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Original Article

The Cesáro Lacunary Ideal bounded linear operator of χ^2- of ϕ-statistical vector valued defined by a bounded linear operator of interval numbers

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Abstract

Let $\left(A_{mn}^{uv}\right)$ be a sequence of bounded linear operators from a separable Banach metric space of $\left(X,0\right)$ into a Banach metric space $\left(Y,0\right)$. Suppose that $\phi \in \Phi$ is a countable fundamental set of X and the ideal $I-$ of subsets $\mathbb{N}\times\mathbb{N}$ has property (AP). The sequence $\left(A_{mn}^{uv}\right)$ is said to be $b^{*}I-$convergent if it is pointwise $I-$convergent and there exists an index set K such that $\mathbb{N}\times\mathbb{N}/K \in I$ and $\left(A_{mn}^{uv}x\right)_{m,n\in K}$ is bounded for any $x \in X$, the concept of lacunary vector valued of χ^2 and the concept of $\Delta_{11}-$lacunary statistical convergent vector valued of χ^2 of difference sequences have been introduced. In addition, we introduce interval numbers of asymptotically ideal equivalent sequences of vector valued difference by Musielak fuzzy real numbers and established some relations related to this concept.

Finally we introduce the notion of interval numbers of Cesáro Orlicz asymptotically equivalent sequences vector valued difference of Musielak Orlicz function and establish their relationship with other classes.

Keywords: Banach metric, bounded linear operator, ideal, $I-$convergence, analytic sequence, Museialk-Orlicz function, double sequences, chi sequence, Lambda, Riesz space, strongly, statistical convergent, lacunary refinement

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in (Bromwich, 1965). Later on it was investigated by (Hardy, 1917; Moricz, 1991; Moricz & Rhoades, 1988) (Basarir & Solankan, 1999; Tripathy *et al.*, 2003, 2004, 2006, 2007, 2008, 2009, 2010, 2011, 2013) (Mishra *et al.*, 2007, 2013, 2014, 2015; Raj *et al.*, 2010, 2011, 2012, 2013; Turkmenoglu, 1999) and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \, (m, n = 1, 2, 3, \ldots).$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, \ldots \right\},$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2. "Let ϕ = {finite sequences}.

Consider a double sequence $x = (x_{mn})$. The $[m,n]^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with 1 in the $[m,n]^{th}$ section and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)!|x_{mn}|\right)^{\frac{1}{m+n}} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2.

2. Concepts of $\Delta_{11}-$ lacunary $\phi-$ statistically convergent and Cauchy χ^2- sequences}

We first introduce the concept of lacunary double sequences as follows:

2.1 Definition

A sequence (A_{mn}) of operators $A_{mn} \in B(X,Y)(m,n \in \mathbb{N})$ is said to be b^*I- convergent if $I-lim_{mn}A_{mn}x_{mn}$ exists $(I-limA_{mn}x_{mn} = Ax_{mn})$ for any $x \in X$ and there is a set $K \in I$ such that $(A_{mn}x_{mn})_{m,n\in K}$ is bounded for every $x \in X$. In the special case $I = I_T$ we get the notion of b^*T- statistical convergence. The b^*I- limit and the b^*T- statistical of (A_{mn}) are denoted, respectively, by $b^*I-lim_{mn}A_{mn}$ and $b^*st_T-lim_{mn}A_{mn}$.

2.2 Definition

By a lacunary double sequence we mean an increasing sequence of positive integers $\theta_{rs} = (m_r n_s)$

$$\begin{pmatrix} m_0n_0 & m_0n_1 & \cdots & & & & & \\ m_1n_0 & m_1n_1 & m_1n_2 & \cdots & & & & \\ m_2n_0 & m_2n_1 & m_2n_2 & m_2n_3 & \cdots & & & \\ m_3n_0 & m_3n_1 & m_3n_2 & m_3n_3 & m_3n_4 & \cdots & & \\ m_4n_0 & m_4n_1 & m_4n_2 & m_4n_3 & m_4n_4 & m_4n_5 & \cdots & \\ m_5n_0 & m_5n_1 & m_5n_2 & m_5n_3 & m_5n_4 & m_5n_5 & m_5n_6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & . & . & & \\ . & . & . & & & & & \\ . & . & . & & & & & \end{pmatrix}$$

such that $m_0n_0 = 0, m_{-1}n_0 = 0, m_0n_{-1} = 0, m_rn_0 = 0.m_0n_s = 0$ for $r \geq 1, s \geq 1,$

$0 < m_in_j < m_kn_j$ if $i < k$

$0 < m_pn_i < m_pn_k$ if $i < k$

$0 < m_pn_j < m_{p+1}n_1$ for $p = 1,2,\cdots$ and $j = 1,2,\cdots$

and $\Delta_{01}h_{rs} = (m_rn_s - m_rn_{s-1}) \to \infty ass \to \infty, \Delta_{10}h_{rs} = (m_rn_s - m_{r-1}n_s) \to \infty asr \to \infty.$ The corresponding intervals are denoted by $\Delta_{01}I_{rs}$ and $\Delta_{10}I_{rs}$ where $\Delta_{01}I_{rs} = (m_rn_{s-1} - m_rn_s]$ and $\Delta_{10}I_{rs} = (m_{r-1}n_s - m_rn_s]$.

We define

$$\Delta h_{rs} \equiv \Delta_{10}h_{rs} \times \Delta_{01}h_{rs} = \underbrace{(m_rn_s - m_{r-1}n_s)}_{\Delta 10^{h}rs} \times \underbrace{(m_rn_s - m_rn_{s-1})}_{\Delta 01^{h}rs}.$$

So $\Delta h_{rs} \to \infty$ as $r,s \to \infty.$

The intervals determined by θ_{rs} are denoted by ΔI_{rs} where

$$\Delta I_{rs} \equiv \Delta_{10}I_{rs} \times \Delta_{01}I_{rs} = \underbrace{(m_{r-1}n_s - m_rn_s]}_{\Delta 10^{I}rs} \times \underbrace{(m_rn_{s-1} - m_rn_s]}_{\Delta 01^{I}rs}.$$

and the ratios $\frac{m_rn_s}{m_{r-1}n_s}, \frac{m_rn_s}{m_rn_{s-1}}$ are denoted by $\Delta_{10}q_{rs}$ and $\Delta_{01}q_{rs}$ respectively. We will denote the set of all double lacunary sequences by $N_{\theta_{rs}}$.

Now we defined the backward differences of the double sequences $x = (x_{mn})$ as follows:

$\Delta_{11} x_{mn} = x_{mn} - x_{m-1,n} - x_{m,n-1} + x_{m-1,n-1}$,

$\Delta_{10} x_{mn} = x_{mn} - x_{m-1,n}$

$\Delta_{01} x_{mn} = x_{mn} - x_{m,n-1}$.

We now define Δ_{11} – statistical convergence, Δ_{11} – lacunary statistically Cauchy sequence for double sequences in the following manner.

3. Definitions and Preliminaries

3.1 Definition

(Lindenstrauss and Tzafriri, 1971) An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function *M* is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function. An Orlicz function *M* is said to satisfy Δ_2 – condition for all values *u*, if there exists $K > 0$ such that $M(2u) \le KM(u), u \ge 0$.

3.2 Lemma

Let *M* be an Orlicz function which satisfies Δ_2 – condition and let $0 < \delta < 1$. Then for each $t \ge \delta$, we have $M(t) < K\delta^{-1} M(2)$ for some constant $K > 0$.

3.3 Definition

Let $n \in \mathbb{N}$ and *X* be a real vector space of dimension *m*, where $n \le m$. A real valued function $d_p(x_1, \ldots, x_n) = \| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p$ on *X* satisfying the following four conditions:

(i) $\| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p = 0$ if and and only if $d_1(x_1,0), \ldots, d_n(x_n,0)$ are linearly dependent,

(ii) $\| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p$ is invariant under permutation,

(iii) $\| (\alpha d_1(x_1,0), \ldots, \alpha d_n(x_n,0)) \|_p = |\alpha| \, \| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p, \alpha \in \mathbb{R}$

(iv) $d_p\big((x_1,y_1),(x_2,y_2)\cdots,(x_n,y_n)\big) = \big(d_X(x_1,x_2,\cdots,x_n)^p + d_Y(y_1,y_2,\cdots,y_n)^p\big)^{1/p}$ for $1 1 \le p < \infty$; (or)

(v) $d\big((x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\big) := \sup\{d_X(x_1,x_2,\cdots,x_n), d_Y(y_1,y_2,\cdots,y_n)\}$,

for $x_1, x_2, \cdots, x_n \in X, y_1, y_2, \cdots, y_n \in Y$ is called the *p* product metric of the Cartesian product of *n* metric spaces is the *p* norm of the *n*-vector of the norms of the *n* subspaces.

A trivial example of *p* product metric of *n* metric space is the *p* norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the *p* norm:

$$\| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_E = sup\left(| \, det(d_{mn}(x_{mn},0)) \, |\right) = sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{mn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$metric. Any complete $p-$metric space is said to be $p-$Banach metric space.

An interval number $\tilde{x}$ is a closed subset of the real numbers and denoted as $\tilde{x} = \left[\Delta_{11}x_{pq}, \Delta_{11}x_{rs}\right]$, where $\Delta_{11}x_{pq} \le \Delta_{11}x_{rs}$ and $\Delta_{11}x_{pq}, \Delta_{11}x_{rs}$ both are real numbers. Let us denote the set of all real valued closed intervals by $R^2(I)$. The set of all interval numbers $R^2(I)$ is a metric space with the metric

$$d(\tilde{x}, \tilde{y}) = max\left\{\left|\Delta_{11}x_{pq} - \Delta_{11}y_{pq}\right|, \left|\Delta_{11}x_{rs} - \Delta_{11}y_{rs}\right|\right\}.$$

Let us define transformation $f : N \times N \to R^2(I) \times R^2(I)$ by $(m,n) \to f(mn) = (\Delta_{11}\tilde{x}_{mn})$. Then $(\Delta_{11}\tilde{x}_{mn})$ is called the sequence of interval numbers. The $\Delta_{11}\tilde{x}_{mn}$ is called the $(m,n)^{th}$ term of sequence $(\Delta_{11}\tilde{x}_{mn})$.

3.4 Definition

Let $I, \Im$ be operator ideals and τ be an ideal topology. We say that a Banach metric space E has the

(i) $(I, \Im, \tau)-$ approximation property, $(I, \Im, \tau)-$ AP for short if $I(F;E) \subseteq \overline{\Im(F;E)}^{\tau}$ for every Banach metric space F.

(i) $(I, \Im, \tau)-$ weak approximation property, $(I, \Im, \tau)-$ WAP for short if $I(E;E) \subseteq \overline{\Im(E;E)}^{\tau}$.

Example: The classical approximation property coincides with the $(K, F, d(x,y))-$ AP, with $(L, F, \tau_c)-$ AP and hence with the $(L, F, \tau_c)-$ Wap.

Let X and Y be two Banach metric spaces, where X has a countable fundamental set $\phi \in \Phi$. If the ideal I has property (AP). A sequence $\left(A_{mn}^{uv}\right)$ of operators $A_{mn}^{uv} \in B(X,Y)$ is $b^{*}I-$ convergent and $\left(A_{mn}^{uv}\phi\right)$ is $I-$ convergent for every $\phi \in \Phi$ and M be a sequence of Orlicz functions and then consider the following definitions and the theorems are obtained:

3.5 Definition

A vector valued sequence $(\Delta_{11}\tilde{x}_{mn})$ of $\left(R^2(I), d\right)$ is said to be convergent to the interval number $\overline{0}$ and we denote it by writing

$$\left[\left\|\chi^{2I}_{A\phi M}, \left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right] = lim_{m,n\to\infty}$$
$$\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|\right)^{1/m+n}, (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p)\right)\right] = 0\right\}.$$

Thus

$$lim_{m,n\to\infty}\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|\right)^{1/m+n}, (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p)\right)\right] = 0\right\} \Leftrightarrow$$

$$lim_{p,q\to\infty}\left\{\left[M \| A_{pq}^{uv}\phi\left(\left((p+q)!\left|\Delta_{11}\tilde{x}_{pq}, \overline{0}\right|\right)^{1/p+q}, (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p)\right)\right] = 0\right\}$$

and

$$lim_{r,s\to\infty}\left\{\left[M\left(\| A_{rs}^{uv}\phi\left((r+s)!\left|\Delta_{11}\tilde{x}_{rs}, \overline{0}\right|\right)^{1/r+s}, (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p)\right)\right] = 0\right\}.$$

A four dimensional interval vector is an ordered 4-tuple of intervals,

$$\tilde{x} = \left(\Delta_{11}\tilde{x}_{11}, \Delta_{11}\tilde{x}_{12}, \Delta_{11}\tilde{x}_{21}, \Delta_{11}\tilde{x}_{22}\right) = \left(\left[\Delta_{11}x_{11pq}, \Delta_{11}x_{12pq}\right], \left[\Delta_{11}x_{21rs}, \Delta_{11}x_{22rs}\right]\right).$$

If the absolute value of each element of $\tilde{x}$ is zero, then $\tilde{x}$ is called zero interval vector and is denoted by $\tilde{\theta} = \left(\tilde{0}, \tilde{0}, \tilde{0}, \tilde{0}\right) = \left([0,0],[0,0]\right)$.

Let $R^2(I_4)$ be the set of all four dimensional interval vectors. The scalar multiplication and addition of two vectors in $R^2(I_4)$ are defined as follows:

$$\alpha\Delta_{11}\tilde{x} = \left(\alpha\Delta_{11}\tilde{x}_{11}, \alpha\Delta_{11}\tilde{x}_{12}, \alpha\Delta_{11}\tilde{x}_{21}, \alpha\Delta_{11}\tilde{x}_{22}\right) =$$

$$\begin{cases} \left(\left[\Delta_{11}x_{11pq}, \Delta_{11}x_{12pq}\right], \left[\Delta_{11}x_{21rs}, \Delta_{11}x_{22rs}\right]\right), & \text{if } \alpha \geq 0 \\ \left(\left[\Delta_{11}x_{12pq}, \Delta_{11}x_{11pq}\right], \left[\Delta_{11}x_{22rs}, \Delta_{11}x_{21rs}\right]\right), & \text{if } \alpha < 0 \end{cases}$$

$$\Delta_{11}\tilde{x} + \Delta_{11}\tilde{y} = \Delta_{11}\left(\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}\right) + \Delta_{11}\left(\tilde{y}_{11}, \tilde{y}_{12}, \tilde{y}_{21}, \tilde{y}_{22}\right) =$$

$$\left(\left[\Delta x_{11pq} + \Delta_{11}y_{11pq}, \Delta_{11}x_{12pq} + \Delta_{11}y_{12pq}\right], \left[\Delta_{11}x_{21rs} + \Delta_{11}y_{21rs}, \Delta_{11}x_{22rs} + \Delta_{11}y_{22rs}\right]\right).$$

Now, we introduce a distance of four vectors in $R^2(I_4)$, which is defined as

$$d(\tilde{x}, \tilde{y}) = max\left\{\left|\Delta_{11}x_{11pq} - \Delta_{11}y_{11pq}\right|, \left|\Delta_{11}x_{12rs} - \Delta_{11}y_{12rs}\right|, \left|\Delta_{11}x_{21pq} - \Delta_{11}y_{21pq}\right|, \left|\Delta_{11}x_{22rs} - \Delta_{11}y_{22rs}\right|\right\}$$

where $\Delta_{11}\tilde{x} = \Delta_{11}\left(\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}\right), \Delta_{11}\tilde{y} = \left(\tilde{y}_{11}, \tilde{y}_{12}, \tilde{y}_{21}, \tilde{y}_{22}\right) \in R^2(I_4)$.

3.6 Definition

Consider two non-negative vector valued sequences of interval numbers $\Delta_{11}x = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}y = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are asymptotically equivalent $\overline{\theta}$ if

$$lim_{mn} \frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}} = \overline{\theta} = \left(\tilde{0}, \tilde{0}, \tilde{0}, \tilde{0}\right) = \left([0,0],[0,0]\right).$$

and it is denoted by $\Delta_{11}\tilde{x} \equiv \overline{\theta}$.

3.7 Definition

Let K be the subset of $\mathbb{N} \times \mathbb{N}$, the set of natural numbers. Then the asymptotically density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = lim_{k,\ell} \frac{1}{k\ell}\left|\{m, n \leq k, \ell : m, n \in K\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set.

3.8 Definition

Consider vector valued sequence of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ is said to be statistically convergent to the number $\overline{\theta}$ if for each $\epsilon > 0$, the set

$$K(\epsilon) = \left\{m \leq k, n \leq \ell : (m+n)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|^{1/m+n} \geq \epsilon\right\}$$ has asympototic density $\overline{\theta}$.

$$lim_{k\ell} \frac{1}{k\ell}\left|\left\{m \leq k, n \leq \ell : \left((m+n)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|\right)^{1/m+n} \geq \epsilon\right\}\right| = \overline{\theta}.$$

In this case we write $St - limx = \overline{\theta}$.

3.9 Definition

Let two non-negative double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be asymptotically double equivalent of multiple L provided that for every $\epsilon > 0$,

$$lim_{k,\ell} \frac{1}{k,\ell}\left|\left\{(m,n): m \le k, n \le \ell, \left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}} - L\right| \ge \epsilon\right\}\right| = \overline{\theta}.$$

and simply asymptotically double statistical equivalent if $L = 1$. Furthermore, let $S^L_{\theta_{rs}}$ denote the set of all vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ such that $\Delta_{11}\tilde{x} = \Delta_{11}\tilde{y}$.

3.10 Definition

Let $\theta_{rs} = \{(m_r, n_s)\}$ be a double lacunary sequence; the two double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ are said to be asymptotically double lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$lim_{r,s} \frac{1}{\Delta h_{r,s}}\left|\left\{(m,n) \in I_{r,s} : \left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}} - L\right| \ge \epsilon\right\}\right| = \overline{\theta}$$

and simply asymptotically double lacunary statistical equivalent if L=1.Furthermore, let $S^L_{\theta_{rs}}$ denote the set of all vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ such that $\Delta_{11}\tilde{x} = \Delta_{11}\tilde{y}$.

3.11 Definition

Consider $\theta_{rs} = \{(m_r, n_s)\}$ be a double lacunary sequence; the two double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ are said to be strong asymptotically double lacunary equivalent of multiple L provided that

$$lim_{r,s} \frac{1}{\Delta h_{r,s}} \sum_{(m,n)\in I_{r,s}} \left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}} - L\right| = \overline{\theta},$$

that is $\Delta_{11}\tilde{x}$ is equivalent to $\Delta_{11}\tilde{y}$ and it is denoted by $N^L_{\theta_{rs}}$ and simply strong asymptotically double lacunary equivalent if L=1. In addition, let $N^L_{\theta_{rs}}$ denote the set of all vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ such that $\Delta_{11}\tilde{x}$ is equivalent to $\Delta_{11}\tilde{y}$.

3.12 Definition

An ideal I has property (AP). A sequence (A^{uv}_{mn}) of operators $A^{uv}_{mn} \in B(X,Y)$ is $b^* I -$ convergent and $(A^{uv}_{mn}\phi)$ is $I -$ convergent for every $\phi \in \Phi$ and the double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$m_\circ = 0, h_r = m_r - m_{r-1} \to \infty$ as $r \to \infty$ and $n_\circ = 0, \overline{h}_s = n_s - n_{s-1} \to \infty$ as $s \to \infty$.

Notations: $m_{r,s} = m_r m_s, h_{r,s} = h_r \overline{h}_s$ and θ_{rs} is determined by

$$I_{rs} = \{(m,n): m_{r-1} < m \le m_r \text{ and } n_{s-1} < n \le n_s\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q}_s = \frac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \overline{q}_s.$$

3.13 Definition

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $P(\sigma)$ the sequence $\{P_{ab}(\sigma)\}$ such that $P_{ab}(\sigma)=1$ for $a,b\in\sigma$ and $P_{ab}(\sigma)=0$ otherwise. Further

$$P_{rs}=\left\{\sigma\in P:\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}P_{ab}(\sigma)\le r,s\right\}$$

that is P_{rs} is the set of those σ whose support has cardinality at most r,s and we get

$\Phi=\left\{\phi=(\phi_{ab}):0<\phi_{11}\le\phi_{ab}\le\phi_{a+1,b+1}a,b\phi_{a+1,b+1}\le(a+1,b+1)\phi_{ab}\right\}$.

We define

$$\tau_{rs}=\frac{1}{\Delta\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}.$$

3.14 Definition

Let $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two non-negative double vector valued sequences of interval numbers $\Delta_{11}\tilde{x}=(\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y}=(\Delta_{11}\tilde{y}_{mn})$ are said to be $\phi-$summable to $\overline{0}$ that is

$$\left[\left\|\chi^{2I}_{A\phi M},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right]=lim_{r,s\to\infty}\frac{1}{\Delta\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}$$

$$\left\{\left[M\left(\|A^{uv}_{mn}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]=\overline{\theta}\right\}.$$

In this we write $\Delta_{11}\chi^2\to\overline{0}$ and the set of all strongly $\phi-$summable sequences is denoted by $[\phi]$.

3.15 Definition

Consider two double non-negative vector valued sequences $\Delta_{11}x=(\Delta_{11}x_{mn})$ and $\Delta_{11}y=(\Delta_{11}y_{mn})$ and let $E\subseteq\mathbb{N}$ is said to be the $\phi-$density of E.

$$\delta_\phi(E)=lim_{r,s\to\infty}\frac{1}{\Delta\phi_{rs}}\left|\{m,n\in\sigma,\sigma\in P_{rs}:m,n\in E\}\right|.$$

It is clear that $\delta_\phi(E)\le\delta(E)$.

3.16 Definition

Let $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative vector valued sequences of interval numbers $\Delta_{11}\tilde{x}=(\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y}=(\Delta_{11}\tilde{y}_{mn})$ are said to be $\phi-$statistical convergent summable to $\overline{0}\in\mathbb{R}$ if for each $\epsilon>0$

$$lim_{r,s\to\infty}\frac{1}{\Delta\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}$$

$$\left|\left\{\left[M\left(A^{uv}_{mn}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\right\}\right|=\overline{\theta}.$$

In this we write $\Delta\chi^2\to\overline{0}$ and it is denoted by St_ϕ.

3.17 Definition

The double non-negative vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be cesáro strong *M*–asymptotically double lacunary of multiple $\overline{0}$

$$\left[\left\|\chi^{2I}_{A\phi M}, \left(d\left(x_1,0\right), d\left(x_2,0\right), \cdots, d\left(x_{n-1},0\right)\right)\right\|_p\right] = lim_{a,b\to\infty} \frac{1}{\Delta ab} \sum_{m=1}^{a} \sum_{n=1}^{b}$$

$$\left\{\left[M\left(\| A^{uv}_{mn}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, \left(d\left(x_1,0\right), d\left(x_2,0\right), \cdots, d\left(x_{n-1},0\right)\|_p\right)\right)\right] = \overline{\theta}\right\}$$

denoted by $\left(\Delta_{11}\tilde{x}_{mn}\right) \cong \left(\Delta_{11}\tilde{y}_{mn}\right)$ and simply cesáro Orlicz asymptotically equivalent.

3.18 Definition

Consider $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double non-negative vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be cesáro strong *M*–asymptotically double lacunary $\Delta I -$ of multiple $\overline{0}$, provided that for every $\delta > 0$

$$\sum_{m=1}^{a} \sum_{n=1}^{b}$$

$$\left\{\left[M\left(\| A^{uv}_{mn}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, \left(d\left(x_1,0\right), d\left(x_2,0\right), \cdots, d\left(x_{n-1},0\right)\|_p\right)\right)\right] \geq \delta\right\} \in \Delta I,$$

where $a, b \in \mathbb{N}$. Simply cesáro asymptotically $\Delta I -$ equivalent.

3.19 Definition

The two non-negative double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be double lacunary ideal $\Delta\phi -$ of multiple $\overline{0}$, provided that

$$lim_{r,s\to\infty} \frac{1}{\Delta\phi_{rs}} \sum_{m\in\sigma} \sum_{n\in\sigma, \sigma\in P_{rs}}$$

$$\left\{\left[M\left(\| A^{uv}_{mn}\phi\left(\left(m+n\right)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|\right)^{1/m+n}, \left(d\left(x_1,0\right), d\left(x_2,0\right), \cdots, d\left(x_{n-1},0\right)\|_p\right)\right)\right] = \overline{\theta}\right\}.$$

Simply cesáro asymptotically $\Delta\phi -$ equivalent.

$$lim_{r,s\to\infty} \frac{1}{\Delta\phi_{rs}} \sum_{m\in\sigma} \sum_{n\in\sigma, \sigma\in P_{rs}}$$

$$\left\{\left\{\left[M\left(\| A^{uv}_{mn}\phi\left(\left(m+n\right)!\left|\Delta_{11}\tilde{x}_{mn}, \overline{0}\right|\right)^{1/m+n}, \left(d\left(x_1,0\right), d\left(x_2,0\right), \cdots, d\left(x_{n-1},0\right)\|_p\right)\right)\right] \geq \delta\right\}\right\} \in \Delta I.$$

Simply cesáro asymptotically $\Delta I - \Delta\phi -$ equivalent.

3.20 Definition

Let $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double non-negative vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be asymptotically double lacunary $\Delta\phi -$ of multiple $\overline{0} \in \mathbb{R}$, provided that for every $\epsilon > 0$

$$lim_{r,s\to\infty} \frac{1}{\Delta\phi_{rs}}$$

$$\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\geq\epsilon\right\}\right|=\overline{\theta},$$

where $m,n\in\sigma,\sigma\in P_{rs}$. Simply asymptotically $\Delta\phi-$ equivalent.

3.21 Definition

Consider two double non-negative vector valued sequences of interval numbers $\Delta_{11}\tilde{x}=\left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y}=\left(\Delta_{11}\tilde{y}_{mn}\right)$ are said to be asymptotically double lacunary $\phi-$ of multiple $\overline{0}\in\mathbb{R},$ provided that for every $\epsilon>0$ and for every $\delta>0$

$$lim_{r,s\to\infty} \frac{1}{\Delta\phi_{rs}}$$

$$\left|\left\{\left[M\left(\| A_{mn}^{uv}\left((m+n)!\left|\frac{\Delta\tilde{x}_{mn}}{\Delta\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\geq\epsilon\right\}\right|\geq\delta\in I,$$

where $m,n\in\sigma,\sigma\in P_{rs}$. Simply asymptotically $\Delta I-\Delta\phi-$ equivalent.

4. Main Results

4.1 Theorem

If $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double vector valued sequences of interval numbers $\Delta_{11}\tilde{x}=\left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y}=\left(\Delta_{11}\tilde{y}_{mn}\right)$ then the two sequences are

(a) $\Delta I-$equivalent $\Rightarrow\Delta I-$statistically equivalent

(b) $\Delta I-$equivalent $\Rightarrow\Delta I-$equivalent, if M is finite.

Proof: Consider $\Delta I-$equivalent and $\epsilon>0$ be given we write

$$\frac{1}{\Delta ab}\sum_{m=1}^{a}\sum_{n=1}^{b}$$

$$\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\right\}\geq$$

$$\frac{M(\epsilon)}{\Delta ab}\left|\left\{\left[\left(\left(\| A_{mn}^{uv}\phi(m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\epsilon\right\}\right|,$$

where $m\leq a,n\leq b$

Consequently for any $\gamma>\overline{\theta},$ we have

$$\frac{1}{\Delta ab}\left|\left\{\left[\left(\left(\| A_{mn}^{uv}\phi(m+n)!\left|\frac{\Delta\tilde{x}_{mn}}{\Delta\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\epsilon\right\}\right|\geq$$

$$\frac{\gamma}{M(\epsilon)}\subseteq\frac{1}{\Delta ab}\sum m=1a\sum n=1b\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\geq$$

$\gamma\in\Delta I,$ where $a,b\in\mathbb{N}$ and $m\leq a,n\leq b$

Hence $\Delta I-$equivalent.

(b) Consider M is finite and ΔI −statistically equivalent then there exists a real number $N > \overline{\theta}$ such that $sup_t M(t) \le N$. Moreover for any $\epsilon > \overline{\theta}$ we can write

$$\frac{1}{\Delta ab}\sum_{m=1}^{a}\sum_{n=1}^{b}$$
$$\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\right\} \le$$
$$\frac{N}{\Delta ab}\left|\left\{\left[\left(\left(\| A_{mn}^{uv}\phi(m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \ge \epsilon\right\}\right| + M(\epsilon),$$

where $m \le a, n \le b$. Now applying $\epsilon \to \overline{\theta}$, then the result follows.

4.2 Theorem

Let $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ then the two sequences and $(\Delta\phi_{rs})$ be a non-decreasing sequence of positive real numbers such that $\Delta\phi_{rs} \to \infty$ as $r, s \to \infty$ and $\Delta\phi_{rs} \le r, s$ for every $r, s \in \mathbb{N}$. Then statistically equivalent $\Rightarrow \Delta\phi -$statistically equivalent.

Proof: By definition of the sequences $\Delta\phi_{rs}$ it follows that $inf_{rs}\,\frac{rs}{rs - \Delta\phi_{rs}} \ge 1$. Then there exists a $t > 0$ such that

$$\frac{rs}{\Delta\phi_{rs}} \le \frac{1+t}{t}$$

suppose that two sequences are statistically equivalent then for every $\epsilon > \overline{\theta}$ and sufficiently large r, s we have

$$\frac{1}{\Delta\phi_{rs}}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta\tilde{x}_{mn}}{\Delta\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right] \ge \epsilon\right\}\right| =$$

$$\frac{1}{rs}\frac{rs}{\Delta\phi_{rs}}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right] \ge \epsilon\right\}\right| -$$

$$\frac{1}{\Delta\phi_{rs}}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right] \ge \epsilon\right\}\right| \le$$

$$\frac{1+t}{t}\frac{1}{rs}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right] \ge \epsilon\right\}\right| -$$

$$\frac{1}{\Delta\phi_{rs}}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right] \ge \epsilon\right\}\right|.$$

where $m, n \in \sigma, \sigma \in P_{rs}, m \in \{1, 2, \cdots, r\} - \sigma, n \in \{1, 2, \cdots, s\} - \sigma, \sigma \in P_{rs}$ and
$m_0 \in \{1, 2, \cdots, r\} - \sigma, n_0 \in \{1, 2, \cdots, s\} - \sigma, \sigma \in P_{rs}$.
This completes the proof.

4.3 Theorem

If two double vector lacunary valued sequences of interval numbers $\Delta\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ and $m, n \in \mathbb{Z}$ such that $\Delta\phi_{rs} \le \left[\Delta\phi_{rs}\right] + mn, sup_{rs} \frac{\left[\Delta\phi_{rs}\right] + mn}{\Delta\phi_{r-1s-1}} < \infty$, then the two vector valued sequences are $\Delta\phi -$ statistically equivalent $\Rightarrow$ statistically equivalent.

Proof: If $sup_{rs} \frac{\left[\Delta\phi_{rs}\right] + mn}{\Delta\phi_{r-1s-1}} < \infty$, then there exists $N > \overline{\theta}$ such that $\frac{\left[\Delta\phi_{rs}\right] + mn}{\Delta\phi_{rs} - 1} < N$ for all $r, s \ge 1$. Let a, b be an integers such that $\Delta\phi_{r-1,s-1} < a, b \le \Delta\phi_{rs}$. Then for every $\epsilon > \overline{\theta}$ we have

$$\frac{1}{\Delta ab}\left|\left\{\left[\left(\| A_{mn}^{uv}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, (d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right) \|_p)\right)\right] \ge \epsilon\right\}\right|$$

$$\le \frac{1}{\Delta ab}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, (d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right) \|_p)\right)\right] \ge M\left(\epsilon\right)\right\}\right|$$

$$\le \frac{1}{\left[\Delta\phi_{rs}\right] + mn} \frac{\left[\Delta\phi_{rs}\right] + mn}{\Delta\phi_{r-1s-1}}$$

$$\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, (d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right) \|_p)\right)\right] \ge M\left(\epsilon\right)\right\}\right|$$

$$\le \frac{1}{\left[\Delta\phi_{rs}\right] + mn} \frac{\left[\Delta\phi_{rs}\right] + mn}{\Delta\phi_{r-1s-1}}$$

$$\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, (d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right) \|_p)\right)\right] \ge M\left(\epsilon\right)\right\}\right|$$

$$\le \frac{N}{\left[\Delta\phi_{rs}\right] + mn}$$

$$\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left(\left(m+n\right)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}}, \overline{0}\right|\right)^{1/m+n}, (d\left(x_1, 0\right), d\left(x_2, 0\right), \cdots, d\left(x_{n-1}, 0\right) \|_p)\right)\right] \ge M\left(\epsilon\right)\right\}\right|,$$

where $m \le a, n \le b, m, n \le \Delta\phi_{rs}, m, n \in \sigma, \sigma \in P_{|\Delta\phi_{rs}|+mn}, m, n \in \sigma, \sigma \in P_{|\phi_{rs}|+mn}$.

4.4 Theorem

If $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double vector valued sequences of interval numbers $\Delta_{11}\tilde{x} = \left(\Delta_{11}\tilde{x}_{mn}\right)$ and $\Delta_{11}\tilde{y} = \left(\Delta_{11}\tilde{y}_{mn}\right)$ and $m, n \in \mathbb{Z}$, then the two vector valued sequences of interval numbers are Cesáro equivalent $\Rightarrow \Delta\phi -$ equivalent

Proof: From the definition of sequence $\left(\Delta\phi_{rs}\right)$ it follows that $inf_{rs} \frac{rs}{rs - \Delta\phi_{rs}} \ge 1$. Then there exists $t > \overline{\theta}$ such that

$$\frac{rs}{\Delta\phi_{rs}} \leq \frac{1+t}{t}.$$

Then the following relation

$$\frac{1}{\Delta\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}$$

$$\left[M\left(\| A_{mn}^{uv}\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]=$$

$$\frac{rs}{\Delta\phi_{rs}}\frac{1}{rs}\sum_{m=1}^{a}\sum_{n=1}^{b}$$

$$\left[M\left(\| A_{mn}^{uv}\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]-$$

$$\frac{1}{\Delta\phi_{rs}}\sum_{m\in\{1,2,\cdots r\}\cdots\sigma}\sum_{n\in\{1,2,\cdots s\}\cdots\sigma,\sigma\in P_{rs}}$$

$$\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\leq$$

$$\frac{1+t}{t}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}$$

$$\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]-$$

$$\frac{1}{\Delta\phi_{rs}}\sum_{m_0\in\{1,2,\cdots r\}\cdots\sigma}\sum_{n_0\in\{1,2,\cdots s\}\cdots\sigma,\sigma\in P_{rs}}$$

$$\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right].$$

Since the two sequences are Cesáro equivalent and M is continuous letting $r,s\to\infty$ we get

$$\frac{1}{\Delta\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}$$

$$\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\bar{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\to\bar{\theta}.$$

Hence two sequences are $\Delta\phi-$ equivalent.

4.5 Theorem

If two double lacunary vector valued sequences of interval numbers $\Delta_{11}\tilde{x}=(\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y}=(\Delta_{11}\tilde{y}_{mn})$ and $m,n\in\mathbb{Z}$, then the two vector valued sequences of interval numbers are

(a) Cesáro equivalent $\Rightarrow$ statistcally equivalent.

(b) If M satisfies the Δ_2-condition and $(\Delta_{11}x_{mn})\in\left[\Lambda_M^2,\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]$ then the two vector valued sequences of interval numbers are statistically equivalent $\Rightarrow$ Cesáro equivalent.

Proof (a): Consider two vector valued sequences are Cesáro equivalent. Then for every $\epsilon > \overline{\theta}$ we have

$$\frac{1}{\Delta ab}\left|\left\{\left[\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\geq\epsilon\right\}\right|\leq$$

$$\frac{1}{\Delta ab}\left|\left\{\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right]\geq M(\epsilon)\right\}\right|\leq$$

$$\frac{1}{\Delta ab}\Sigma_{m=1}^{a}\Sigma_{n=1}^{b}\left[M\left(\| A_{mn}^{uv}\phi\left((m+n)!\left|\frac{\Delta_{11}\tilde{x}_{mn}}{\Delta_{11}\tilde{y}_{mn}},\overline{0}\right|\right)^{1/m+n},(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p)\right)\right],$$

where $m \leq a, n \leq b.$ This completes the proof.

Proof (b): Follows from the same technique of Theorem 4.1 and Theorem 4.4.

Theorem If two double vector lacunary valued sequences of interval numbers $\Delta_{11}\tilde{x} = (\Delta_{11}\tilde{x}_{mn})$ and $\Delta_{11}\tilde{y} = (\Delta_{11}\tilde{y}_{mn})$ and $m, n \in \mathbb{Z}$ then the two sequences of interval numbers are

(a) $\Delta\phi -$ equivalent $\Rightarrow \Delta\phi -$statistcally equivalent

(b) If M satisfies the Δ_2 $-$condition and $(\Delta_{11}\tilde{x}_{mn}) \in \left[\left\|\Lambda_M^2,(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right]$ then the two vector valued sequences of interval numbers are statistically equivalent $\Rightarrow \Delta\phi -$statistically equivalent.

(c) If M satisfies the Δ_2 $-$\$condition, then $\Delta\phi -$ equivalent $\bigcap\left[\left\|\Lambda_M^2,(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right] = \Delta\phi -$ statistically equivalent $\bigcap\left[\left\|\Lambda_M^2,(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\right\|_p\right].$

Proof: Follows from the same technique of Theorem 4.1 and Theorem 4.5.

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