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Original Article

Topp-Leone generalized Rayleigh distribution and its applications

Pimwarat Nanthaprut, Mena Patummasut, and Winai Bodhisuwan*

Department of Statistics, Faculty of Science, Kasetsart University,
Chatuchak, Bangkok, 10900 Thailand

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Abstract

A new three-parameter lifetime distribution called the Topp-Leone generalized Rayleigh (TLGR) distribution obtained by the Topp-Leone generator based on the generalized Rayleigh (GR) distribution is proposed. Some of the proposed distribution's mathematical properties such as its survival function, hazard function, moments, and moment generating function are investigated. Furthermore, the expansion of the probability density function is derived in terms of a linear combination of the GR distribution; this function is used to obtain its moments, and the maximum likelihood method is applied to estimate its parameters. Using real-life datasets, a comparative analysis is carried out to fit them to the TLGR, GR, and Rayleigh distributions based on the Anderson-Darling test and the Akaike information and Bayesian information criteria. The results from these datasets exhibit that the TLGR distribution is more appropriate than the other distributions.

Keywords: lifetime distribution, Rayleigh distribution, maximum likelihood estimation, Topp-Leone generator

1. Introduction

Lifetime analysis refers to survival time or failure time and plays an important role in various fields including engineering, biological sciences, finance, and medicine in predicting, for example, the time to failure of equipment, time to occurrence of events, time of death, time of the next earthquake, and so on. These phenomena can be explained by the characteristics of a lifetime distribution in a statistical framework, and there are many well-known distributions suitable for lifetime analysis, such as exponential, Weibull, and chi-squared.

In the late 19th century, a number of interesting distributions were discovered, one of which is the Rayleigh distribution (Rayleigh, 1880), which is a special case of the Weibull distribution and is often encountered in a number of areas: in particular, lifetime testing and reliability. For example, Siddiqui (1962) discussed datasets connected with the Rayleigh distribution, Hoffman and Karst (1975) studied properties and the bivariate of the Rayleigh distribution and dealt with its application to a targeted problem, and Ali and Woo (2005) proposed inference on reliability $P(Y < X)$ in a *p*-dimensional Rayleigh distribution. However, the Rayleigh

*Corresponding author
Email address: fsciwnb@ku.ac.th

distribution only deals with increasing failure rate, which poses a weakness for modeling phenomena with other failure rate shapes. Vodă (1976a) proposed the generalized Rayleigh (GR) distribution with two parameters, and also discussed parameter estimation for a two-component mixture of the GR distribution (Vodă, 1976b) to accommodate better goodness of fit in real-life applications.

Besides, the GR distribution has been applied to lifetime testing and reliability; for instance, Tsai and Wu (2006) developed an acceptance sampling plan for a truncated lifetime test when the data followed a GR distribution, and Aslam (2008) developed an economic reliability acceptance sampling plan for a GR distribution when the value of the shape parameter is known.

The extension of several families of distributions have been discussed in the last decade in order to generate a more flexible family of distributions, and the method for creating this consisted of two main components: a generator and the parent distributions (Alzaatreh, Lee, & Famoye, 2013; Lee, Famoye, & Alzaatreh, 2013). Following this approach, Sangsanit and Bodhisuwan (2016) recently proposed the Topp-Leone generator (TLG) for distributions by using a one-parameter Topp-Leone distribution (Topp & Leone, 1955) as a generator to establish a new family of distributions. The TLG for the distributions not only adds one parameter to the parent distribution but also creates an advantage for the parent distribution. The authors also demonstrated one of its special cases, called the Topp-Leone generalized exponential distribution, and suggested its flexibility for fitting real-life data to a generalized exponential distribution. The cumulative distribution function (cdf) $F(x)$ of the TLG random variable X is defined as

$$\begin{aligned} F(x) &= \int_0^{G(x)} 2\beta t^{\beta-1}(1-t)(2-t)^{\beta-1}\mathrm{d}t; \quad 0<t<1,\ \beta>0 \\ &= G(x)^{\beta}\left(2-G(x)\right)^{\beta}; \qquad x>0, \end{aligned} \tag{1}$$

where $G(x)$ is the cdf of the parent distribution and β is the shape parameter of the TL distribution. The corresponding probability density function (pdf) of the new distribution is

$$f(x) = 2\beta g(x)\left(1-G(x)\right)G(x)^{\beta-1}\left(2-G(x)\right)^{\beta-1},$$

where $g(x)$ is the pdf of the parent distribution.

In this research, we attempt to produce a new three-parameter lifetime distribution, namely the Topp-Leone generalized Rayleigh (TLGR) distribution, and some mathematical properties are also studied. Furthermore, the TLGR random variable is obtained by generating GR random variates from the TLG framework.

The rest of this paper is composed as follows. In Section 2, we present the GR distribution, while a new lifetime distribution, the TLGR distribution, is proposed in Section 3. After this, some mathematical properties including the quantile function, expansion of the pdf, moments, and moment generating function (mgf) are derived in Section 4. The parameter estimation according to the maximum likelihood method is discussed in Section 5 and more detail on the information matrix is included in the Appendix. Moreover, the fitting results of real-life data with the TLGR, GR, and Rayleigh distributions are verified by the Anderson-Darling (AD) test and the Akaike information and Bayesian information criteria in the application section.

2. GR Distribution

The GR distribution was first presented by Vodă (1976a), who also provided various mathematical properties such as non-central and central moments. Moreover, the GR distribution was used to solve a variety of lifetime and reliability problems. It has two parameters, namely scale parameter θ and shape parameter α.

Definition 1:

Let X be a random variable of the GR distribution with $\theta>0$ and $\alpha>-1$, then the pdf and cdf of X are given by

$$g(x) = \frac{2\theta^{\alpha+1}}{\Gamma(\alpha+1)}x^{2\alpha+1}e^{-\theta x^2}; \quad x>0,\ \theta>0,\ \alpha>-1 \tag{2}$$

and

$$G(x)=\frac{\gamma\left(\alpha+1,\theta x^2\right)}{\Gamma\left(\alpha+1\right)}, \qquad (3)$$

respectively, where $\gamma\left(a,b\right)=\int_0^b t^{a-1}e^{-t}\mathrm{d}t$ is the lower incomeplete gamma function and $\Gamma\left(a\right)=\int_0^\infty t^{a-1}e^{-t}\mathrm{d}t$ is the incomplete gamma function.

The GR distribution reduces to the Rayleigh distribution when $\alpha=0$ and $\theta=1/(2\lambda)^2$. If we take $\alpha=1/2$ and $\theta=1/(2\lambda^2)$, then we obtain the Maxwell distribution (Bekker & Roux, 2005), and if $\alpha=-1/2$ and $\theta=1/(2\sigma^2)$, the half-normal distribution is realized (Tanis & Hogg, 1993). In the case of $\alpha\geq 0$, we suppose that $\alpha=\nu/2-1$ and $\theta=1/2$, where $\nu\in N$, and the GR distribution becomes a Chi-squared distribution with ν degrees of freedom. Vodă (1976a) also derived the rth moment of the GR distribution by applying the integral formula provided by Grandshteyn and Ryzhik (2007). Hence,

$$E(X^r)=\frac{\Gamma(r/2+\alpha+1)}{\Gamma(\alpha+1)\theta^{r/2}}. \qquad (4)$$

Furthermore, the expectation and variance of the GR distribution are $\Gamma(3/2+\alpha)\big/\left(\Gamma(\alpha+1)\theta^{1/2}\right)$ and $\left((\alpha+1)/\theta\right)-\left(\Gamma^2\left(3/2+\alpha\right)\big/\left(\theta\,\Gamma^2(\alpha+1)\right)\right)$, respectively.

3. New Lifetime Distribution

In this section, we propose a new lifetime distribution called the TLGR distribution. Its pdf and cdf are obtained from the TLG and a parent distribution, in this case the GR distribution.

Theorem 1:

Let X be a positive continuous random variable of the TLGR distribution with parameters $\theta,\beta>0,$ and $\alpha>-1,$ denoted as $X\sim TLGR(\theta,\alpha,\beta).$ The cdf and pdf of X are

$$F(x)=\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta}\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta} \qquad (5)$$

and

$$f(x)=\frac{4\beta\theta^{\alpha+1}}{\Gamma(\alpha+1)}x^{2\alpha+1}e^{-\theta x^2}\left(1-\gamma_1\left(\alpha+1,\theta x^2\right)\right)\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}$$
$$\times\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1};\quad x>0, \qquad (6)$$

respectively, where $\gamma_1(a,b)=\gamma(a,b)/\Gamma(a)$ is the incomplete gamma ratio function.

Proof:

The cdf of X can be obtained by substituting Equation (3) into Equation (1). By differentiating the cdf of X with respect to X, the pdf is obtained, i.e.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(X)&=\frac{\mathrm{d}}{\mathrm{d}x}\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta}\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta}\\
&=\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta}\left(\beta\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}\left(-\frac{2\theta x(\theta x^2)^{\alpha}e^{-\theta x^2}}{\Gamma(\alpha+1)}\right)\right)\\
&\quad+\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta}\left(\beta\gamma_1\left(\alpha+1,\theta x^2\right)^{\beta-1}\left(\frac{2\theta x(\theta x^2)^{\alpha}e^{-\theta x^2}}{\Gamma(\alpha+1)}\right)\right)\\
&=\beta\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}\left(2-2\gamma_1\left(\alpha+1,\theta x^2\right)\right)\\
&\quad\times\left(\frac{2\theta x(\theta x^2)^{\alpha}e^{-\theta x^2}}{\Gamma(\alpha+1)}\right)\\
&=\frac{4\beta\theta^{\alpha+1}}{\Gamma(\alpha+1)}x^{2\alpha+1}e^{-\theta x^2}\left(1-\gamma_1\left(\alpha+1,\theta x^2\right)\right)\left(\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}\\
&\quad\times\left(2-\gamma_1\left(\alpha+1,\theta x^2\right)\right)^{\beta-1}.
\end{aligned}$$

Figure 1 shows plots of the pdf and cdf of the TLGR distribution. The pdf can be classified into two cases; one is a decreasing function while the other is unimodal and right-tailed and depends on the α and β parameter values.

4. Mathematical Properties of the TLGR Distribution

Some of the mathematical properties for the TLGR distribution are derived, such as its quantile, survival, and hazard functions, the expansion of the probability density function, moments, and mgf.

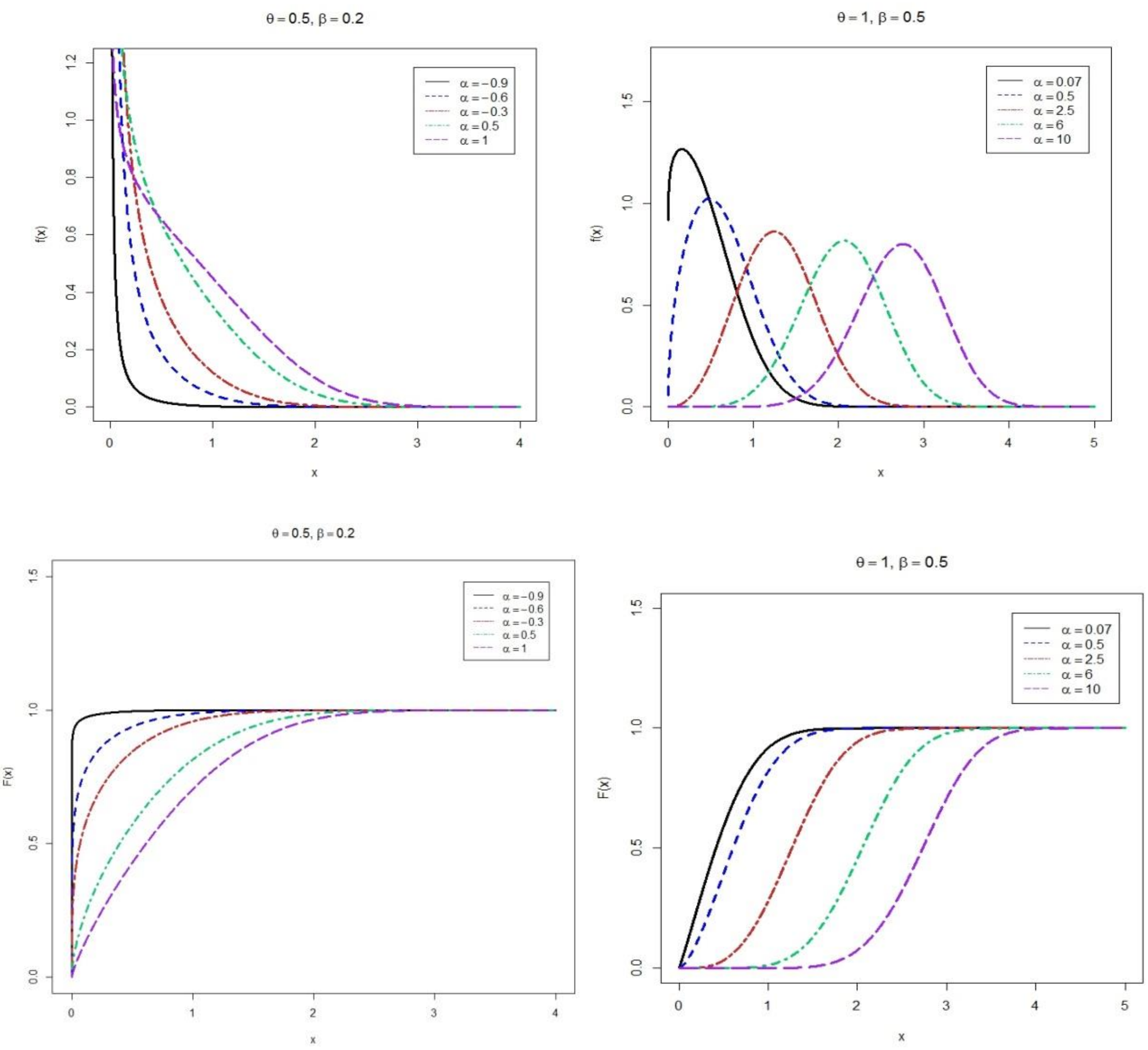


Figure 1. Pdf and cdf plots of the TLGR distribution for different parameter values.

4.1 Quantile function

The quantile function is obtained from the inversion method as

$$X = \sqrt{\theta^{-1}Q_G\left(1-\sqrt{1-U^{1/\beta}}\right)},$$

where U is a uniform (0,1) distribution and $Q_G(\cdot)$ is the standardized gamma quantile function with shape parameter $\alpha+1$. The standardized gamma quantile function is available in most statistical packages, such as the `zipfR` package (Evert & Baroni, 2007) in the R language (R Core Team, 2016). This quantile function can be generated from a random sample of the TLGR distribution. We demonstrate two cases of 50 random variates from the quantile function in Figure 2.

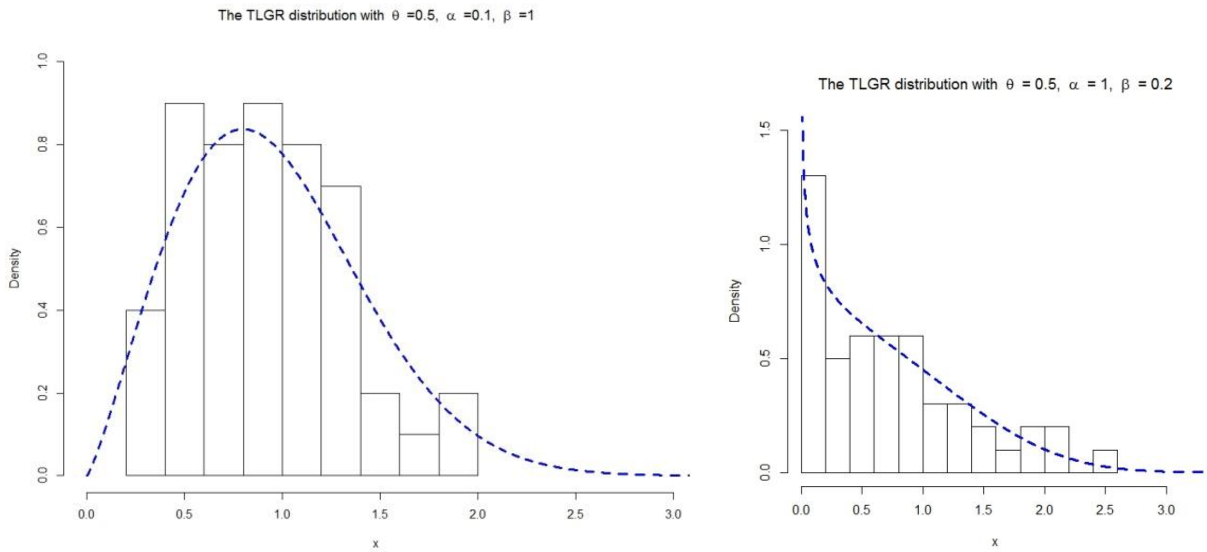


Figure 2. Generated samples and fitted density function of the TLGR distribution for different values of $\theta, \alpha,$ and β.

4.2 Survival and hazard functions

The survival function, $S(x)$, also known as the reliability function, is usually related to the mortality of specimens or failure of equipment or systems, and is the probability that the system will survive beyond a specified time. Similarly, alternative characterization of the distribution results in the hazard function, $h(x)$, which is sometimes known as the failure rate or hazard rate. The hazard function refers to the instantaneous rate of death or failure at a specified time and is the ratio of the pdf to the survival function. These are obtained respectively as

$$S(x) = 1 - \left(\left(\gamma_1\left(\alpha+1, \theta x^2\right) \right)^{\beta} \left(2 - \gamma_1\left(\alpha+1, \theta x^2\right) \right)^{\beta} \right)$$

and

$$h(x) = \frac{4\beta\theta^{\alpha+1} x^{2\alpha+1} e^{-\theta x^2} \left(1-\gamma_1\left(\alpha+1, \theta x^2\right)\right)\left(\gamma_1\left(\alpha+1, \theta x^2\right)\right)^{\beta-1}\left(2-\gamma_1\left(\alpha+1, \theta x^2\right)\right)^{\beta-1}}{\Gamma(\alpha+1)\left[1-\left(\gamma_1\left(\alpha+1, \theta x^2\right)^{\beta}\left(2-\gamma_1\left(\alpha+1, \theta x^2\right)\right)^{\beta}\right)\right]}.$$

Figure 3 shows the hazard function of the TLGR distribution. We can see that the TLGR hazard function can be bathtub-shaped or monotonically increasing.

4.3 Expansion of the probability density function

We consider the pdf in the form of a series expansion. The pdf in Equation (6) can be provided by a simple expansion, and by using binomial, lower incomplete gamma, and power series expansions, that of the TLGR distribution becomes convenient to use. First, we consider the term $\left(1-G(x)\right)\left(G(x)\right)^{\beta-1}\left(2-G(x)\right)^{\beta-1}$ to be applied by a binomial series expansion as follows:

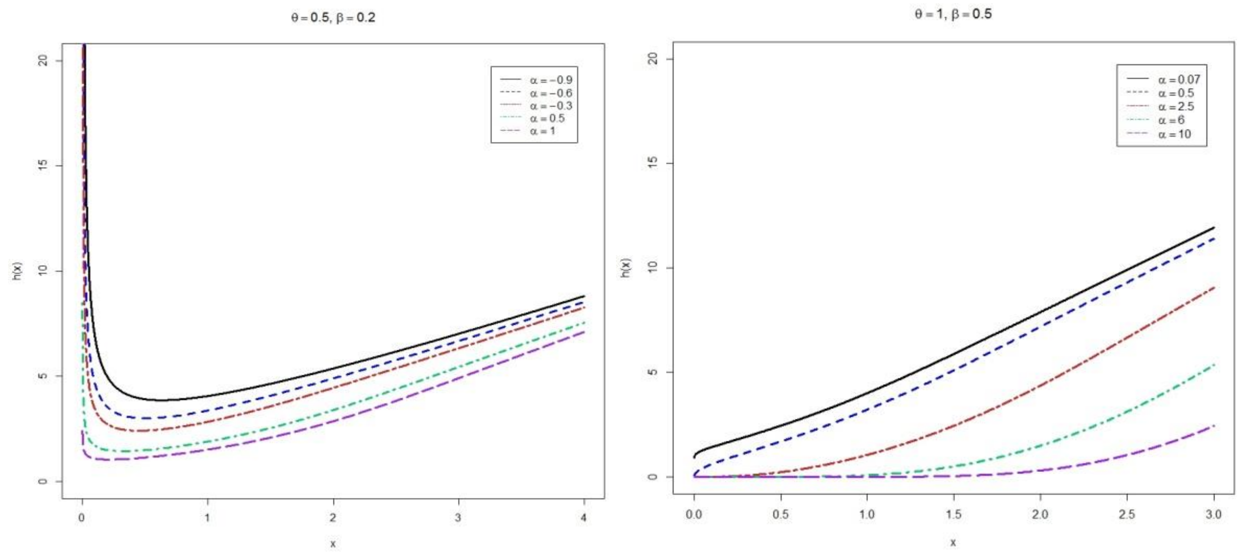


Figure 3. Hazard plots of the TLGR distribution for different parameter values.

$$\begin{aligned}(1-G(x))\left(G(x)\right)^{\beta-1}\left(2-G(x)\right)^{\beta-1} &= \sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}2^{\beta-1-i}\left(1-G(x)\right)G(x)^{\beta-1+i}\\ &= \sum_{i=0}^{\infty}(-1)^i\, 2^{\beta-1-i}\binom{\beta-1}{i}\left(1-G(x)\right)\left(1-\left(1-G(x)\right)\right)^{\beta-1+i}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\, 2^{\beta-1-i}\binom{\beta-1}{i}\binom{\beta-1+i}{j}\left(1-G(x)\right)^{j+1}\\ &= \sum_{i,j=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{i+j+k}\, 2^{\beta-1-i}\binom{\beta-1}{i}\binom{\beta-1+i}{j}\binom{j+1}{k}G(x)^k.\end{aligned} \quad (7)$$

Substituting Equation (7) into Equation (6) results in

$$f(x)=\frac{4\beta\theta^{\alpha+1}x^{2\alpha+1}e^{-\theta x^2}}{\Gamma(\alpha+1)}\sum_{i,j=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{i+j+k}\, 2^{\beta-1-i}\binom{\beta-1}{i}\binom{\beta-1+i}{j}\binom{j+1}{k}G(x)^k. \quad (8)$$

In Equation (8), β is a non-integer, but on the other hand, if it is a positive integer, the index i in Equation (8) stops at $\beta-1$. The cdf of the GR distribution (Equation (3)) has a lower incomplete gamma term, thus we use the series representation for the incomplete gamma function (Gradshteyn & Ryzhik, 2007) resulting in

$$\gamma(\alpha,x)=x^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!(\alpha+n)}.$$

Therefore, $G_{\theta,\alpha}(x)$ can be rewritten as

$$G_{\theta,\alpha}(x) = \frac{\gamma(\alpha+1,\theta x^2)}{\Gamma(\alpha+1)} = \frac{(\theta x^2)^{\alpha+1}}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\theta x^2)^n}{n!(\alpha+1+n)},$$

then

$$G_{\theta,\alpha}(x)^k = \left(\frac{(\theta x^2)^{\alpha+1}}{\Gamma(\alpha+1)}\right)^k \left(\sum_{n=0}^{\infty} \frac{(-1)^n (\theta x^2)^n}{n!(\alpha+1+n)}\right)^k.$$

The power series (Gradshteyn & Ryzhik, 2007) is applied in $G_{\theta,\alpha}(x)^k$. The power series is

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty} c_{n,k} x^n,$$

where the coefficients $c_{n,k}$ are obtained from the relationship $c_{0,k} = a_0^k$, $n = 0$, and

$$c_{n,k} = \frac{1}{na_0} \sum_{m=1}^{n} (km + m - n)\, a_m c_{n-m,k}; \quad n = 1, 2, \ldots, \quad a_m = \left((-1)^m \theta^m\right) / \left(m!(\alpha+1+m)\right).$$

Consequently, $G_{\theta,\alpha}(x)^k$ can be written as

$$G_{\theta,\alpha}(x)^k = \frac{\theta^{(\alpha+1)k}}{\left(\Gamma(\alpha+1)\right)^k} \sum_{n=0}^{\infty} c_{n,k} x^{2n+2k(\alpha+1)}. \tag{9}$$

Thus, the pdf in Equation (9) becomes

$$f(x) = \frac{4\beta\theta^{\alpha+1} x^{2\alpha+1} e^{-\theta x^2}}{\Gamma(\alpha+1)} \times \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} \frac{(-1)^{i+j+k}\, 2^{\beta-i-1} \theta^{(\alpha+1)k} c_{n,k}}{\left(\Gamma(\alpha+1)\right)^k} \binom{\beta-1}{i} \binom{\beta+i-1}{j} \binom{j+1}{k} x^{2n+2k(\alpha+1)}. \tag{10}$$

By taking Equation (10) into account, a convenient form can be expressed as

$$f(x) = \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n}\, g_{\theta,\alpha^*}(x), \tag{11}$$

where $\alpha^* = n + k\alpha + k + \alpha$ and

$$b_{i,j,k,n} = \frac{(-1)^{i+j+k}\, 2^{\beta-i-1} c_{n,k} \Gamma(\alpha^*+1)}{\theta^n \left(\Gamma(\alpha+1)\right)^k} \binom{\beta-1}{i} \binom{\beta+i-1}{j} \binom{j+1}{k}.$$

Consequently, the pdf of the TLGR distribution in Equation (11) can be expressed in terms of a linear combination of the GR distribution. Moreover, it is useful to derive several properties of the TLGR distribution.

4.4 Moments

Many interesting characteristics of the TLGR distribution can be considered through its moments, thus we derive the rth moment of X when $X \sim \text{TLGR}(\theta, \alpha, \beta)$. In addition, the rth moment of X can be expressed as

$$\begin{aligned} E(X^r) &= \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} \int_0^{\infty} x^r g_{\theta,\alpha^*}(x)\, \mathrm{d}\, x \\ &= \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} E_{\theta,\alpha^*}(X^r), \end{aligned} \tag{12}$$

which is an important result since it presents the moments of the TLGR distribution as a linear combination of GR moments with θ, α^* parameters. Substituting Equation (4) into $E_{\theta,\alpha^*}(X^r)$ in Equation (12) results in

$$E(X^r) = \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} \frac{\Gamma(r/2+\alpha^*+1)}{\Gamma(\alpha^*+1)\theta^{r/2}}.$$

Consequently, we can use the moments to find the expectation, variance, skewness, and kurtosis.

4.5 Mgf

The moments of the TLGR distribution were discussed earlier and at this point, the mgf of TLGR distribution is derived. Let $X \sim \text{TLGR}(\theta, \alpha, \beta)$, then the mgf can be obtained from Equation (11) as

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} g_{\theta,\alpha^*}(x)\, \mathrm{d}x \\ &= \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} \frac{2\theta^{\alpha^*+1}}{\Gamma(\alpha^*+1)} \int_0^{\infty} x^{2\alpha^*+1} e^{-\theta x^2} \mathrm{d}x. \end{aligned} \tag{13}$$

Considering the term $\int_0^{\infty} x^{2\alpha^*+1} e^{-\theta x^2} \mathrm{d}x$, this integral can be calculated in an accessible form based on Prudnikov, Brychkov, and Marichev's (1986) suggestion into Equation (13). Hence,

$$\begin{aligned} M_X(t) &= \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} \frac{2\theta^{\alpha^*+1}}{\Gamma(\alpha^*+1)} \frac{\Gamma(2\alpha^*+2)}{(2\theta)^{\alpha^*+1}} e^{t^2/8\theta} \mathrm{D}_{-(2\alpha^*+2)}(-t/\sqrt{2\theta}) \\ &= \frac{2\beta}{\Gamma(\alpha+1)} \sum_{i,j,n=0}^{\infty} \sum_{k=0}^{j+1} b_{i,j,k,n} \frac{\Gamma(2\alpha^*+2)}{2^{\alpha^*}} e^{t^2/8\theta} \mathrm{D}_{-(2\alpha^*+2)}(-t/\sqrt{2\theta}), \end{aligned}$$

where $\mathrm{D}_p(z)$ is a parabolic cylinder function (Gradshteyn & Ryzhik, 2007) given by

$$\mathrm{D}_p(z)=\frac{e^{-z^2/4}}{\Gamma(-p)}\int_0^\infty e^{-wz-\frac{w^2}{2}}w^{-p-1}\,\mathrm{d}w;\quad p<0.$$

We obtain the rth moment about the origin by differentiating the mgf *r* times with respect to *t* and setting *t* = 0 as it can then be used to calculate the expectation, variance, skewness, and kurtosis.

5. Parameter Estimation

The most widely used method for estimating parameters of a probability distribution is maximum likelihood estimation (MLE), and so in this research, we use it to estimate the unknown parameters of the TLGR distribution. Suppose $\Theta=(\theta,\alpha,\beta)^T$ is the unknown parameter vector of the TLGR distribution, and let $x_1,x_2,\ldots,x_n$ be an observed random sample of size n from the TLGR distribution, then the likelihood function can be expressed as

$$L(\Theta)=\left[\frac{4\beta\theta^{\alpha+1}}{\Gamma(\alpha+1)}\right]^n\prod_{i=1}^n x_i^{2\alpha+1}e^{-\theta x_i^2}\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^{\beta-1}$$
$$\times\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^{\beta-1},$$

and the log-likelihood function of Θ is given by

$$\begin{aligned}\ell(\Theta)&=\log\left(L(\Theta)\right)\\&=n\log(4)+n\log(\beta)+n(\alpha+1)\log(\theta)-n\log\left(\Gamma(\alpha+1)\right)\\&+(2\alpha+1)\sum_{i=1}^n\log(x_i)-\theta\sum_{i=1}^n x_i^2+\sum_{i=1}^n\log\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)\\&+(\beta-1)\sum_{i=1}^n\log\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)+(\beta-1)\sum_{i=1}^n\log\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right).\end{aligned}$$

By taking the partial derivatives of $\ell(\Theta)$ with respect to $\theta,\alpha,$ and β, the components of the unit score vector are

$$U_\theta(\Theta)=\frac{n(\alpha+1)}{\theta}-\sum_{i=1}^n x_i^2-\frac{\theta^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n\frac{e^{-\theta x_i^2}x_i^{2(\alpha+1)}}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}+\frac{(\beta-1)\theta^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n\frac{e^{-\theta x_i^2}x_i^{2(\alpha+1)}}{\gamma_1\left(\alpha+1,\theta x_i^2\right)}$$
$$-\frac{(\beta-1)\theta^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^n\frac{e^{-\theta x_i^2}x_i^{2(\alpha+1)}}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)},$$

$$U_\alpha(\Theta)=n\log(\theta)-n\psi(\alpha+1)+2\sum_{i=1}^n\log(x_i)+\sum_{i=1}^n\left\{\frac{\gamma_1\left(\alpha+1,\theta x_i^2\right)\psi(\alpha+1)}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}\right\}$$
$$-\sum_{i=1}^n\frac{\gamma'\left(\alpha+1,\theta x_i^2\right)|_\alpha}{\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)\Gamma(\alpha+1)}+(\beta-1)\sum_{i=1}^n\left\{\frac{\gamma'\left(\alpha+1,\theta x_i^2\right)|_\alpha}{\gamma_1\left(\alpha+1,\theta x_i^2\right)\Gamma(\alpha+1)}-\psi(\alpha+1)\right\}$$

$$+(\beta-1)\sum_{i=1}^{n}\left\{\frac{\gamma_1\left(\alpha+1,\theta x_i^2\right)\psi\left(\alpha+1\right)}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)}-\frac{\gamma'\left(\alpha+1,\theta x_i^2\right)|_{\alpha}}{\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)\Gamma\left(\alpha+1\right)}\right\},$$

$$U_{\beta}\left(\Theta\right)=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)+\sum_{i=1}^{n}\log\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right),$$

where $\psi(\cdot)=\Gamma'(\cdot)/\Gamma(\cdot)$ is a digamma function. A simplified form of $\gamma'\left(\alpha+1,\theta x_i^2\right)|_{\alpha}=\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\gamma\left(\alpha+1,\theta x_i^2\right)\right)$ is shown in the Appendix.

In addition, we derive a system of non-linear equations by setting $U_{\theta}(\Theta)=U_{\alpha}(\Theta)=U_{\beta}(\Theta)=0$. Subsequently, the maximum likelihood estimate $\hat{\Theta}=(\hat{\theta},\hat{\alpha},\hat{\beta})^T$ was obtained by numerically solving the system of non-linear equations through the stats package of the R language (R Core Team, 2016). In order to determine the confidence intervals and hypothesis tests on the model parameters, we needed to use the Fisher information matrix, the elements of which as information matrix $J(\Theta)$ are given in the Appendix.

6. Applications

In applications study, we fitted the TLGR model to three real-life datasets and compare the fitness with GR and Rayleigh distributions. The first dataset taken from Barlow, Toland, and Freeman (1984) contains data on the stress-rupture lifetime of Kevlar 49 epoxy strands at 90% stress level until all had failed; the complete data with the exact times of failure are presented in Table 1. Besides, Table 2 shows a dataset consisting of 63 observations of the tensile strength in GPa of single carbon fibers and impregnated 1000-carbon fiber tows (Bader & Priest, 1982). For the last dataset, the time-to-failure (10^3h) for 40 turbocharger suits (Xu, Xie, Tang, & Ho, 2003) are shown in Table 3.

Table 1. Epoxy strand stress-rupture lifetime dataset.

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06
0.07	0.07	0.08	0.09	0.09	0.10	0.10	0.11	0.11	0.12
0.13	0.18	0.19	0.2	0.23	0.24	0.24	0.29	0.34	0.35
0.36	0.38	0.4	0.42	0.43	0.52	0.54	0.56	0.60	0.6
0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73	0.79	0.79
0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29
1.31	1.33	1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53
1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69
7.89									

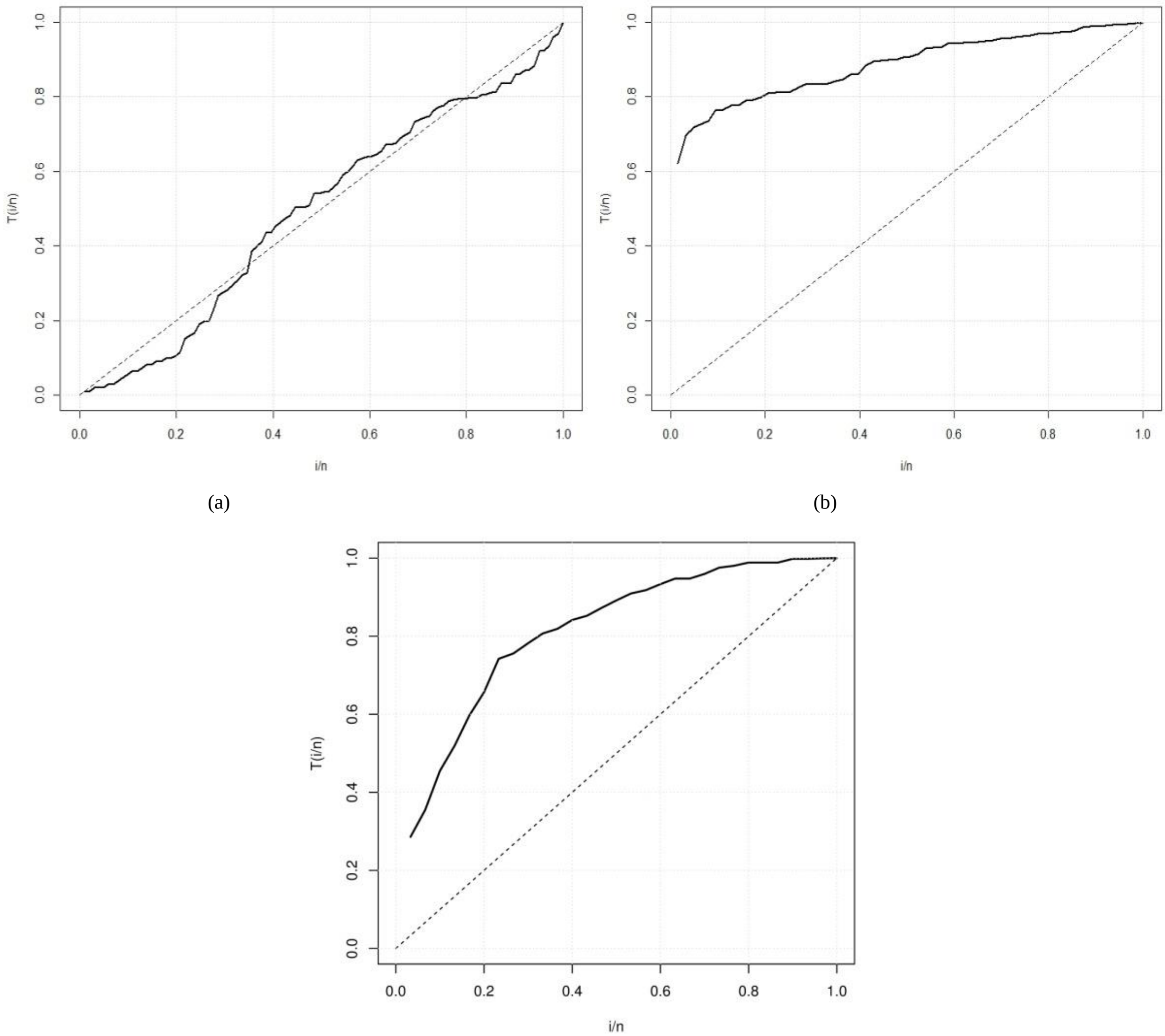
Table 2. Carbon fiber tensile strength dataset.

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445	2.454	2.474
2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659	2.675	2.738
2.740	2.856	2.917	2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435
3.493	3.501	3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020									

Table 3. Turbocharger suit time-to-failure (10^3h) dataset.

1.6	2.0	2.6	3.0	3.5	3.9	4.5	4.6	4.8	5.0
5.1	5.3	5.4	5.6	5.8	6.0	6.0	6.1	6.3	6.5
6.5	6.7	7.0	7.1	7.3	7.3	7.3	7.7	7.7	7.8
7.9	8.0	8.1	8.3	8.4	8.4	8.5	8.7	8.8	9.0

We plotted the total time on test (TTT) for these datasets as shown in Figure 4 to provide information on the hazard rate shape. Figure 4(a) indicates that the hazard function of the first dataset has a bathtub shape, while Figure 4(b) and (c) show the increasing monotone shape of their hazard functions. Therefore, the TLGR distribution is a possible candidate for fitting the data from these datasets.



(a) (b)

(c)

Figure 4. TTT plots of (a) the epoxy strand stress-rupture lifetime, (b) the carbon fiber tensile strength, and (c) the turbocharger suit time-to-failure (10^3h) datasets.

Comparison of the TLGR, GR, and Rayleigh distributions was performed using the AD test, for which the test values for comparing the fitting of the distributions were calculated using the ADGofTest package (Bellosta, 2011) in the R language. The other criteria used for comparing the modeling for the three distributions were the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Consequently, the estimates, the AD test results, and the AIC and BIC values are shown in Tables 4, 5, and 6 for datasets 1, 2, and 3, respectively.

Table 4. Summary of fitting and AD test results, and AIC and BIC values for the stress-rupture lifetime dataset.

	MLE	AD test (p-value)	AIC	BIC
TLGR	$\hat{\theta}$ = 0.0309 $\hat{\alpha}$ = -0.8625 $\hat{\beta}$ = 3.8194	1.0602 (0.3270)	212.9762	220.8216
GR	$\hat{\theta}$ = 0.1400 $\hat{\alpha}$ = -0.6790	1.1542 (0.2855)	216.3660	221.5962
Rayleigh	$\hat{\theta}$ = 1.0703	35.0624 (<0.0001)	362.4596	365.0748

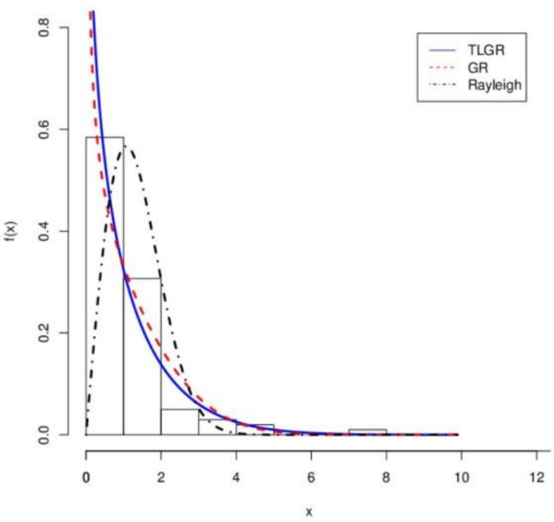
Table 5. Summary of fitting, AD test results, AIC, and BIC values for the carbon fiber tensile strength dataset.

	MLE	AD test (p-value)	AIC	BIC
TLGR	$\hat{\theta}$ = 0.1535 $\hat{\alpha}$ = - 0.0743 $\hat{\beta}$ = 13.1262	0.3396 (0.9054)	119.0136	125.4430
GR	$\hat{\theta}$ = 0.6658 $\hat{\alpha}$ = 5.4843	0.4468 (0.8008)	119.3114	123.5976
Rayleigh	$\hat{\theta}$ = 2.2067	11.0175 (<0.0001)	189.0399	191.1830

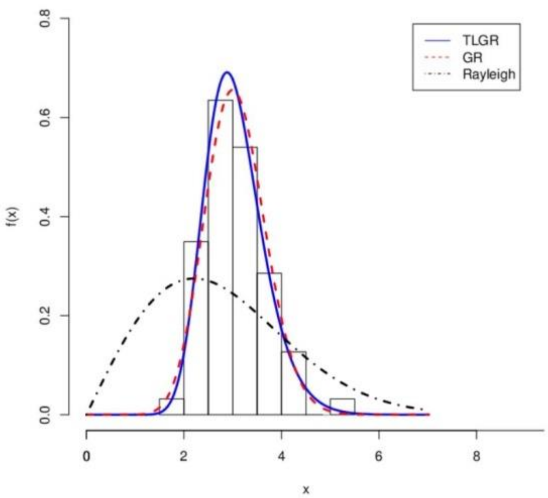
Table 6. Summary of fitting, AD test results, AIC, and BIC values for the turbocharger suit time-to-failure (10^3h) of dataset.

	MLE	AD test (p-value)	AIC	BIC
TLGR	$\hat{\theta}$ = 0.2675 $\hat{\alpha}$ = 18.1572 $\hat{\beta}$ = 0.1015	0.8160 (0.4686)	122.3212	126.5248
GR	$\hat{\theta}$ = 0.0690 $\hat{\alpha}$ = 1.3680	0.8734 (0.4300)	125.2414	128.0437
Rayleigh	$\hat{\theta}$ = 4.1451	2.6033 (0.0441)	133.3049	134.7061

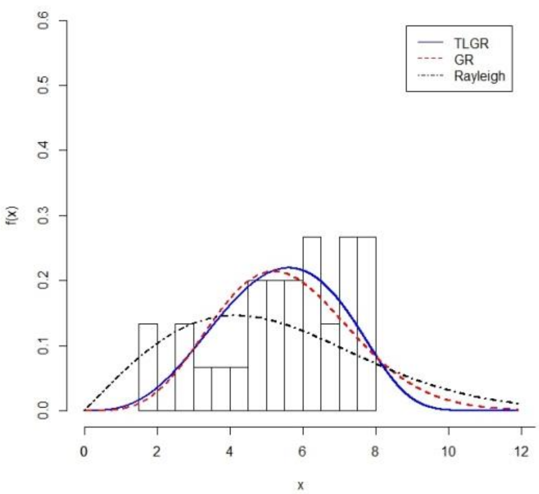
Considering the results of fitting the real-life datasets, the p-value of the AD test under the TLGR distribution was greater than those of the other distributions. In addition, the AIC and BIC values of the TLGR distribution were smaller than those of the others. Consequently, the TLGR distribution was better at fitting these datasets than the GR and Rayleigh distributions. Furthermore, the graphical analysis in Figures 5 and 6 were another way to help us verify the fitting of the distributions. Figure 5 displays the histograms of the datasets with the fitted TLGR, GR, and Rayleigh density functions, and a comparison of the empirical and estimated cdfs are presented in Figure 6. Clearly, the TLGR distribution provided the best results.



(a)



(b)



(c)

Figure 5. Empirical and fitted distributions of the TLGR, GR, and Rayleigh distributions

7. Conclusions

In this research, the TLGR distribution is proposed. Some properties of the TLGR distribution are discussed, including the cdf, pdf, and hazard, survival, and quantile functions. Moreover, the expansion of the TLGR pdf was accomplished by using binomial, lower incomplete gamma, and power series expansions, and the moments and mgf were also derived. Parameter estimation and the observed Fisher information matrix of the TLGR distribution were provided using the maximum likelihood method. Afterwards, we also applied the TLGR distribution in a comparative analysis with the GR and Rayleigh distributions to real-life datasets and compared the fitting results. Referring to the values of the AD test, AIC, and BIC in Section 6, the TLGR distribution was the best at fitting data from these real-life datasets. In practice, the TLGR distribution is likely to attract wide application in real-life data for lifetime and failure analysis, and it could also become an alternative distribution to the current methods of describing various kinds of lifetime data.

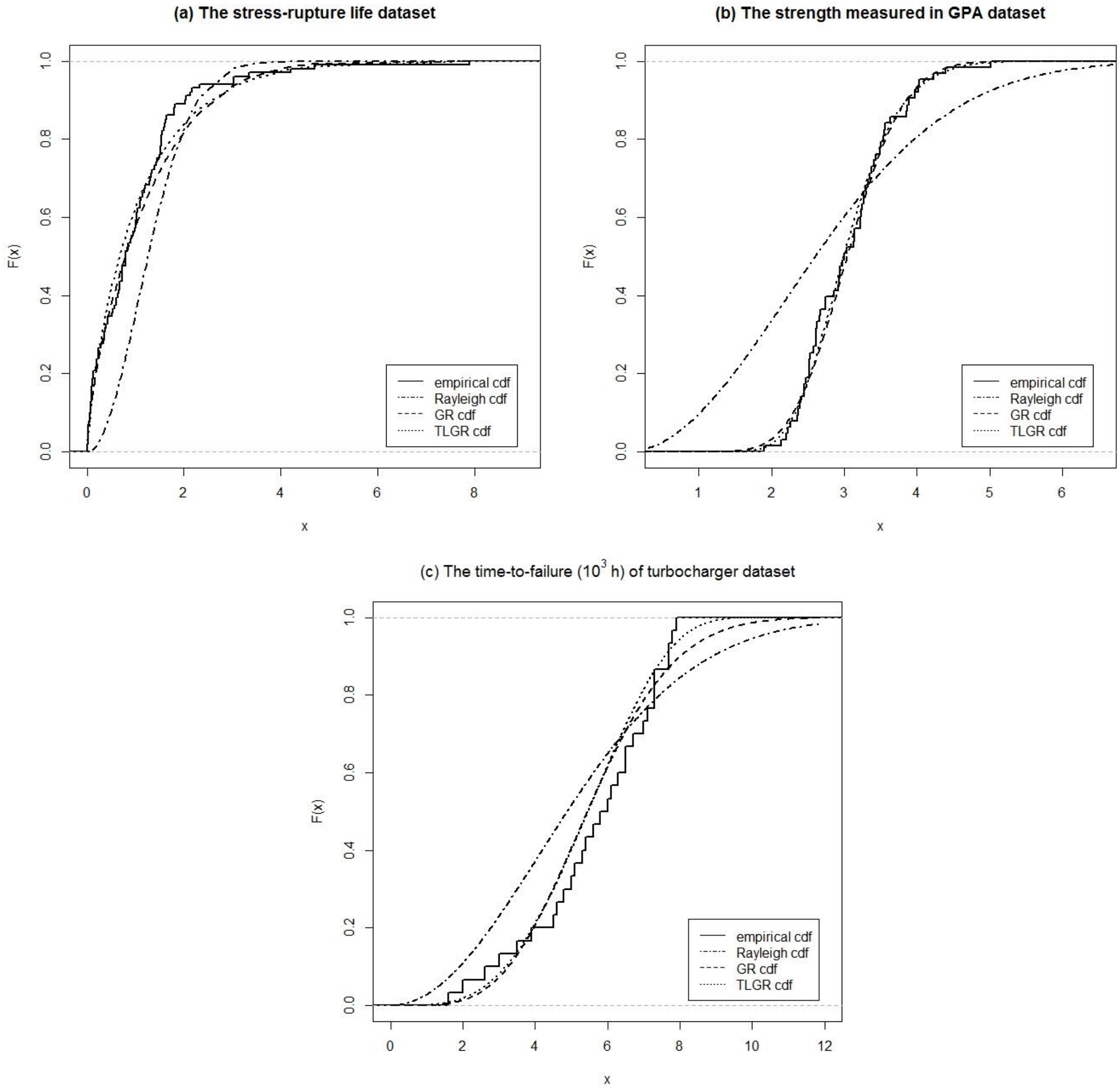


Figure 6. Empirical and theoretical cdfs of the Rayleigh, GR, and TLGR distributions.

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Appendix

The 3×3 total information matrix along with its elements is given by the elements of the observed information matrix $J(\Theta)$ for the parameters (θ,α,β) as

$$J_{\theta\theta}(\Theta)=-\frac{n(\alpha+1)}{\theta^2}-\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\left\{\frac{v_i\left(\alpha-\theta x_i^2\right)}{\theta\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)}+\frac{v_i^2}{\Gamma(\alpha+1)\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}\right\}$$

$$+\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\left\{\frac{v_i\left(\alpha-\theta x_i^2\right)}{\theta\gamma_1\left(\alpha+1,\theta x_i^2\right)}-\frac{v_i^2}{\Gamma(\alpha+1)\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}\right\}$$

$$-\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\left\{\frac{v_i\left(\alpha-\theta x_i^2\right)}{\theta\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)}+\frac{v_i^2}{\Gamma(\alpha+1)\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}\right\},$$

$$J_{\theta\alpha}(\Theta)=\frac{n}{\theta}-\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i\left(\log\left(\theta x_i^2\right)-\psi(\alpha+1)\right)}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}-\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i(\mathrm{A}-\mathrm{B})}{\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}$$

$$-\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i\left(\log\left(\theta x_i^2\right)-\psi(\alpha+1)\right)}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)}+\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i(\mathrm{B}-\mathrm{A})}{\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}$$

$$+\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\left\{\frac{v_i\left(\log\left(\theta x_i^2\right)-\psi(\alpha+1)\right)}{\gamma_1\left(\alpha+1,\theta x_i^2\right)}-\frac{v_i(\mathrm{A}-\mathrm{B})}{\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}\right\},$$

$$J_{\theta\beta}(\Theta)=\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i}{\gamma_1\left(\alpha+1,\theta x_i^2\right)}-\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{v_i}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)},$$

$$J_{\alpha\alpha}(\Theta)=-n\psi'(\alpha+1)-(\beta-1)\psi'(\alpha+1)-\sum_{i=1}^{n}\frac{\mathrm{B}(\mathrm{B}-\mathrm{A})}{\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}$$

$$+\sum_{i=1}^{n}\left\{\frac{\gamma_1\left(\alpha+1,\theta x_i^2\right)\psi'(\alpha+1)+\psi(\alpha+1)(\mathrm{A}-\mathrm{B})}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}\right\}-\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{\gamma''\left(\alpha+1,\theta x_i^2\right)\big|_{\alpha}}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}$$

$$+\sum_{i=1}^{n}\frac{\mathrm{A}\psi(\alpha+1)}{1-\gamma_1\left(\alpha+1,\theta x_i^2\right)}+\sum_{i=1}^{n}\frac{\mathrm{A}(\mathrm{B}-\mathrm{A})}{\left(1-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}+\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{\gamma''\left(\alpha+1,\theta x_i^2\right)\big|_{\alpha}}{\gamma_1\left(\alpha+1,\theta x_i^2\right)}$$

$$-(\beta-1)\sum_{i=1}^{n}\frac{\mathrm{A}\psi(\alpha+1)}{\gamma_1\left(\alpha+1,\theta x_i^2\right)}-(\beta-1)\sum_{i=1}^{n}\frac{\mathrm{A}(\mathrm{A}-\mathrm{B})}{\left(\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}-(\beta-1)\sum_{i=1}^{n}\frac{\mathrm{B}(\mathrm{B}-\mathrm{A})}{\left(2-\gamma_1\left(\alpha+1,\theta x_i^2\right)\right)^2}$$

$$+(\beta-1)\sum_{i=1}^{n}\left\{\frac{\gamma_1\left(\alpha+1,\theta x_i^2\right)\psi'(\alpha+1)+\psi(\alpha+1)(\mathrm{A}-\mathrm{B})}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)}\right\}-\frac{(\beta-1)}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\frac{\gamma''\left(\alpha+1,\theta x_i^2\right)\big|_{\alpha}}{2-\gamma_1\left(\alpha+1,\theta x_i^2\right)}$$

$$+(\beta-1)\sum_{i=1}^{n}\frac{\mathrm{A}\psi(\alpha+1)}{2-\gamma_1(\alpha+1,\theta x_i^2)}+(\beta-1)\sum_{i=1}^{n}\frac{\mathrm{A}(\mathrm{B}-\mathrm{A})}{\left(2-\gamma_1(\alpha+1,\theta x_i^2)\right)^2},$$

$$J_{\alpha\beta}(\Theta)=\sum_{i=1}^{n}\left\{\frac{A}{\gamma_1(\alpha+1,\theta x_i^2)}-\psi(\alpha+1)\right\}+\sum_{i=1}^{n}\frac{B-A}{2-\gamma_1(\alpha+1,\theta x_i^2)},$$

$$J_{\beta\beta}(\Theta)=-\frac{n}{\beta^2},$$

where, $\psi'(\cdot)$ is the trigamma function, and so

$$v_i=\theta^{\alpha}e^{-\theta x_i^2}x_i^{2(\alpha+1)},$$

$$\mathrm{A}=\frac{\gamma'(\alpha+1,\theta x_i^2)\big|_{\alpha}}{\Gamma(\alpha+1)},$$

$$\mathrm{B}=\gamma_1(\alpha+1,\theta x_i^2)\psi(\alpha+1),$$

$$\begin{aligned}\gamma'(\alpha+1,\theta x_i^2)\big|_{\alpha}&=\int_0^{\theta x_i^2}t^{\alpha}e^{-t}\log(t)\,\mathrm{d}t\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\int_0^{\theta x_i^2}t^{\alpha+s}\log(t)\,\mathrm{d}t\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}J(\theta x_i^2,\alpha+s,1),\end{aligned}$$

$$\begin{aligned}\gamma''(\alpha+1,\theta x_i^2)\big|_{\alpha}&=\int_0^{\theta x_i^2}t^{\alpha}e^{-t}\log^2(t)\,\mathrm{d}t\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\int_0^{\theta x_i^2}t^{\alpha+s}\log^2(t)\,\mathrm{d}t\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}J(\theta x_i^2,\alpha+s,2),\end{aligned}$$

We can use the integrals of the logarithmic and power functions from Abramowitz and Stegun (1964) to help calculate $J(\theta x_i^2,\alpha+s,1)$ and $J(\theta x_i^2,\alpha+s,2)$ more easily, thus

$$J(a,p,1)=\int_0^{a}x^p\log(x)\,\mathrm{d}x=\frac{a^{p+1}}{p+1}\left\{\log(a)-\frac{1}{p+1}\right\}$$

and

$$J(a,p,2)=\int_0^{a}x^p\log^2(x)\,\mathrm{d}x=\frac{a^{p+1}\log^2(a)}{p+1}-\frac{2a^{p+1}}{(p+1)^2}\left\{\log(a)-\frac{1}{p+1}\right\}.$$