Original Article

Asymptotic Approximation of the Norms of Monomials in Weighted Segal-Bargmann Spaces

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Abstract

We study radial weighted Segal-Bargmann spaces

$$H_0 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dz < \infty \right\},$$

$$H_1 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{|z|} e^{-|z|^2} dz < \infty \right\},$$

$$H_{-1} := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|} e^{-|z|^2} dz < \infty \right\}$$

and investigate the norms of monomials in these spaces. It is well-known that $\|z^k\|_0^2 = k!$. However, we cannot find a closed form the norms $\|z^k\|_1$ and $\|z^k\|_{-1}$. The purpose of this work is to establish an upper bound for $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$.

Keywords: weighted Segal-Bargmann, asymtotic, norm.

1. Introduction

The Segal-Bargmann space (also called a Fock space) is the holomorphic function space $HL^2(\mathbb{C},\mu)$ where $\mu(z)=\frac{1}{\pi}e^{-|z|^2}$. It is a Hilbert space of holomorphic functions on $\mathbb{C}$ with inner product given by

$$\langle f,g\rangle=\int_{\mathbb{C}} f(z)\overline{g(z)}\mu(z)dz.$$

(See (Bargmann, 1961), (Hall, 2000), (Le, 2017), (Soltani, 2006).) The norm of z^k in this space can be calculated using polar coordinates as follows:

$$\left\|z^k\right\|^2=\int_{\mathbb{C}}\left|z^k\right|^2\mu(z)dz=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}r^{2k+1}e^{-r^2}drd\theta=k!.$$

Therefore, the set $\left\{\frac{z^k}{\sqrt{k!}}\right\}_{k=0}^{\infty}$ forms an orthonormal basis for this space. (See (Hall, 2000).)

We commonly weight the measure by multiplying a nonnegative function in weighted Segal-Bargmann space or weighted Fock space. However, there are different varieties of these spaces. For example, the author of (Soltani, 2006) defined and investigated a weighted Fock space associated with the perturbed Dunkl operator. The inner product in this space is given by

$$\langle f,g\rangle_Q=\int_{\mathbb{C}} f_e(z)\overline{g_e(z)}dm_\alpha^Q(z)+2(\alpha+1)\int_{\mathbb{C}} f_o(z)\overline{g_o(z)}|z|^{-2}dm_{\alpha+1}^Q(z)$$

where $\alpha > -1/2$, $f_e(z) = \dfrac{f(z)+f(-z)}{2}$, $f_o(z) = \dfrac{f(z)-f(-z)}{2}$ and a measure $dm_\alpha^Q(z)$ associated with a function Q. In (Bergman, 2017) and (Escudero, Haimi & Romero, 2021), a weighted Fock space is defined as $HL^2\left(\mathbb{C}, e^{\phi(z)}\right)$ for some plurisubharmonic function $\phi(z)$. In (Choe & Nam, 2019), the t-weighted α-Fock space is introduced as a space consisting of all holomorphic functions f on $\mathbb{C}^n$ such that the integral

$$\int_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p \frac{1}{\left(1+|z|\right)^t} dV(z) < \infty$$

where $\alpha > 0, 0 < p < \infty$ and $dV(z)$ is the volume measure on $\mathbb{C}^n$. The radial weighted Segal-Bargmann space is the variant that we employ in this paper. For $h(z) := h(|z|)$, this weighted Segal-Bargmann space consists of all holomorphic functions on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \left| f(z) \right|^2 e^{-h(z)} dz < \infty .$$

(See (Baranov, Belov & Borichev, 2018).)

In this paper, we let $\mu(z) = \dfrac{1}{\pi} e^{-|z|^2}$ and denote the classical Segal-Bargmann space by $H_0 := HL^2(\mathbb{C}, \mu) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \| f \|_0^2 = \dfrac{1}{\pi} \int_{\mathbb{C}} \left| f(z) \right|^2 e^{-|z|^2} dz < \infty \right\}$.

By multiplying a positive function $\phi(z)$ to the measure $d\mu(z)$, we obtain another holomorphic function space $HL^2(\mathbb{C}, \phi\mu)$. This new space will be referred to as a weighted Segal-Bargmann space. To make use of polar coordinates as we compute the norm $\left\|z^k\right\|_0$, one may assume that the function ϕ is rotation invariant as $\phi=\phi(|z|)$. Since the function $\mu(z)=\frac{1}{\pi}e^{-|z|^2}$ depends only on $|z|$, the space $HL^2(\mathbb{C},\phi\mu)$ is a radial weighted Segal-Bargmann space.

Let $\phi_1=e^{|z|}$ and $\phi_{-1}=e^{-|z|}$. Then we define the spaces H_1 and H_{-1} as follows.

$$H_1 := HL^2(\mathbb{C},\phi_1\mu)=\left\{ f:\mathbb{C}\to\mathbb{C} \,\middle|\, \|f\|_1^2=\frac{1}{\pi}\int_{\mathbb{C}}|f(z)|^2 e^{|z|}e^{-|z|^2}dz<\infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C},\phi_{-1}\mu)=\left\{ f:\mathbb{C}\to\mathbb{C} \,\middle|\, \|f\|_{-1}^2=\frac{1}{\pi}\int_{\mathbb{C}}|f(z)|^2 e^{-|z|}e^{-|z|^2}dz<\infty \right\}.$$

Consider

$$\frac{1}{\pi}\int_{\mathbb{C}}\left|z^k\right|^2 e^{a|z|}e^{-|z|^2}dz=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} r^{2k+1}e^{ar-r^2}drd\theta$$

where $a=\pm1$. Now, the integral $\int r^{2k+1}e^{ar-r^2}dr$ no longer produces a basic function. However, the integral $\int_{\mathbb{C}}\left|z^k\right|^2 e^{a|z|}e^{-|z|^2}dz$ is still finite because the term $\mu(z)=e^{-|z|^2}$ dominates all other terms.

Despite the fact that the formula for $\left\|z^k\right\|_a^2 := \frac{1}{\pi}\int_{\mathbb{C}}\left|z^k\right|^2 e^{a|z|}e^{-|z|^2}dz$ is implicit, the behavior of the growth of $\left\|z^k\right\|_a^2$ in terms of k is remarkably similar to that of $\left\|z^k\right\|_0^2$. We shall show in Section 2 that the functions $r^{2k+1}e^{-r^2}$, $r^{2k+1}e^{r-r^2}$ and $r^{2k+1}e^{-r-r^2}$ are all concentrated towards the peaks of these functions. As a result, the norm $\|z^k\|_0^2$, $\|z^k\|_1^2$ and $\|z^k\|_{-1}^2$ can be approximated asymptotically by definite integrals.

In (Chailuek & Senmoh, 2020), the authors shows that the boundedness of $\dfrac{\|z^k\|_\alpha^2 \|z^k\|_\beta^2}{\|z^k\|_\gamma^4}$ plays an important role in a proof of the dual of a generalized Bergman spaces, $HL^2\left(\mathrm{B}^d,\alpha\right)^* = HL^2\left(\mathrm{B}^d,\beta\right)$ under the integral pairing

$$\left\langle f,g\right\rangle_\gamma = \int_{\mathrm{B}^d} f\left(z\right)\overline{g\left(z\right)}c_\lambda\left(1-|z|^2\right)^{\lambda-(d+1)}dz$$

for $f \in H\left(B^d,\alpha\right)$, $g \in H\left(B^d,\beta\right)$.

Despite the fact that the formulas for $\|z^k\|_1^2$ and $\|z^k\|_{-1}^2$ are implicit , we will show in Section 3 that $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically less than a constant.

2. Norms of Monomials in Segal-Bargmann Spaces

In the classical Segal-Bargamann space, the norm of a monomial can be computed explicitly as $\|z^k\|_0^2 = \dfrac{1}{\pi}\int_0^{2\pi}\int_0^\infty r^{2k+1}e^{-r^2}drd\theta = 2\int_0^\infty r^{2k+1}e^{-r^2}dr = k!$. Consider the graph of $f_k(r) = r^{2k+1}e^{-r^2}$. It resembles a Gaussian-shaped wave function that propagates to the right as k increases. (See figure 1.)

In this section, we will show that the function f_k behaves like a Gaussian-shaped wave function in the sense that it is concentrated towards its peak and likely to have a finite width which is measured from where the function is somehow cut off. Consequently, the integral $\int_0^{\infty} r^{2k+1}e^{-r^2}dr$ can be estimated by a definite integral $\int_0^{\infty} r^{2k+1}e^{-r^2}dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}dr$ for some $r_0 > 0$.

As previously stated, explicit formulas for $\left\|z^k\right\|_1$ and $\left\|z^k\right\|_{-1}$ are unavailable. However, when we compare the graphs of $f_{k,-1}(r) = r^{2k+1}e^{-r-r^2}$

Figure 1 The graphs of $f_k\left(r\right) = r^{2k+1}e^{-r^2}$ for different k's.

and $f_{k,1}(r) = r^{2k+1}e^{r-r^2}$ to that of $f_k(r) = r^{2k+1}e^{-r^2}$. We can see that they are similarly concentrated towards their peaks and have finite widths. (See figure 2.)

Figure 2 The graphs of $f_{k,-1}\left(r\right)$, $f_{k,1}\left(r\right)$ and $f_k\left(r\right)$.

So it makes sense to estimate those integrals by definite integrals. Therefore, the goals of this section are to compare $\left\|z^k\right\|_{-1}^2$ and $\left\|z^k\right\|_1^2$ with $\left\|z^k\right\|_0^2$ as we obtain

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr} \text{ and } \frac{\left\|z^k\right\|_1^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}$$

for some $\hat{r}_0, \tilde{r}_0 > 0$. We begin by generating some relevant identities as follows.

Lemma 2.1. $\left\| z^k \right\|_0^2 = k!$ where k is a nonnegative integer.

Proof. We compute $\left\| z^k \right\|_0^2$ by induction on k. For $k = 0$,

$$\int_0^\infty r e^{-r^2} dr = -\frac{1}{2} \lim_{t \to \infty} e^{-r^2} \Big|_0^t = \frac{1}{2}.$$

For $k \ge 1$, integrating by parts gives

$$\begin{aligned}\int_0^\infty r^{2k+1} e^{-r^2} dr &= -\int_0^\infty \left(2kr^{2k-1}\right)\left(-\frac{e^{-r^2}}{2}\right) dr \\ &= k \int_0^\infty r^{2(k-1)+1} e^{-r^2} dr.\end{aligned}$$

Therefore, $\int_0^\infty r^{2k+1} e^{-r^2} dr = \frac{k!}{2}$ and hence $\left\| z^k \right\|_0^2 = k!$.

Lemma 2.2. For a nonnegative integer n and $a, b > 0$.

$$\int_0^b x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \left(1 - e^{-ab} \sum_{i=0}^{n} \frac{(ab)^i}{i!}\right). \tag{2.1}$$

Proof. Integration by parts gives

$$\begin{aligned}\int_0^b x^n e^{-ax} dx &= -\frac{x^n e^{-ax}}{a} - \frac{n x^{n-1} e^{-ax}}{a^2} - \cdots - \frac{n! x e^{-ax}}{a^n} - \frac{n! e^{-ax}}{a^{n+1}} \Bigg|_0^b \\ &= \frac{n!}{a^{n+1}} \left(1 - e^{-ab} \sum_{i=0}^{n} \frac{(ab)^i}{i!}\right).\end{aligned}$$

Lemma 2.3. For $r_0 = \sqrt{\dfrac{2k+1}{2}}, \lim_{k\to\infty} e^{-4r_0^2} \sum_{i=0}^{k} \dfrac{\left(4r_0^2\right)^i}{i!} = 0$.

Proof. For $i = 0,1,2,\ldots,k$, we have $i+1 < 4k+2$ for all positive integer k.

Thus $\dfrac{(4r_0^2)^i}{i!} < \dfrac{(4r_0^2)^{i+1}}{(i+1)!}$ and hence

$$0 < e^{-4r_0^2} \sum_{i=0}^{k} \frac{\left(4r_0^2\right)^i}{i!} < e^{-4r_0^2}(k+1)\frac{\left(4r_0^2\right)^k}{k!} = e^{-(4k+2)}(k+1)\frac{(4k+2)^k}{k!}.$$

It's not difficult to understand that the last quantity's limit is zero.

Next, we will show that $\left\|z^k\right\|_0^2$ is asymptotically equal to a definite integral as follows.

Proposition 2.4. Let $k = 0,1,2,3,\ldots$ and $r_0 = \sqrt{\dfrac{2k+1}{2}}$ be the critical point of $f_k(r) = r^{2k+1}e^{-r^2}$. Then $\left\|z^k\right\|_0^2 \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}dr$.

Proof. From Lemma 2.1, we obtain

$$\int_0^{\infty} r^{2k+1}e^{-r^2}dr = \frac{k!}{2}. \qquad (2.2)$$

Substitute $n = k, a = 1$, and $b = 4r_0^2$ in the equation (2.1), we obtain

$$\int_0^{2r_0} r^{2k+1}e^{-r^2}dr = \frac{1}{2}\int_0^{4r_0^2} s^k e^{-s}ds = \frac{k!}{2}\left(1 - e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}\right). \qquad (2.3)$$

From equations (2.2) and (2.3), we obtain

$$\frac{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}{\int_0^{\infty} r^{2k+1}e^{-r^2}dr} = 1 - e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}.$$

From Lemma 2.3, we obtain

$$\lim_{k\to\infty} e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!} = 0.$$

Thus,

$$\lim_{k\to\infty}\frac{\int_0^{2r_0} r^{2k+1}e^{-r^2}dr}{\int_0^{\infty} r^{2k+1}e^{-r^2}dr} = \lim_{k\to\infty}\left(1 - e^{-4r_0^2}\sum_{i=0}^{k}\frac{\left(4r_0^2\right)^i}{i!}\right) = 1.$$

Therefore, $\int_0^{\infty} r^{2k+1}e^{-r^2}dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}dr$. Hence, $\left\|z^k\right\|_0^2 = 2\int_0^{\infty} r^{2k+1}e^{-r^2}dr \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}dr$.

Recall that $\left\|z^k\right\|_1^2 = 2\int_0^{\infty} r^{2k+1}e^{r-r^2}dr$ and $\left\|z^k\right\|_{-1}^2 = 2\int_0^{\infty} r^{2k+1}e^{-r-r^2}dr$.

Although we can derive the closed form of the integral $\int_0^{\infty} r^{2k+1}e^{-r^2}dr$ using integration by substitution and induction, there is no elementary function whose derivative is $r^{2k+1}e^{-r-r^2}$ or $r^{2k+1}e^{r-r^2}$. The function $r^{2k+1}e^{-r-r^2}$ or $r^{2k+1}e^{r-r^2}$ behave similar to the function $f_k\left(r\right) = r^{2k+1}e^{-r^2}$ when k is fixed.

As a result, we focus our attention on the asymptotic approximation of $\left\|z^k\right\|_1^2 \big/ \left\|z^k\right\|_0^2$ and $\left\|z^k\right\|_{-1}^2 \big/ \left\|z^k\right\|_0^2$.

Proposition 2.5. Let $k=0,1,2,3,\ldots$. Then

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \quad \text{and} \quad \frac{\left\|z^k\right\|_{1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}$$

where $r_0=\sqrt{\dfrac{2k+1}{2}}$, $\hat{r}_0=\dfrac{-1+\sqrt{16k+9}}{4}$ and $\tilde{r}_0=\dfrac{1+\sqrt{16k+9}}{4}$ are the critical points of $f_k(r)=r^{2k+1}e^{-r^2}$, $f_{k,-1}(r)=r^{2k+1}e^{-r-r^2}$ and $f_{k,1}(r)=r^{2k+1}e^{r-r^2}$, respectively.

Proof. Consider

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2} \sim \frac{\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} + \frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}.$$

Since $0<r^{2k+1}e^{-r-r^2}<r^{2k+1}e^{-r^2}$,

$$\frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \le \frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} = \frac{\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr - \int_0^{2\hat{r}_0} r^{2k+1}e^{-r^2}\,dr}{\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}.$$

By using integration by substitution and substituting $n=k$, $a=1$, and $b=4r_0^2$ into the equation (2.1), we obtain

$$\int_0^{2\hat{r}_0} r^{2k+1}e^{-r^2}\,dr = \frac{k!}{2}\left(1-e^{-4\hat{r}_0^2}\sum_{i=0}^{k}\frac{\left(4\hat{r}_0^2\right)^i}{i!}\right). \tag{2.4}$$

From equations (2.2), (2.3) and (2.4), we obtain

$$\lim_{k\to\infty}\frac{\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}dr}{\int_{0}^{2r_0} r^{2k+1}e^{-r^2}dr}=0\,.$$

Therefore,

$$\frac{\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^2}\sim\frac{\int_{0}^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\int_{0}^{2r_0} r^{2k+1}e^{-r^2}dr}\,.$$

Now, consider

$$\frac{\left\|z^k\right\|_{1}^2}{\left\|z^k\right\|_0^2}\sim\frac{\int_{0}^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}dr}{\int_{0}^{2r_0} r^{2k+1}e^{-r^2}dr}+\frac{\int_{2\tilde{r}_0}^{\infty} r^{2k+1}e^{r-r^2}dr}{\int_{0}^{2r_0} r^{2k+1}e^{-r^2}dr}\,.$$

Let r be an element in an interval $\left(2\tilde{r}_0,\infty\right)$. The function e/e^r is decreasing and $e/e^r\to 0$ as $r\to\infty$; on the other hand, the function $\left(r-1\right)^{2k+1}/r^{2k+1}$ is increasing and $\left(r-1\right)^{2k+1}/r^{2k+1}\to 1$ as $r\to\infty$. Consider $r=2r_0$. We see that $\frac{e}{e^{2r_0}}\leq\left(\frac{2r_0-1}{2r_0}\right)^{2k+1}$ for all k. Thus, we obtain $\frac{e}{e^r}\leq\left(\frac{r-1}{r}\right)^{2k+1}$ and hence $r^{2k+1}\leq\left(r-1\right)^{2k+1}e^{r-1}$ for all $r\geq 2\tilde{r}_0$.

Therefore,

$$\int_{2\tilde{r}_0}^{\infty} r^{2k+1}e^{r-r^2}dr\leq\int_{2\tilde{r}_0}^{\infty}\left(r-1\right)^{2k+1}e^{-\left(r-1\right)^2}dr\,.$$

By using integration by substitution and equations (2.1) and (2.2), we have

$$\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2} dr = \frac{k!}{2} - \frac{k!}{2}\left(1 - e^{-(2\tilde{r}_0-1)^2} \sum_{i=0}^{k} \frac{(2\tilde{r}_0-1)^{2i}}{i!}\right). \qquad (2.5)$$

From equations (2.3) and (2.5), we obtain

$$\lim_{k\to\infty} \frac{\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr} = 0.$$

Therefore, we obtain $\dfrac{\|z^k\|_1^2}{\|z^k\|_0^2} \sim \dfrac{\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr}{\int_0^{2r_0} r^{2k+1} e^{-r^2} dr}$.

3. The boundedness of $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$

It is easy to see that $\|z^k\|_{-1}^2 \le \|z^k\|_0^2 \le \|z^k\|_1^2$. This implies that the ratio $\|z^k\|_{-1}^2 / \|z^k\|_0^2$ may decrease, whilst the ratio $\|z^k\|_1^2 / \|z^k\|_0^2$ may increase. We shall demonstrate in this section that these two quantities are mutually compensated resulting in the boundedness of $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$. In addition, the upper bound is involved in the peaks of f_k, $f_{k,-1}$ and $f_{k,1}$. Since $\hat{r}_0 \sim r_0 \sim \tilde{r}_0$ and $|\tilde{r}_0 - r_0| \sim |r_0 - \hat{r}_0|$, it should come as no surprise that the values $f_k(r_0)$, $f_{k,-1}(\hat{r}_0)$ and $f_{k,1}(\tilde{r}_0)$ are somehow offset.

Theorem 3.1. Let $k = 0,1,2,3,\ldots$. Then

$$\frac{\left\|z^k\right\|_1^2\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4} \sim e^{\frac{1}{4}}.$$

Proof. From the previous section, we have

$$\frac{\left\|z^k\right\|_1^2\left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4} \sim \frac{\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}dr\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}dr}{\left(\int_0^{2r_0} r^{2k+1}e^{-r^2}dr\right)^2}. \qquad (2.6)$$

First, we consider the definite integral

$$\int_0^{2r_0} r^{2k+1}e^{-r^2}dr = \int_0^{2r_0} e^{-r^2+(2k+1)\ln r}dr = \int_0^{2r_0} e^{f(r)}dr$$

where $f(r) = -r^2 + (2k+1)\ln r$.

The Taylor series expansion of $f(r)$ about a point $r = r_0$ is given by

$$f(r) = \sum_{n=0}^{\infty}\frac{f^n(r_0)}{n!}(r-r_0)^n$$

with the interval of convergence $(0, 2r_0)$. Thus,

$$\int_0^{2r_0} e^{f(r)}dr = \int_0^{2r_0} e^{f(r_0)+f'(r_0)(r-r_0)+\frac{f''(r_0)(r-r_0)^2}{2!}+\sum_{n=3}^{\infty}\frac{f^n(r_0)}{n!}(r-r_0)^n}dr.$$

We have $f'(r_0) = 0$ and $f''(r_0) = -4$. If we consider $k \to \infty$, then $f^m(r_0) \to 0$ for all $m \geq 3$. Therefore,

$$\int_0^{2r_0} e^{f(r)}dr = e^{f(r_0)}\int_0^{2r_0} e^{-2(r-r_0)^2}dr = e^{f(r_0)}\int_{-r_0}^{r_0} e^{-2u^2}du \qquad (2.7)$$

where $u = r - r_0$.

Next, we consider the definite integral

$$\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2} dr = \int_0^{2\tilde{r}_0} e^{r-r^2+(2k+1)\ln r} dr = \int_0^{2\tilde{r}_0} e^{\tilde{f}(r)} dr$$

where $\tilde{f}(r) = r - r^2 + (2k+1)\ln r$.

Similarly, we have

$$\int_0^{2\tilde{r}_0} e^{\tilde{f}(r)} dr \sim e^{\tilde{f}(\tilde{r}_0)} \int_0^{2\tilde{r}_0} e^{-2(r-\tilde{r}_0)^2} dr = e^{\tilde{f}(\tilde{r}_0)} \int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2} d\tilde{u} \qquad (2.8)$$

where $\tilde{u} = r - \tilde{r}_0$ and

$$\int_0^{2\hat{r}_0} e^{\hat{f}(r)} dr \sim e^{\hat{f}(\hat{r}_0)} \int_0^{2\hat{r}_0} e^{-2(r-\hat{r}_0)^2} dr = e^{\hat{f}(\hat{r}_0)} \int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2} d\tilde{u} \qquad (2.9)$$

where $\hat{f}(r) = -r - r^2 + (2k+1)\ln r$ and $\hat{u} = r - \hat{r}_0$.

Observe that $r_0 \sim \tilde{r}_0 \sim \hat{r}_0$ as $k \to \infty$. Thus,

$$\int_{-r_0}^{r_0} e^{-2u^2} du \sim \int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2} d\tilde{u} \sim \int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2} d\hat{u}.$$

Therefore,

$$\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4} \sim e^{\tilde{f}(\tilde{r}_0)+\hat{f}(\hat{r}_0)-2f(r_0)}.$$

Next, we compute

$$2f(r_0) = -2k - 1 + (2k+1)\ln\left(k + \frac{1}{2}\right)$$

and

$$\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) = \frac{1}{4} - 2k - 1 + (2k+1)\ln\left(k + \frac{1}{2}\right).$$

Therefore $\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) = 2f(r_0) + \frac{1}{4}$. This yields

$$\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4} \sim e^{\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) - 2f(r_0)} = e^{\frac{1}{4}}.$$

Finally, we obtain that $\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4}$ is asymptotically less than a constant $e^{\frac{1}{4}}$.

We notice that the estimation in (2.7), (2.8) and (2.9) looks similar to the integral asymptotic $\int_a^b f(t)e^{-\lambda g(t)}dt \sim e^{-\lambda g(c)} f(c)\sqrt{\frac{2\pi}{\lambda g''(c)}}$ as $\lambda \to \infty$ where c represents the critical point of g. Using Taylor's expansion and Laplace's method, the integral involves in the value at the critical point.

Conclusion

In this paper, we obtained that $\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4}$ is asymptotically less than a constant $e^{\frac{1}{4}}$. It implies that $\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4}$ is bounded and independent of k. Future research could use the boundedness of $\frac{\left\|z^k\right\|_1^2 \left\|z^k\right\|_{-1}^2}{\left\|z^k\right\|_0^4}$ to prove the dual of reciprocal weighted Segal-Bargmann spaces, $H_1{}^* = H_{-1}$ under the integral pairing

$$\langle F, S\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{S(z)}e^{-|z|^2}dz$$

where $F \in H_1$ and $S \in H_{-1}$.

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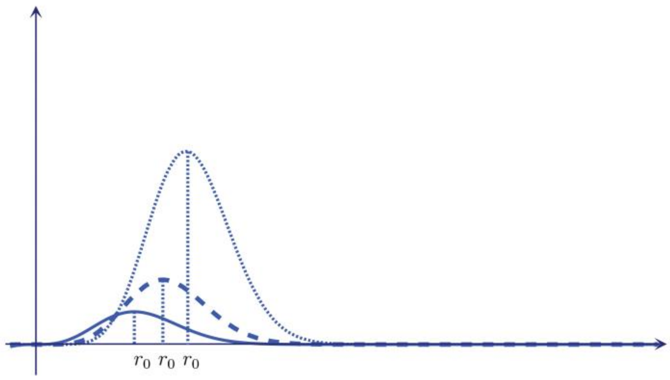

Figure 1 The graphs of $f_k(r) = r^{2k+1}e^{-r^2}$ for different k's.

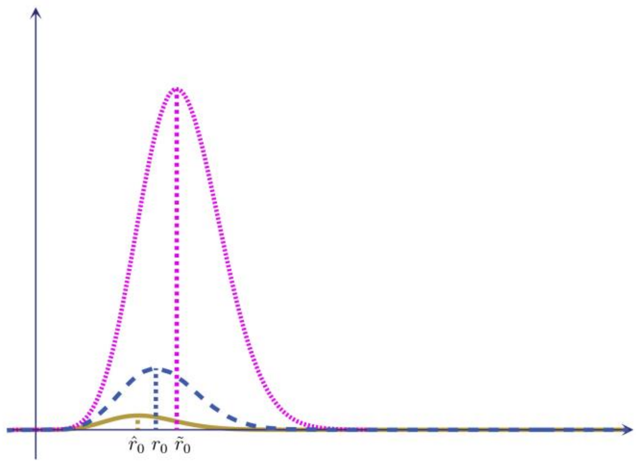

Figure 2 The graphs of $f_{k,-1}(r)$, $f_{k,1}(r)$ and $f_k(r)$.