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*Original Article*

# The zero inflated negative binomial – Crack distribution: some properties and parameter estimation

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**Abstract**

The zero inflated negative binomial-Crack (ZINB-CR) distribution is a mixture of Bernoulli distribution and negative binomial-Crack (NB-CR) distribution, which is an alternative distribution for the excessive zero counts and overdispersion. In this paper, some properties of the ZINB-CR distribution are discussed. Statistical inference of the parameters is derived by maximum likelihood estimation (MLE) and the method of moments (MM). Monte Carlo Simulations are used to evaluate the performance of parameter estimation methods in term of mean squared error (MSE). An application of the distribution is carried out on a sample of excess zero-count data. Simulation results show that the MLE method outperforms the MM method in specific parameter values. Furthermore, the ZINB-CR provides a better fit compared to the zero inflated Poisson (ZIP), the zero inflated negative binomial (ZINB) and the negative binomial-Crack (NB-CR) distributions.

**Keywords:** zero inflated negative binomial-Crack distribution, count data, overdispersion, excessive zero, maximum likelihood estimation, method of moments

## 1. Introduction

In many situations, count data are characterized by overdispersion and excess zeros. Consequently, researchers have attempted to develop a probabilistic model that is more flexible than the Poisson and the negative binomial (NB) distributions. Mixed-distribution models. In particular, zero inflated (ZI) count models, are defined as alternative models which often have desirable properties for modeling count data with excess zeros. The ZI count models contain a mixture of a point mass at zero and an untruncated count distribution, for instance, the zero inflated Poisson (ZIP) and zero inflated negative binomial (ZINB) distributions (Neelon *et al.*, 2010).

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The distribution of ZIP was introduced by Lambert (1992), who applied a logit model in order to capture the influence of covariates on the probability of excess zeros. The ZIP can be used efficiently in the model for count data with excess zeros only if the data are without overdispersion. In practice, count data are often overdispersed, which will result in severe bias in parameter estimates for the ZIP distribution, and an alternative distribution, such as the ZINB distribution, is more appropriate (Greene, 2010; Ridout *et al.*, 1998; Phang and Ong, 2006; Doyle, 2009). Later, Yip and Yau (2005) discussed several parametric zero-inflated count distributions including ZIP, ZINB, zero inflated generalized Poisson (ZIGP), and zero inflated double Poisson (ZIDP) distribution to incorporate the situation of the excess zeros for insurance claim count data, and subsequently Famoye and Singh (2006) developed a ZIGP model with an application to domestic violence data.

In this work, we propose an alternative distribution for the excessive zero counts and overdispersion, the zero inflated negative binomial-Crack (ZINB-CR) distribution, which is obtained by mixing Bernoulli distribution and negative binomial-Crack (NB-CR) distribution. The latter distribution is another discrete distribution for count data which is a mixture of the NB and Crack distributions. The Crack distribution is a mixture of inverse Gaussian (IG) and length biased inverse Gaussian (LBIG) distributions, which has recently been studied by Jorgensen *et al*. (1991), Balakrishnan *et al*. (2009) and Bowonrattanaset and Budsaba (2011). This formulation distribution provides powerful and popular tools for generating flexible distributions with attractive statistical and probabilistic properties because the pdf of the Crack distribution, $f_{CR}\left(x;\lambda,\theta,\gamma\right)$, is expressed by the mixture between IG distribution and LBIG distribution as the form: $f_{CR}\left(x;\lambda,\theta,\gamma\right)=\left(1-\gamma\right)f_{IG}\left(x;\lambda,\theta\right)+\gamma f_{LBIG}\left(x;\lambda,\theta\right)$ for $x>0$, where $0<\gamma<1$ is a mixing parameter, $f_{IG}(x;\lambda,\theta)$ and $f_{LBIG}(x;\lambda,\theta)$ are the densities of the IG and LBIG distributions, respectively. The Crack distribution can be useful for statistical analysis in prospective studies connected with the engineering problem of a fatigue crack development in a metalic plate under some kind of pressure loading.

The NB-CR distribution represents generalization of distribution for count data, including the negative binomial-inverse Gaussian (NB-IG), negative binomial-Birnbaum-Saunders (NB-BS) and negative binomial-length biased inverse Gaussian (NB-LBIG). Saengthong and Bodhisuwan (2013) showed that the NB-CR distribution provided a better fit compared to the Poisson and the NB distributions.

This paper is organized as follows: in Section 2, we consider the characteristics, an algorithm for generating random data and the parameter estimation of the ZINB-CR distribution. In Sections 3 and 4, a simulation study is carried out to evaluate the performance of parameter estimation methods, and the usefulness of the ZINB-CR distribution is illustrated by a real data set. Finally, conclusions are presented in Section 5.

## 2. The Zero Inflated Negative Binomial-Crack Distribution

### 2.1 Zero inflated count models

The probability mass function (pmf) of zero inflated count models can be written as

$$g(x)=\begin{cases}\omega+(1-\omega)h\left(0\right) & ,x=0\\ (1-\omega)h\left(x\right) & ,x=1,2,\ldots,\end{cases} \tag{1}$$

where $\omega$ is a zero inflation parameter $\left(0<\omega<1\right)$, and $h(.)$ is the pmf of the parent count model.

Now, we consider zero inflated count models corresponding to the Poisson and the NB distributions. For the ZIP distribution, which is a mixing of Bernoulli distribution and Poisson distribution, the pmf has the form

$$g(x)=\begin{cases}\omega+(1-\omega)\exp\left(-\lambda\right) & ,x=0\\ (1-\omega)\dfrac{\exp\left(-\lambda\right)\lambda^{x}}{x!} & ,x=1,2,\ldots,\end{cases} \tag{2}$$

where $\lambda>0$ and $0<\omega<1$.

Another mixed probability distribution related to the ZIP distribution is the ZINB distribution. The ZINB is a mixture of Bernoulli and NB distributions. The pmf for the ZINB is given by

$$g(x)=\begin{cases}\omega+(1-\omega)p^{r} & ,x=0\\ (1-\omega)\dbinom{r+x-1}{x}p^{r}(1-p)^{x} & ,x=1,2,\ldots,\end{cases} \tag{3}$$

where $r>0, 0<p<1$ and $0<\omega<1$.

### 2.2 A new zero inflated distribution

As mentioned earlier, the ZINB distribution have been developed to cope with zero inflated outcome data with overdispersion. In order to provide another competition to the ZINB distribution, a new mixed distribution is considered. We propose the ZINB-CR distribution, which is a mixture of Bernoulli and NB-CR distributions. We first provide a general definition of the NB-CR distribution which will subsequently reveal its pmf.

**Definition 1:**

Let $X$ be a random variable which follows the negative binomial-Crack distribution with parameters $r$, $\lambda$, $\theta$ and $\gamma$, $X\sim$ NB-CR$(r,\lambda,\theta,\gamma)$, when $X$ has a NB distribution with parameters $r>0$ and $p=\exp(-a)$ where $a$ is distributed as CR with the positive parameters $\lambda,\theta$ and $\gamma$, i.e., $X|a\sim$ NB $\left(r,p=\exp\left(-a\right)\right)$ and $a\sim$ CR$(\lambda,\theta,\gamma)$.

The next theorem provides a closed form for the pmf and for the factorial moments.

**Theorem 1:**

If $X\sim$ NB-CR$(r,\lambda,\theta,\gamma)$, then the pmf and the factorial moments of order $k$ of the distribution are given as (4) and (5) respectively:

$$h\left(x\right)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}\left(-1\right)^{j}\frac{\exp\left(\lambda\left(1-\sqrt{1+2\theta\left(r+j\right)}\right)\right)}{\sqrt{1+2\theta\left(r+j\right)}}\times\left(1-\gamma\left(1-\sqrt{1+2\theta\left(r+j\right)}\right)\right),\qquad x=0,1,2,..., \tag{4}$$

and

$$\mu_{(k)}\left(X\right)=\frac{\Gamma\left(r+k\right)}{\Gamma\left(r\right)}\sum_{j=0}^{k}\binom{k}{j}\left(-1\right)^{j}\frac{\exp\left(\lambda\left(1-\sqrt{1-2\theta\left(k-j\right)}\right)\right)}{\sqrt{1-2\theta\left(k-j\right)}}\times\left(1-\gamma\left(1-\sqrt{1-2\theta\left(k-j\right)}\right)\right),\qquad x=0,1,2,..., \tag{5}$$

where $r,\lambda,\theta > 0$ and $0 \le \gamma \le 1$.

***Proof:*** (see Saengthong and Bodhisuwan (2013)).

**Definition 2:**

Let $X$ be a random variable of NB-CR distribution, with pmf $h(x)$ defined as in (4) and $g(x)$ in (1). Then $g(x)$ is a pmf of ZINB-CR distribution with parameters $r,\lambda,\theta,\gamma$ and $\omega$, $X \sim$ ZINB-CR$(r,\lambda,\theta,\gamma,\omega)$.

**Theorem 2:**

If $X \sim$ ZINB-CR$(r,\lambda,\theta,\gamma,\omega)$, the pmf of the distribution is given by

$$g(x)=\begin{cases}\omega+(1-\omega)\dfrac{\exp\left(\lambda\left(1-\sqrt{1+2\theta r}\right)\right)\left(1-\gamma\left(1-\sqrt{1+2\theta r}\right)\right)}{\sqrt{1+2\theta r}}, & x=0\\[2ex] (1-\omega)\dbinom{r+x-1}{x}\displaystyle\sum_{j=0}^{x}\dbinom{x}{j}(-1)^j\dfrac{\exp\left(\lambda\left(1-\sqrt{1+2\theta(r+j)}\right)\right)}{\sqrt{1+2\theta(r+j)}} & \\ \qquad\times\left(1-\gamma\left(1-\sqrt{1+2\theta(r+j)}\right)\right), & x=1,2,\ldots,\end{cases} \tag{6}$$

where $r,\lambda,\theta > 0, 0 \le \gamma \le 1$ and $0 < \omega < 1$.

***Proof:*** By Definition 2, substitute (4) into (1), then the pmf for the ZINB-CR distribution can be expressed as (6).

Some specified parameters of ZINB-CR distribution and their graphs are displayed in Figure 1. From the graphs it can be shown that the distribution has a positive skew and tends to be flatter when $\omega$ is increasing.

**Theorem 3:**

If $X \sim$ ZINB-CR$(r,\lambda,\theta,\gamma,\omega)$, some basic properties are:

(a) The $k^{th}$ moment of $X$ is given by

$$E(X^k)=(1-\omega)\sum_{x=1}^{\infty}x^k\left(\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{\exp\left(\lambda\left(1-\sqrt{1+2\theta(r+j)}\right)\right)}{\sqrt{1+2\theta(r+j)}}\right.$$
$$\left.\times\left(1-\gamma\left(1-\sqrt{1+2\theta(r+j)}\right)\right)\right). \quad k=1,2,\ldots \tag{7}$$

(b) The moment generating function of $X$ is given by

$$M_X(t)=\left(\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-\sqrt{1+2\theta r}\right)\right)\left(1-\gamma\left(1-\sqrt{1+2\theta r}\right)\right)}{\sqrt{1+2\theta r}}\right)$$
$$+(1-\omega)\sum_{x=1}^{\infty}\left(\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{\exp\left(\lambda tx\left(1-\sqrt{1+2\theta(r+j)}\right)\right)}{\sqrt{1+2\theta(r+j)}}\right.$$

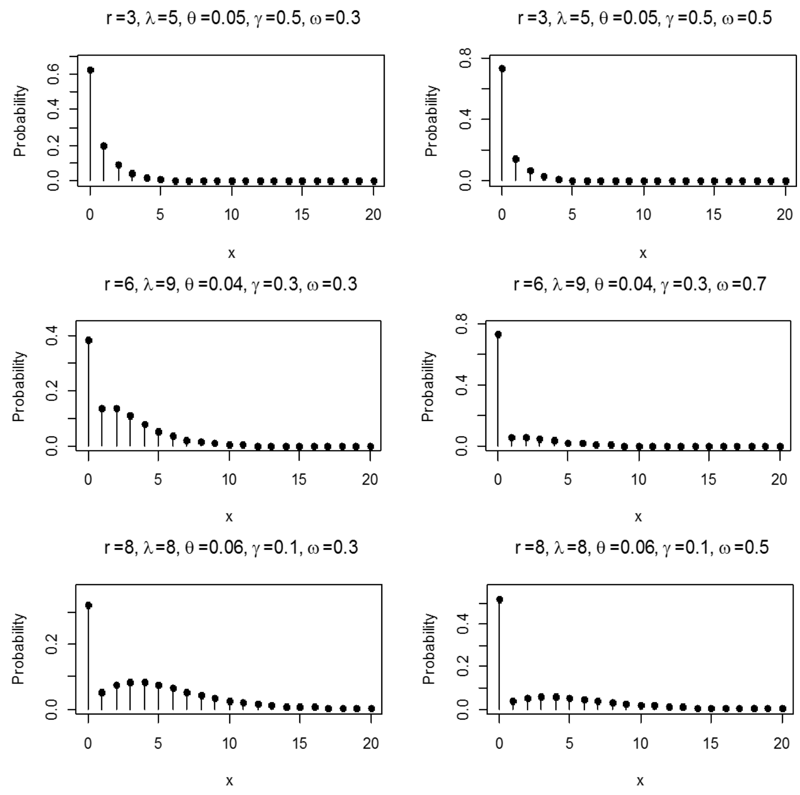


Figure 1. Some examples of probability mass functions of the ZINB-CR random variable with some specified values of $r,\lambda,\theta,\gamma$ and $\omega$.

$$\left.\times\left(1-\gamma\left(1-\sqrt{1+2\theta(r+j)}\right)\right)\right]. \tag{8}$$

(c) The mean and variance are given by

$$E(X)=(1-\omega)r\left(\frac{(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}-1\right), \tag{9}$$

and

$$Var(X)=(1-\omega)\left(\frac{(r^2+r)(1-\gamma(1-\varsigma))\exp(\lambda(1-\varsigma))}{\varsigma}\right.$$

$$\left.-\frac{r(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}\right)$$

$$-\omega(1-\omega)\left(\frac{2r^2(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}-r^2\right)$$

$$-\left(\frac{(1-\omega)r(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}\right)^2, \tag{10}$$

where $\delta=\sqrt{1-2\theta}$ and $\varsigma=\sqrt{1-4\theta}$.

### 2.3 Algorithm for generating random data

The objective of this part is to generate a random variable $X$ from the ZINB-CR $(r,\lambda,\theta,\gamma,\omega)$ distribution using the following step.

1) Fix the parameters $\lambda>0, \theta>0$ and $0\le\gamma\le 1$.

2) We use Acceptance-Rejection Method to generate random numbers from Crack distribution since the functional form of this distribution makes it difficult to generate random numbers using direct or inversion methods.

First, we generate a random number $\alpha$ from standard normal distribution, N(0,1). Next, we obtain a BS-distributed random variate by using

$$b=\theta\left(\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}+\lambda}\right)^2.$$

3) By the Acceptance-Rejection Method, we use BS distribution to generate random variable with Crack distribution. Hence, we have to calculate BS distribution from the density function for this distribution as

$$f_{BS}(x;\lambda,\theta)=\frac{1}{2\theta\sqrt{2\pi}}\left[\lambda\left(\frac{\theta}{x}\right)^{3/2}+\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right].$$

4) Calculate Crack distribution from the density function for this distribution as

$$f_{CR}(x;\lambda,\theta,\gamma)=\frac{1}{\theta\sqrt{2\pi}}\left[\gamma\lambda\left(\frac{\theta}{x}\right)^{3/2}+(1-\gamma)\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right].$$

5) Generate $U_1$ from uniform distribution, U(0,1).

6) If $U_1<\dfrac{f_{CR}(x;\lambda,\theta,\gamma)}{cf_{BS}(x;\lambda,\theta)}$ where $c=2\max[\gamma,(1-\gamma)]$, then set $a=b$, otherwise go to step 2.

7) Generate $Y$ from NB$(r, p=\exp(-a))$ distribution.

8) Generate $U_2$ from uniform distribution, U(0,1).

9) If $U_2\le\omega$, then set $X=0$, otherwise set $X=Y$.

Figure 2 show the example of the theoretical pmf of ZINB-CR distribution and the experimental pmf from the ZINB-CR distributed random number with $r=6, \lambda=9$, $\theta=0.04, \gamma=0.3, \omega=0.3$ and samples of size n = 50, 200 and 500. By looking at the pattern, we can see that the experimental pmf with large n will be very close to the theoretical pmf of ZINB-CR distribution.

### 2.4 Parameter estimation

In this part, the estimation of parameters for ZINB-CR $(r,\lambda,\theta,\gamma,\omega)$ via the maximum likelihood estimation and the method of moments are provided.

#### 2.4.1 Maximum likelihood estimation (MLE)

The estimation of parameters for ZINB-CR distribution via MLE method procedure will be discussed. For convenience we let $Z_j=\sqrt{1+2\theta(r+j)}, j=0,1,2,...,x_i$, then the likelihood function of the ZINB-CR$(r,\lambda,\theta,\gamma,\omega)$ can be obtained by

$$L(r,\lambda,\theta,\gamma,\omega)=\prod_{i=1}^{n}\left[I_{(x_i=0)}\left(\omega+(1-\omega)\frac{\exp(\lambda(1-Z_j))(1-\gamma(1-Z_j))}{Z_j}\right)\right.$$

$$\left.+I_{(x_i>0)}\left((1-\omega)\binom{r+x_i-1}{x_i}\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp(\lambda(1-Z_j))(1-\gamma(1-Z_j))}{Z_j}\right)\right],$$

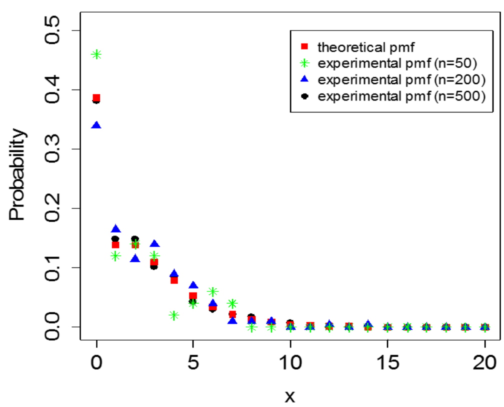


Figure 2. The theoretical probability mass function and the experimental probability mass function of the ZINB-CR distribution with parameters $r=6, \lambda=9, \theta=0.04, \gamma=0.3$, $\omega=0.3$ and samples of size n = 50, 200 and 500.

from which we calculate the log-likelihood function

$$\ell(r,\lambda,\theta,\gamma,\omega)=\log L(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\log\left(\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}\right)\right.$$

$$+I_{(x_i>0)}\left(\log(1-\omega)+\log(r+x_i-1)!-\log(r-1)!-\log x_i!\right.$$

$$\left.\left.+\log\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}\right)\right].$$

It can be verified that the first partial derivatives $\ell(r,\lambda,\theta,\gamma,\omega)$ with respect to $r,\lambda,\theta,\gamma$ and $\omega$ are given by the following differential equations:

$$\frac{\partial}{\partial r}\ell(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\frac{\theta\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j^2}\left(\gamma-\frac{\left(1-\gamma\left(1-Z_j\right)\right)\left(\lambda Z_j+1\right)}{Z_j}\right)\right)}{\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)\right.$$

$$\left.+I_{(x_i>0)}\left(\psi(r+x_i)-\psi(r)+\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial r}\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)\right],$$

where $\psi(k)=\frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function.

$$\frac{\partial}{\partial\lambda}\ell(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)\left(1-Z_j\right)}{Z_j}\right)}{\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)\right.$$

$$\left.+I_{(x_i>0)}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}\left(1-Z_j\right)\exp\left(\lambda\left(1-Z_j\right)\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j}\left(1-\gamma\left(1-Z_j\right)\right)}\right)\right],$$

$$\frac{\partial}{\partial\theta}\ell(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\frac{r\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j^2}\left(\gamma-\frac{\left(1-\gamma\left(1-Z_j\right)\right)\left(\lambda Z_j+1\right)}{Z_j}\right)\right)}{\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)\right.$$

$$\left.+I_{(x_i>0)}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial\theta}\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j}\left(1-\gamma\left(1-Z_j\right)\right)\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j}\left(1-\gamma\left(1-Z_j\right)\right)}\right)\right],$$

$$\frac{\partial}{\partial\gamma}\ell(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(Z_j-1\right)}{Z_j}\right)}{\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)\right.$$

$$\left.+I_{(x_i>0)}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j+1}\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j}\left(1-Z_j\right)\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\exp\left(\lambda\left(1-Z_j\right)\right)}{Z_j}\left(1-\gamma\left(1-Z_j\right)\right)}\right)\right],$$

and

$$\frac{\partial}{\partial\omega}\ell(r,\lambda,\theta,\gamma,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{1-\left(\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}\right)}{\omega+(1-\omega)\frac{\exp\left(\lambda\left(1-Z_j\right)\right)\left(1-\gamma\left(1-Z_j\right)\right)}{Z_j}}\right)-I_{(x_i>0)}\left(\frac{1}{1-\omega}\right)\right],$$

With the above equating to zero, the five derivative equations cannot be solved analytically, and therefore need to rely on Newton-Raphson technique: a simple iterative numerical method to approximate MLE. The MLE solution of $\hat{r},\hat{\lambda},\hat{\theta},\hat{\gamma}$ and $\hat{\omega}$ can be obtained by solving the resulting equations simultaneously using *nlm* function in the R program (R Core Team, 2012).

### 2.4.2 Method of moments (MM)

For the method of moments estimation, the parameters can be obtained by equating the sample and population moments. Because we have five parameters, we need the first five moments of the ZINB-CR distribution, which are given by

$$E(X)=(1-\omega)r\left(\frac{(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}-1\right),$$

$$E(X^2)=(1-\omega)\left(\frac{(r^2+r)(1-\gamma(1-\varsigma))\exp(\lambda(1-\varsigma))}{\varsigma}-\frac{(2r^2+r)(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}+r^2\right),$$

$$E(X^3)=(1-\omega)\left(\frac{(r^3+3r^2+2r)(1-\gamma(1-\vartheta))\exp(\lambda(1-\vartheta))}{\vartheta}-\frac{(3r^3+6r^2+3r)(1-\gamma(1-\varsigma))\exp(\lambda(1-\varsigma))}{\varsigma}+\frac{(3r^3+3r^2+r)(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}-r^3\right),$$

$$E(X^4)=(1-\omega)\left(\frac{(r^4+6r^3+11r^2+6r)(1-\gamma(1-\kappa))\exp(\lambda(1-\kappa))}{\kappa}-\frac{(4r^4+18r^3+26r^2+12r)(1-\gamma(1-\vartheta))\exp(\lambda(1-\vartheta))}{\vartheta}+\frac{(6r^4+18r^3+19r^2+7r)(1-\gamma(1-\varsigma))\exp(\lambda(1-\varsigma))}{\varsigma}-\frac{(4r^4+6r^3+4r^2+r)(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}+r^4\right),$$

$$E(X^5)=(1-\omega)\left(\frac{(r^5+10r^4+35r^3+50r^2+24r)(1-\gamma(1-\upsilon))\exp(\lambda(1-\upsilon))}{\upsilon}-\frac{(5r^5+40r^4+115r^3+140r^2+60r)(1-\gamma(1-\kappa))\exp(\lambda(1-\kappa))}{\kappa}+\frac{(10r^5+60r^4+135r^3+135r^2+50r)(1-\gamma(1-\vartheta))\exp(\lambda(1-\vartheta))}{\vartheta}-\frac{(10r^5+40r^4+65r^3+50r^2+15r)(1-\gamma(1-\varsigma))\exp(\lambda(1-\varsigma))}{\varsigma}+\frac{(5r^5+10r^4+10r^3+5r^2+r)(1-\gamma(1-\delta))\exp(\lambda(1-\delta))}{\delta}-r^5\right),$$

where $\delta=\sqrt{1-2\theta}$, $\varsigma=\sqrt{1-4\theta}$, $\vartheta=\sqrt{1-6\theta}$, $\kappa=\sqrt{1-8\theta}$ and $\upsilon=\sqrt{1-10\theta}$.

The $i^{th}$ moment for the sample, $m_i$, is equated as $m_i=\frac{1}{n}\sum_{j=1}^{n}x_j^i$. Then, the method of moments estimator is derived by solving equation $m_1=E(X)$, $m_2=E(X^2)$, $m_3=E(X^3)$, $m_4=E(X^4)$ and $m_5=E(X^5)$, using *gmm* function in the R program (Chausse, 2010).

## 3. Simulation Study

This section presents the performance of parameter estimation methods both MLE and MM with some specified values of parameter using simulated data. Using the above algorithm to generate random samples from the ZINB-CR distribution, the simulations were performed to compare the true value of the parameters of the ZINB-CR $(r,\lambda,\theta,\gamma,\omega)$ distribution and their estimates using MLE and MM method. The true parameter values are fixed in all 9 cases, which are shown parameters, mean, variance and the index of dispersion in Table 1. All cases of simulation study were then generated from this ZINB-CR distribution with samples of size n = 50, 100, 200 and 500 respectively. We used the R

Table 1. All cases simulation study for estimated parameters of ZINB-CR distribution

| Case | $r$ | $\lambda$ | $\theta$ | $\gamma$ | $\omega$ | Mean | Variance | Index of Dispersion |
|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 0.05 | 0.5 | 0.3 | 0.6877 | 1.3944 | 2.0276 |
| 2 | 3 | 5 | 0.05 | 0.5 | 0.5 | 0.4912 | 1.0925 | 2.2241 |
| 3 | 3 | 5 | 0.05 | 0.5 | 0.7 | 0.2947 | 0.7134 | 2.4208 |
| 4 | 6 | 9 | 0.04 | 0.3 | 0.3 | 2.0461 | 6.1134 | 2.9878 |
| 5 | 6 | 9 | 0.04 | 0.3 | 0.5 | 1.4615 | 5.2211 | 3.5724 |
| 6 | 6 | 9 | 0.04 | 0.3 | 0.7 | 0.8769 | 3.6453 | 4.1570 |
| 7 | 8 | 8 | 0.06 | 0.1 | 0.3 | 4.1358 | 21.3721 | 5.1676 |
| 8 | 8 | 8 | 0.06 | 0.1 | 0.5 | 2.9541 | 18.7565 | 6.3493 |
| 9 | 8 | 8 | 0.06 | 0.1 | 0.7 | 1.7725 | 13.3484 | 7.5308 |

program to generate each sample of a fixed size and repeated this for 500 trials. The bias is defined as the difference between the estimated and true parameter values. The sample averages of the estimated parameter $(\bar{\hat{\theta}})$, bias $(\mathrm{Bias}(\bar{\hat{\theta}}))$, variance $(\mathrm{Var}(\bar{\hat{\theta}}))$, and mean squared error $(\mathrm{MSE}(\bar{\hat{\theta}}))$ are calculated by the following measures: $\bar{\hat{\theta}} = \frac{1}{500}\sum_{t=1}^{500}\hat{\theta}_t$, $\mathrm{Bias}(\bar{\hat{\theta}}) = \bar{\hat{\theta}} - \theta$, $\mathrm{Var}(\bar{\hat{\theta}}) = \frac{1}{500-1}\sum_{t=1}^{500}(\hat{\theta}_t - \bar{\hat{\theta}})^2$ and $\mathrm{MSE}(\bar{\hat{\theta}}) = \mathrm{Var}(\bar{\hat{\theta}}) + \mathrm{Bias}^2(\bar{\hat{\theta}})$.

The results from our simulations are summarised in Tables 2-4, which contain the sample averages of the estimated parameters and MSE for each of the ML and MM

Table 2. The average estimates and the corresponding MSE in parentheses for the ZINB-CR distribution (True parameters : $r = 3$, $\lambda = 5$, $\theta = 0.05$, $\gamma = 0.5$ and $\omega = 0.3, 0.5, 0.7$)

| n | Parameter $r$ | MLE | | | Method of Moments | | |
|---|---|---|---|---|---|---|---|
| | | $\omega$=0.3 | $\omega$=0.5 | $\omega$=0.7 | $\omega$=0.3 | $\omega$=0.5 | $\omega$=0.7 |
| 50 | $r$ | 20.8026 | 24.8667 | 35.0068 | 32.5242 | 39.5194 | 56.2362 |
| | | (693.5831) | (996.5406) | (1931.1911) | (1885.4068) | (2774.9833) | (5364.6542) |
| | $\lambda$ | 30.046 | 34.6136 | 41.7421 | 46.9392 | 53.2187 | 63.3303 |
| | | (1192.0561) | (1609.5616) | (2535.0841) | (3128.4993) | (4073.1336) | (6104.8559) |
| | $\theta$ | 0.0144 | 0.0142 | 0.0104 | 0.0057 | 0.0053 | 0.0037 |
| | | (0.0018) | (0.0019) | (0.0020) | (0.0020) | (0.0021) | (0.0022) |
| | $\gamma$ | 0.1079 | 0.0875 | 0.0560 | 0.7174 | 0.7162 | 0.6468 |
| | | (0.2036) | (0.2135) | (0.2211) | (0.1957) | (0.1888) | (0.1631) |
| | $\omega$ | 0.3826 | 0.5207 | 0.6982 | 0.1493 | 0.2249 | 0.3997 |
| | | (0.0331) | (0.0310) | (0.0269) | (0.0557) | (0.1213) | (0.1501) |
| 100 | $r$ | 13.4802 | 15.4109 | 19.7970 | 20.9434 | 24.4301 | 32.0074 |
| | | (274.0356) | (416.6340) | (689.4874) | (730.9701) | (1169.5845) | (1976.0065) |
| | $\lambda$ | 19.5111 | 22.6923 | 25.7423 | 31.4251 | 35.5302 | 39.3452 |
| | | (498.2003) | (727.9759) | (973.0517) | (1427.4538) | (1931.8167) | (2412.1643) |
| | $\theta$ | 0.0224 | 0.0226 | 0.0210 | 0.0097 | 0.0091 | 0.0080 |
| | | (0.0015) | (0.0017) | (0.0017) | (0.0018) | (0.0018) | (0.0019) |
| | $\gamma$ | 0.1809 | 0.1622 | 0.1242 | 0.6874 | 0.7126 | 0.6731 |
| | | (0.1737) | (0.1876) | (0.2011) | (0.1547) | (0.1554) | (0.1439) |
| | $\omega$ | 0.3604 | 0.5038 | 0.6756 | 0.2115 | 0.2967 | 0.4397 |
| | | (0.0243) | (0.0268) | (0.0257) | (0.0462) | (0.0910) | (0.1261) |
| 200 | $r$ | 7.4350 | 9.2306 | 12.3486 | 11.7925 | 14.3856 | 19.4347 |
| | | (72.4392) | (130.9781) | (256.9596) | (207.7966) | (358.4074) | (714.9175) |
| | $\lambda$ | 12.9305 | 14.5895 | 16.9362 | 21.5065 | 23.1955 | 26.2326 |
| | | (209.9968) | (291.9971) | (410.4352) | (699.6490) | (825.8080) | (1063.5563) |
| | $\theta$ | 0.0314 | 0.0310 | 0.0292 | 0.0149 | 0.0141 | 0.0122 |
| | | (0.0011) | (0.0013) | (0.0016) | (0.0014) | (0.0015) | (0.0016) |
| | $\gamma$ | 0.2325 | 0.2134 | 0.1716 | 0.6452 | 0.6832 | 0.6871 |
| | | (0.1513) | (0.1615) | (0.1778) | (0.1159) | (0.1238) | (0.1257) |
| | $\omega$ | 0.3165 | 0.4798 | 0.6685 | 0.2812 | 0.3603 | 0.4951 |
| | | (0.0183) | (0.0243) | (0.0230) | (0.0437) | (0.0709) | (0.0956) |
| 500 | $r$ | 4.6980 | 5.2142 | 6.7689 | 7.6731 | 8.1930 | 10.5295 |
| | | (9.7023) | (22.0971) | (52.5528) | (38.7817) | (67.1388) | (154.5054) |
| | $\lambda$ | 6.9862 | 9.3031 | 10.4776 | 11.3627 | 15.2902 | 16.6283 |
| | | (33.3647) | (96.5675) | (127.0873) | (119.0248) | (327.7339) | (368.4699) |
| | $\theta$ | 0.0414 | 0.0388 | 0.0363 | 0.0227 | 0.0204 | 0.0179 |
| | | (0.0006) | (0.0008) | (0.0011) | (0.0010) | (0.0011) | (0.0013) |
| | $\gamma$ | 0.2939 | 0.2544 | 0.2212 | 0.6336 | 0.6436 | 0.6632 |
| | | (0.1044) | (0.1238) | (0.1459) | (0.0664) | (0.0788) | (0.0928) |
| | $\omega$ | 0.2936 | 0.4738 | 0.6748 | 0.3981 | 0.4548 | 0.5879 |
| | | (0.0115) | (0.0154) | (0.0130) | (0.0480) | (0.0506) | (0.0561) |

Table 3. The average estimates and the corresponding MSE in parentheses for the ZINB-CR distribution (True parameters: $r = 6$, $\lambda = 9$, $\theta = 0.04$, $\gamma = 0.3$ and $\omega = 0.3, 0.5, 0.7$)

| n | Parameter $r$ | MLE | | | Method of Moments | | |
|---|---|---|---|---|---|---|---|
| | | $\omega$=0.3 | $\omega$=0.5 | $\omega$=0.7 | $\omega$=0.3 | $\omega$=0.5 | $\omega$=0.7 |
| 50 | $r$ | 10.0056 | 11.8090 | 17.3398 | 15.7802 | 18.3659 | 27.7652 |
| | | (89.3894) | (161.2464) | (410.1144) | (273.9664) | (470.5508) | (1259.0436) |
| | $\lambda$ | 24.7603 | 31.6682 | 37.1857 | 39.6714 | 49.3284 | 56.5315 |
| | | (751.7713) | (1313.8136) | (1734.6542) | (2324.4299) | (3637.6055) | (4459.9239) |
| | $\theta$ | 0.0285 | 0.0258 | 0.0218 | 0.0132 | 0.0118 | 0.0094 |
| | | (0.0009) | (0.0011) | (0.0013) | (0.0009) | (0.0010) | (0.0011) |
| | $\gamma$ | 0.1649 | 0.1284 | 0.0989 | 0.6551 | 0.6874 | 0.7387 |
| | | (0.0494) | (0.0640) | (0.0729) | (0.2472) | (0.2645) | (0.2981) |
| | $\omega$ | 0.2837 | 0.4839 | 0.6811 | 0.2060 | 0.2943 | 0.4345 |
| | | (0.0098) | (0.0116) | (0.0094) | (0.0402) | (0.0850) | (0.1207) |
| 100 | $r$ | 7.2908 | 8.4865 | 10.8187 | 11.8463 | 13.0894 | 16.8953 |
| | | (26.4415) | (48.0479) | (122.0891) | (96.8123) | (142.4844) | (378.4028) |
| | $\lambda$ | 20.6594 | 22.3712 | 28.4398 | 33.4103 | 35.3081 | 43.5653 |
| | | (447.0301) | (631.0660) | (1084.7831) | (1476.0023) | (1853.5099) | (2893.1389) |
| | $\theta$ | 0.0308 | 0.0303 | 0.0270 | 0.0156 | 0.0149 | 0.0128 |
| | | (0.0006) | (0.0007) | (0.0009) | (0.0007) | (0.0008) | (0.0009) |
| | $\gamma$ | 0.1519 | 0.1725 | 0.1294 | 0.6232 | 0.6507 | 0.7132 |
| | | (0.0468) | (0.0446) | (0.0568) | (0.1942) | (0.2121) | (0.2616) |
| | $\omega$ | 0.2925 | 0.4908 | 0.6891 | 0.2911 | 0.3736 | 0.5165 |
| | | (0.0054) | (0.0054) | (0.0049) | (0.0394) | (0.0590) | (0.0738) |
| 200 | $r$ | 6.0473 | 6.2124 | 8.1813 | 9.8916 | 10.1097 | 12.5662 |
| | | (7.8735) | (11.8616) | (42.7475) | (38.5876) | (42.5334) | (136.2966) |
| | $\lambda$ | 15.7839 | 18.6851 | 21.4896 | 25.2538 | 29.5959 | 32.9602 |
| | | (180.0129) | (306.4642) | (579.6301) | (653.6727) | (993.5747) | (1595.8258) |
| | $\theta$ | 0.0350 | 0.0329 | 0.0307 | 0.0185 | 0.0174 | 0.0158 |
| | | (0.0004) | (0.0005) | (0.0006) | (0.0006) | (0.0007) | (0.0008) |
| | $\gamma$ | 0.1371 | 0.1154 | 0.1611 | 0.6064 | 0.5836 | 0.6956 |
| | | (0.0430) | (0.0503) | (0.0422) | (0.1575) | (0.1441) | (0.2270) |
| | $\omega$ | 0.2952 | 0.4936 | 0.6942 | 0.3602 | 0.4717 | 0.6004 |
| | | (0.0025) | (0.0025) | (0.0020) | (0.0413) | (0.0380) | (0.0365) |
| 500 | $r$ | 5.7041 | 5.9002 | 6.7194 | 9.7274 | 9.7881 | 10.3938 |
| | | (1.4992) | (2.1900) | (9.1385) | (18.8916) | (20.2171) | (39.5173) |
| | $\lambda$ | 10.8689 | 11.3716 | 13.4305 | 16.7964 | 17.6928 | 20.6586 |
| | | (37.9674) | (48.8545) | (183.6736) | (157.7526) | (192.9932) | (525.9799) |
| | $\theta$ | 0.0399 | 0.0388 | 0.0382 | 0.0222 | 0.0215 | 0.0207 |
| | | (0.0002) | (0.0003) | (0.0004) | (0.0004) | (0.0004) | (0.0005) |
| | $\gamma$ | 0.1713 | 0.1452 | 0.2074 | 0.5934 | 0.5490 | 0.6155 |
| | | (0.0298) | (0.0354) | (0.0242) | (0.1171) | (0.0929) | (0.1499) |
| | $\omega$ | 0.2979 | 0.4978 | 0.6979 | 0.4695 | 0.5934 | 0.6843 |
| | | (0.0009) | (0.0008) | (0.0007) | (0.0546) | (0.0274) | (0.0143) |

estimators. It is shown that the ML estimator outperforms the MM estimator in all cases for the parameters $r$, $\lambda$, $\theta$ and $\omega$. However, the ML estimator does not perform well overall on the ZINB-CR distribution, especially for small to moderate sample sizes. As to be expected, the MSE in almost all the estimated parameters except $\gamma$ decreases as the sample size increases, which may be the result of a weak parameter ($\gamma$) from the Crack distribution thus making the estimate uncertain. Bowonrattanaset (2011) showed the estimates of $\gamma$ are out of the closed interval [0,1] or are far from true para-

Table 4. The average estimates and the corresponding MSE in parentheses for the ZINB-CR distribution (True parameters : $r = 8$ , $\lambda = 8$, $\theta = 0.06$, $\gamma = 0.1$ and $\omega = 0.3, 0.5, 0.7$)

| n | Parameter $r$ | MLE | | | Method of Moments | | |
|---|---|---|---|---|---|---|---|
| | | $\omega = 0.3$ | $\omega = 0.5$ | $\omega = 0.7$ | $\omega = 0.3$ | $\omega = 0.5$ | $\omega = 0.7$ |
| 50 | $r$ | 7.9017 | 8.2705 | 9.9263 | 12.8557 | 12.9972 | 15.4340 |
| | | (11.5620) | (26.0328) | (69.0230) | (51.7836) | (85.5968) | (230.8282) |
| | $\lambda$ | 14.6170 | 21.9235 | 35.2507 | 23.7135 | 34.4293 | 53.5673 |
| | | (303.0780) | (937.3355) | (2042.5102) | (976.4302) | (2584.7595) | (5098.9238) |
| | $\theta$ | 0.0578 | 0.0505 | 0.0361 | 0.0263 | 0.0232 | 0.0165 |
| | | (0.0012) | (0.0015) | (0.0018) | (0.0014) | (0.0016) | (0.0021) |
| | $\gamma$ | 0.3976 | 0.2629 | 0.3033 | 0.6751 | 0.7005 | 0.7638 |
| | | (0.1224) | (0.0705) | (0.1379) | (0.4184) | (0.4433) | (0.5306) |
| | $\omega$ | 0.3005 | 0.5006 | 0.6952 | 0.2406 | 0.3173 | 0.4953 |
| | | (0.0050) | (0.0051) | (0.0046) | (0.0227) | (0.0607) | (0.0781) |
| 100 | $r$ | 6.9886 | 7.5790 | 8.3589 | 11.6448 | 12.0076 | 12.8470 |
| | | (5.5071) | (9.1322) | (19.0649) | (26.9409) | (35.6681) | (69.5462) |
| | $\lambda$ | 12.6439 | 13.5234 | 15.8022 | 20.0600 | 21.3158 | 23.9725 |
| | | (183.4274) | (248.1196) | (381.0389) | (574.3650) | (738.8352) | (995.8539) |
| | $\theta$ | 0.0622 | 0.0594 | 0.0546 | 0.0292 | 0.0277 | 0.0258 |
| | | (0.0009) | (0.0010) | (0.0012) | (0.0011) | (0.0012) | (0.0014) |
| | $\gamma$ | 0.4787 | 0.4851 | 0.2954 | 0.7251 | 0.7223 | 0.7390 |
| | | (0.1866) | (0.1866) | (0.0820) | (0.4578) | (0.4477) | (0.4664) |
| | $\omega$ | 0.2964 | 0.4959 | 0.6996 | 0.2588 | 0.3638 | 0.5249 |
| | | (0.0024) | (0.0030) | (0.0022) | (0.0230) | (0.0424) | (0.0615) |
| 200 | $r$ | 6.6516 | 6.8952 | 7.1409 | 11.0697 | 11.0800 | 10.8890 |
| | | (3.6876) | (4.3678) | (6.5430) | (15.5145) | (17.8388) | (21.4721) |
| | $\lambda$ | 9.4362 | 10.4761 | 12.9935 | 14.9139 | 16.4798 | 19.8588 |
| | | (33.6238) | (80.9886) | (253.8543) | (134.7750) | (268.5188) | (683.3817) |
| | $\theta$ | 0.0701 | 0.0671 | 0.0619 | 0.0331 | 0.0317 | 0.0299 |
| | | (0.0007) | (0.0008) | (0.0008) | (0.0008) | (0.0010) | (0.0011) |
| | $\gamma$ | 0.4700 | 0.4895 | 0.4927 | 0.7296 | 0.7107 | 0.7484 |
| | | (0.1675) | (0.1898) | (0.1973) | (0.4538) | (0.4230) | (0.4688) |
| | $\omega$ | 0.2962 | 0.4981 | 0.7004 | 0.2588 | 0.3993 | 0.5571 |
| | | (0.0013) | (0.0015) | (0.0011) | (0.0186) | (0.0358) | (0.0462) |
| 500 | $r$ | 6.4721 | 6.5634 | 6.7594 | 10.7464 | 10.5672 | 10.4264 |
| | | (3.0364) | (3.3082) | (3.4636) | (9.8254) | (10.3340) | (10.3449) |
| | $\lambda$ | 8.5674 | 9.0707 | 9.3316 | 13.5395 | 14.3366 | 14.3789 |
| | | (3.8138) | (17.2931) | (17.7374) | (38.6579) | (80.3799) | (84.2035) |
| | $\theta$ | 0.0707 | 0.0696 | 0.0674 | 0.0341 | 0.0334 | 0.0332 |
| | | (0.0003) | (0.0004) | (0.0005) | (0.0007) | (0.0008) | (0.0008) |
| | $\gamma$ | 0.4520 | 0.4659 | 0.4977 | 0.7207 | 0.7122 | 0.7247 |
| | | (0.1533) | (0.1708) | (0.1935) | (0.4378) | (0.4096) | (0.4205) |
| | $\omega$ | 0.2949 | 0.4985 | 0.6987 | 0.2774 | 0.4215 | 0.6039 |
| | | (0.0006) | (0.0006) | (0.0005) | (0.0134) | (0.0238) | (0.0263) |

meter value. In addition, the performance of the estimators also depends on the zero-inflation parameter ($\omega$). We observed that when $\omega$ increases the MSE tends to increase, but it is not consistent for the parameters $\gamma$ and $\omega$, which are bounded on [0,1].

## 4. Application Study

For one application of ZINB-CR, the distribution is applied to the number of major derogatory reports (MDRs) in the credit history of individual credit card applicants (see

Greene (1994)). This data has 80.36 % of zeros and the index of dispersion is 3.298, indicating that there is overdispersion and a high percentage of zeros. We applied the ZIP, ZINB, NB-CR and ZINB-CR distributions to fit the real data set. In order to compare distributions, we consider the Chi-squares test, AIC and BIC statistics for the data. From the results in Table 5, we found that the MLE provides very poor fit for the ZIP and ZINB distributions and acceptable fits for the NB-CR and ZINB-CR distributions. However, the ZINB-CR distribution is a much better model than the NB-CR distribution for the given data.

## 5. Conclusions

This paper offers ZINB-CR distribution, which is obtained by mixing the Bernoulli distribution with a NB-CR distribution. We have derived some properties of the ZINB-CR distribution, including mean, variance and higher order moments. The parameter estimation via the maximum likelihood method and the method of moments are also implemented. We conducted a simulation study to compare the methods. Based on our simulations, we conclude that the MLE is not performing well on the ZINB-CR distribution even though it outperforms the MM method. In our next-step research, we will apply some new estimating procedures (e.g., Bayesian approach) to estimate the parameters and compare their results with MLE and MM methods. Finally, we have compared efficiencies of the ZINB-CR distribution with the ZIP, ZINB and NB-CR distributions, fitting distribution by using a real data. The results show that the ZINB-CR distribution provides a better fit compared to the ZIP, ZINB and NB-CR distributions.

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Table 5. Observed and expected frequencies for the consumer credit behavior data.

| Number of MDRs | Observed | Fitting distributions | | | |
|---|---|---|---|---|---|
| | | ZIP | ZINB | NB-CR | ZINB-CR |
| 0 | 1060 | 1059.99 | 1060.08 | 1057.84 | 1061.64 |
| 1 | 137 | 80.32 | 94.17 | 131.90 | 134.50 |
| 2 | 50 | 80.90 | 72.86 | 55.40 | 54.75 |
| 3 | 24 | 54.31 | 45.17 | 26.38 | 26.98 |
| 4 | 17 | 27.35 | 24.52 | 13.19 | 14.86 |
| 5 | 11 | 11.02 | 12.18 | 9.10 | 8.84 |
| 6 | 5 | 5.11 (6, 7, 8-14) | 10.02 (6, 7, 8-14) | 5.54 | 5.57 |
| 7 | 6 | | | 19.65 (7, 8-14) | 11.86 (7, 8-14) |
| 8-14 | 9 | | | | |
| Estimated parameters | | $\hat{\lambda} = 2.0142$ | $\hat{r} = 3.9566$ | $\hat{r} = 11.2632$ | $\hat{r} = 4.0538$ |
| | | $\hat{\omega} = 0.7734$ | $\hat{p} = 0.6878$ | $\hat{\lambda} = 0.0947$ | $\hat{\lambda} = 0.2783$ |
| | | | $\hat{\omega} = 0.7459$ | $\hat{\theta} = 0.0742$ | $\hat{\theta} = 0.1481$ |
| | | | | $\hat{\gamma} = 0.6226$ | $\hat{\gamma} = 0.2027$ |
| | | | | | $\hat{\omega} = 0.4438$ |
| AIC | | 2285 | 2188 | 2139 | 2122 |
| BIC | | 2296 | 2203 | 2159 | 2148 |
| Chi-squares | | 116.0757 | 48.9360 | 3.5933 | 1.7682 |
| Degree of freedom | | 4 | 3 | 3 | 2 |
| $p$-value | | $<0.0001$ | $<0.0001$ | 0.3089 | 0.4131 |

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