right)} = 0.88\,.$$

Final ranking is mentioned in Table 10. From Table 10, we conclude that the final ranking is made according to descending order as

$$A_3 \succ A_4 \succ A_1 \succ A_2$$

From the above ranking of alternatives, it is clear that the alternative A_3 is considered as the best among all available four alternatives. The ranking of alternatives A_i are carried out in descending order based on the overall value ε_i of each alternative $\tilde{H}_i$. The alternative A_i having the highest overall value is selected as the best alternative. Hence, the alternative A_3 is considered as the best alternative.

6. Conclusions

Varieties of information measures are suggested in the literature but still there is space to improve, modify and to create new ones. Adopting the similarity measures between PFSs based on lower, middle and upper fuzzy sets, we provide a novel way of constructing similarity measures between PFSs. The new similarity measure between PFSs based on the information taken by converting the PFSs into their lower, upper and middle fuzzy sets (FSs) to calculate the degree of similarity between PFSs is very effective and useful. On the basis of numerical analysis results, we found that our novel construction provides very useful and reasonable results. Applications of our proposed methods related to pattern recognition and multicriteria decision making show the usefulness and practical applicability of proposed methods. Finally, Pythagorean fuzzy TODIM method based on our proposed similarity measures is constructed to handle complex problems related to daily life. Holistically, our suggested measures between PFSs can provide a more reasonable and effective way of calculating degree of similarity between PFSs.

Future Direction

In the future, we will consider clustering objects in uncertain and ambiguous environments using our proposed new construction method for similarity measures between PFSs.

Acknowledgements

The authors would like to thank the reviewers and the editor for their valuable comments and feedback.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally. All authors read and approved the final manuscript.

References

Atanassov, K. T. (1999). Intuitionistic Fuzzy Sets, Theory and Applications, Physica-Verlag, Heidelberg.

Atanasssov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.

Akram, M., & Waseem, N. (2019). Similarity Measures for New Hybrid Models: mF Sets and mF Soft Sets, Punjab University Journal of Mathematics, 51 (6), 115-130.

Akram, M., Samirah, A., & Zahid, K. (2021). A hybrid method for complex Pythagorean fuzzy decision making, Mathematical Problems in Engineering, DOI: 10.1155/2021/9915432.

Akram, M., Luqman, A., & Alcantud, J. C. R. (2022). An integrated ELECTRE-I approach for risk evaluation with hesitant Pythagorean fuzzy information, Expert Systems with Applications, 200, 116945.

Akram, M., Zahid, K., Alcantud, J. C. R. (2022). A new outranking method for multicriteria decision making with complex Pythagorean fuzzy information, Neural Computing and Applications, 34(10), 8069-8102.

Dengfeng, L., Chuntian, C. (2002). New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern recognition letters, 23(1-3), 221-225.

Fan, L. & Zhangyan, X. (2001). Similarity measures between vague sets. J. Software, 12(6),922-927.

Firozja, M. A., Agheli, B. & Jamkhaneh, E. B. (2019). A new similarity measure for Pythagorean fuzzy sets. Complex & Intelligent Systems, 1-8.

Garg, H. (2017). Novel intuitionistic fuzzy decision making method based on an improved operation laws and its application. Engineering Applications of Artificial Intelligence, 60, 164-174.

Gen, M., Tsujimura, Y. & Zheng, D. (1997). An application of fuzzy set theory to inventory control models. Computers & Industrial Engineering, 33(3-4), 553-556.

Grzegorzewski, P. (2011). On possible and necessary inclusion of intuitionistic fuzzy sets. Information Sciences, 181(2), 342-350.

Hwang, C. M. & Yang, M. S. (2013). New construction for similarity measures between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets. International Journal of Fuzzy Systems, 15(3).

Hung W. L. & Yang M. S. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern recognition letters, 25(14), 1603-1611.

Hussian, Z. & Yang, M. S. (2019). Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS. International Journal of Intelligent Systems, 34(10), 2633-2654.

Habib, A., Akram, M. & Kahraman, C. (2022). Minimum spanning tree hierarchical clustering algorithm: A new Pythagorean fuzzy similarity measure for the analysis of functional brain networks, Expert Systems with Applications, 201 117016.

Hussain, Z., Abbas, S., Rahman, S. & Hussain, R. (2022) Belief and plausibility measures on

Pythagorean fuzzy sets and its applications with BPI-VIKOR. Journal of Intelligent and Fuzzy Systems DOI:10.3233/JIFS-212098

Kaufman, L. R. & Rousseeuw, P. J. (1990). Finding groups in data: An introduction to cluster analysis. Hoboken NJ John Wiley & Sons Inc, 725.

Li, Z. & Lu, M. (2019). Some novel similarity and distance measures of Pythagorean fuzzy sets and their applications. Journal of Intelligent & Fuzzy Systems, 37(2), 1781-1799.

Liang, Z. & Shi, P. (2003). Similarity measures on intuitionistic fuzzy sets. Pattern recognition letters, 24(15), 2687-2693.

McBratney, A. B. & Odeh, I. O. (1997). Application of fuzzy sets in soil science: fuzzy logic, fuzzy measurements and fuzzy decisions. Geoderma, 77(2-4), 85-113.

Mitchell, H. B. (2003). On the Dengfeng–Chuntian similarity measure and its application to pattern recognition. Pattern Recognition Letters, 24(16), 3101-3104.

Pal, S. K. & King. R. A. (1980). Image enhancement using fuzzy set. Electronic letters, 16(10), 376-378.

Pan, L. Gao, X. Deng, Y. & Cheong, K. H. (2021). The constrained Pythagorean fuzzy sets and its similarity measure. IEEE Transactions on Fuzzy Systems.

Peng, X. & Garg. H, (2019). Multipara metric similarity measures on Pythagorean fuzzy sets with applications to pattern recognition. Applied Intelligence, 49(12), 4058-4096.

Peng, X. Yuan, H. & Yang, Y. (2017). Pythagorean fuzzy information measures and their applications. International Journal of Intelligent Systems, 32(10), 991-1029.

Ullah, K. Mahmood, T. Ali, Z. & Jan, N. (2020). On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex & Intelligent Systems, 6(1), 15-27.

Varol, B. P. & Aygun, H. (2012). Intuitionistic Fuzzy Metric Groups. International Journal of Fuzzy Systems, 14(3), 454-461.

Verma, R. Merigó, J. M. (2019). On generalized similarity measures for Pythagorean fuzzy sets and their applications to multiple attribute decision- making. International Journal of Intelligent Systems, 34(10), 2556-2583.

Xu, Y. & Wang, H. (2012). The induced generalized aggregation operators for intuitionistic fuzzy sets and their application in group decision making. Applied Soft Computing, 12(3), 1168-1179.

Xu, Z. S. & Chen, J. (2008). An overview of distance and similarity measures of intuitionistic fuzzy sets. International Journal of Uncertainty, Fuzziness and Knowledge Based Systems, 16(4), 529-555.

Yager, R. R. (2013) Pythagorean membership grades in multicriteria decision making. IEEE Transactions on Fuzzy Systems, 22(4), 958-965.

Yager, R. R. (2013, June). Pythagorean fuzzy subsets. In 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS), 7- 61.

Yager, R. R. & Abbasov, A. M. (2013). Pythagorean membership grades, complex numbers, and decision making. International Journal of Intelligent Systems, 28(5), 436-452.

Yang, M. S. & Hussain, Z. (2018). Fuzzy Entropy for Pythagorean Fuzzy Sets with Application to Multicriterion Decision Making. Complexity Volume 2018, Article ID 2832839, 14 pages. https://doi.org/10.1155/2018/2832839

Yang, M. S. & Hussain, Z. (2019). Distance and similarity measures of hesitant fuzzy sets based on Hausdorff metric with applications to multi-criteria decision making and clustering. Soft Computing, 23(14), 5835-5848.

Ye. J, (2009). Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. Expert systems with Applications, 36(3), 6899-6902.

Zadeh, L. A. (1965) Fuzzy Sets. Information and control, 8, 338-356.

Zeng, W. Li, D. & Yin, Q. (2018). Distance and similarity measures of Pythagorean fuzzy sets and their applications to multiple criteria group decision making. International Journal of Intelligent Systems, 33(11), 2236-2254.

Zhou, F. & Chen, T. Y. (2019). A novel distance measure for Pythagorean fuzzy sets and its applications to the technique for order preference by similarity to ideal solutions. International Journal of Computational Intelligence Systems, 12(2), 955-969.

Table

Table 1. Calculations of similarity measures of PFSs using Eqs. (1) - (9).

	1	2	3	4	5	6
G	$(0.7, 0.7)$	$(0.6, 0.5)$	$(0.2, 0.5)$	$(0.6, 0.6)$	$(0.6, 0.4)$	$(0.3, 0.4)$
H	$(0.8, 0.4)$	$(0.8, 0.4)$	$(0.4, 0.6)$	$(0.9, 0.2)$	$(0.5, 0.7)$	$(0.2, 0.2)$
$\tilde{S}_c$	0.7200	0.7050	0.7950	0.6150	0.9800	0.9650
$\tilde{S}_L$	0.6000	0.8025	0.8475	0.4425	0.9150	0.9325
$\tilde{S}_H$	0.8799	0.9527	0.9807	0.7964	0.8851	0.9843
$\tilde{S}_O$	0.6534	0.7825	0.8611	0.5487	0.6610	0.8749
$\tilde{S}_{DC}$	0.7600	0.8150	0.9950	0.6150	0.7800	0.9650
$\tilde{S}_s^p$	0.7200	0.8200	0.9000	0.6200	0.7800	0.900
$\tilde{S}_{HB}$	0.7300	0.8200	0.9000	0.6200	0.7800	0.9200
$\tilde{S}_s^p$	0.7300	0.1960	0.7200	0.6200	0.8100	0.9700
$\tilde{S}_h^p$	0.8800	0.9600	0.9600	0.9700	0.9600	0.6200

Table 2. Calculations of similarity measures of PFSs using Eqs. (10) - (18)

	1	2	3	4	5	6
G	$(0.7, 0.7)$	$(0.6, 0.5)$	$(0.2, 0.5)$	$(0.6, 0.6)$	$(0.6, 0.4)$	$(0.3, 0.4)$
H	$(0.8, 0.4)$	$(0.8, 0.4)$	$(0.4, 0.6)$	$(0.9, 0.2)$	$(0.5, 0.7)$	$(0.2, 0.2)$
$\tilde{S}_{lumc}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lumH}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lumL}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lumO}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lumDC}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lumHB}$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{leum}^p$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lsum}^p$	0.73	0.82	0.92	0.62	0.78	0.92
$\tilde{S}_{lhum}^p$	0.82	0.91	0.95	0.74	0.85	0.95

Table 3. Pythagorean fuzzy decision matrix

	C_1	C_2	C_3	C_4
A_1	$(0.60, 0.70)$	$(0.70, 0.30)$	$(0.50, 0.40)$	$(0.60, 0.60)$
A_2	$(0.50, 0.50)$	$(0.60, 0.50)$	$(0.40, 0.30)$	$(0.40, 0.70)$
A_3	$(0.80, 0.20)$	$(0.80, 0.30)$	$(0.40, 0.60)$	$(0.80, 0.40)$
A_4	$(0.70, 0.50)$	$(0.70, 0.70)$	$(0.70, 0.20)$	$(0.50, 0.60)$

Table 4. The normalized decision matrix

	C_1	C_2	C_3	C_4
A_1	$(0.60, 0.70)$	$(0.30, 0.70)$	$(0.50, 0.40)$	$(0.60, 0.60)$
A_2	$(0.50, 0.50)$	$(0.50, 0.60)$	$(0.40, 0.30)$	$(0.40, 0.70)$
A_3	$(0.80, 0.20)$	$(0.30, 0.80)$	$(0.40, 0.60)$	$(0.80, 0.40)$
A_4	$(0.70, 0.50)$	$(0.70, 0.70)$	$(0.70, 0.20)$	$(0.50, 0.60)$

Table 5. The matrix for criteria C_1

	A_1	A_2	A_3	A_4
A_1	0.0000	0.6000	–0.4500	–0.3098
A_2	–0.3098	0.0000	–0.3900	–0.3200
A_3	1.0600	0.9800	0.0000	0.7200
A_4	0.7700	0.7700	0.2890	0.0000

Table 6. The matrix for criteria C_2

	A_1	A_2	A_3	A_4
A_1	0.0000	0.8940	–0.3460	–0.5656
A_2	–0.3577	0.0000	–0.4730	–1.3856
A_3	0.7500	1.183	0.0000	1.4140
A_4	–0.6066	1.0900	–1.4140	0.0000

Table 7. The matrix for criteria C_3

	A_1	A_2	A_3	A_4
A_1	0.0000	0.9480	1.4140	–0.1690
A_2	–0.3794	0.0000	–0.6570	–0.7260
A_3	–0.5656	–0.6570	0.0000	–0.7260
A_4	1.4140	1.8160	1.8160	0.0000

Table 8. The matrix for criteria C_4

	A_1	A_2	A_3	A_4
A_1	0.0000	0.8660	–0.3860	0.6550
A_2	–32650	0.0000	–0.5059	–0.2633
A_3	0.9660	1.2640	0.0000	1.1400
A_4	–0.2422	0.6580	–0.4560	0.0000

Table 9. The overall dominance degree

	A_1	A_2	A_3	A_4	$\sum_{i=1}^{n} \phi_y (A_x, A_t)$
A_1	0.0000	3.3080	0.2320	–0.3894	3.1506
A_2	–1.3734	0.0000	–2.0259	–2.6949	–6.0942
A_3	2.2104	2.7700	0.0000	2.5480	7.5280
A_4	1.3352	4.3340	0.235	0.0000	5.9042

Table 10. Ranking of alternatives

	A_1	A_2	A_3	A_4
ε_i	0.6787	0.0000	1.0000	0.8800