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Original Article

Solutions of Fredholm integro-differential equations by using a hybrid of block-pulse functions and Taylor polynomials

Nattinee Khongnual[1*] and Weerachai Thadee[2]

[1] *Faculty of Science, Nakhon Si Thammarat Rajabhat University, Mueang, Nakhon Si Thammarat, 80280 Thailand*

[2] *Department of General Education, Faculty of Liberal Arts, Rajamangala University of Technology Srivijaya, Mueang, Songkhla, 90000 Thailand*

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Abstract

In this paper, we present numerical method for solving integro-differential equations of fractional order based on a hybrid of block-pulse functions and Taylor polynomials. Fractional derivative is described in the Caputo sense. Some numerical examples are presented to demonstrate the theoretical results.

Keywords: hybrid of block-pulse function, fractional differential equation, integro-differential equations

1. Introduction

The fractional calculus is important in Science and Engineering, including earthquake engineering, biomedical engineering, and fluid mechanics. Some numerical algorithms for solving integro-differential equation of fractional order can be summarized, such as sinc-collocation method (Altan, 2017), Taylor expansion method (Huang, Xian-Fang, Zhao, & Xiang-Yang, 2011), Adomian decomposition method (Mittal & Nigam, 2008), Least Squares Method and Bernstein Polynomials (Oyedepo, Taiwo, Abubakar, & Ogunwobi, 2016), and Second kind Chebyshev wavelets (Setia, Liu, & Vatsala, 2014).

In this paper, we present a numerical method for solving integro-differential equations of fractional order based on a hybrid of block-pulse function and Taylor polynomials. Our study focuses on a class of integro-differential equation of fractional order, of the type

$$D^{\alpha} f(t) = y(t) + \int_0^1 k(t,s) f(s)ds, \quad 0 \le s,t \le 1, \ \alpha > 0 \qquad (1)$$

with the initial condition $f^{(i)}(0) = \beta_i$, for $\beta_i \in R$ and $i \in N \cup \{0\}$ where $D^{\alpha}(\cdot)$ is Caputo's fractional derivative, α is the order of the fractional derivative, $k(t,s)$ is a smooth function, t and s are real variables and $y(t)$ is a given function. This type of equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems.

2. Basic Definitions

In this section, we present the definitions of fractional calculus theory and some of its basics.

Definition 1. Let $f:[a,b] \to R$ be a function, α a positive real number, and Γ the gamma function. The Riemann-Liouville fractional integral of order α is defined by

$$I^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0 \\ f(t), & \alpha = 0. \end{cases} \qquad (2)$$

*Corresponding author
Email address: jeatlala@hotmail.co.th

Definition 2. Let $f:[a,b]\to R$ be a function, α a positive real number, n the integer satisfying $n-1\le\alpha<n$, and Γ the gamma function. The Caputo's fractional derivative of order α is defined by

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds. \tag{3}$$

Riemann-Liouville fractional integral and Caputo's fractional differentiation are linear operators, similar to integer order differentiation, i.e., $I^{\alpha}(\lambda f(t)\pm\mu g(t))=\lambda I^{\alpha}f(t)\pm\mu I^{\alpha}g(t)$ and $D^{\alpha}(\lambda f(t)\pm\mu g(t))=\lambda D^{\alpha}f(t)\pm\mu D^{\alpha}g(t)$ where λ and μ are constants.

Next, we mention properties of the operators I^{α} and D^{α} as follows

$$I^{\alpha}\left(D^{\alpha}f(t)\right)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0)\frac{t^k}{k!} \tag{4}$$

and

$$I^{n-\alpha}\left(D^{n}f(t)\right)=D^{\alpha}f(t) \tag{5}$$

for any positive real number α such that $n-1\le\alpha<n$ and $n\in N$.

Definition 3. The second kind Fredholm integral equation is defined by

$$f(t)=y(t)+\lambda\int_a^b k(t,x)y(x)dx, \tag{6}$$

where λ is a constant and $k(t,x)$ is a function of variables x and t.

3. Hybrid of Block-Pulse Functions and Taylor Polynomials

The hybrid function denoted by $b_{nm}(t)$ is defined by

$$b_{nm}(t)=\begin{cases}T_m\left(Nt-(n-1)t_f\right), & t\in\left[\frac{n-1}{N}t_f,\frac{n}{N}t_f\right),\\ 0, & \text{otherwise}\end{cases} \tag{7}$$

where n is the order of block-pulse functions, $n=1,2,3,\ldots,N$ and $m=0,1,2,\ldots,M-1$. The Taylor polynomial of order m in (7) is defined by $T_m(t)=t^m$. We approximate $f(t)$ by

$$f(t)\cong\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{nm}b_{nm}(t)=C^TB(t), \tag{8}$$

where

$$C^T=\left[\begin{array}{cccccccccc}c_{10} & c_{20} & \cdots & c_{N0} & c_{11} & c_{21} & \cdots & c_{N1} & \cdots c_{1M-1} & \cdots & c_{NM-1}\end{array}\right]$$

and

$$B^T(t)=\left[\begin{array}{l} b_{10}(t)\;\; b_{20}(t)\;\;\cdots\;\; b_{N0}(t)\;\; b_{11}(t) \\ b_{21}(t)\;\;\cdots\;\; b_{N1}(t)\;\;\cdots b_{1M-1}(t)\;\;\cdots\;\; b_{NM-1}(t)\end{array}\right]. \tag{9}$$

Next, we will introduce operator I^{α} on a hybrid function given by

$$I^{\alpha}B(t)=\overline{B}(t,\alpha), \tag{10}$$

where

$$\overline{B}(t,\alpha)=\left[I^{\alpha}b_{10}(t),\ldots,I^{\alpha}b_{N0}(t),I^{\alpha}b_{11}(t),\ldots,I^{\alpha}b_{N1}(t),\ldots,I^{\alpha}b_{NM-1}(t)\right]^T.$$

Similary, operator D^{β} is given by

$$D^{\beta}\overline{B}(t,\alpha)=\overline{B}(t,\alpha-\beta),\qquad 0\le\beta<\alpha. \tag{11}$$

4. Numerical Method

Consider (1) where $D^{\alpha}(\cdot)$ are operators defined as in (3). We assume that β is the smallest integer greater than or equal to α and expand $D^{\beta}f(t)$ with a hybrid of block-pulse functions and Taylor polynomials as

$$D^{\beta}f(t)=C^TB(t). \tag{12}$$

Operating with I^{β} on both sides of the (12) by using (4) and (10). Thus, the approximate solution is given by

$$f(t)=C^T\overline{B}(t,\beta)+\sum_{k=0}^{\beta-1}f^{(k)}(0)\frac{t^k}{k!}. \tag{13}$$

Calculating $D^{\alpha}f(t)$ from (13), we get

$$D^{\alpha}\left(f(t)\right)=C^T\overline{B}(t,\beta-\alpha)+\sum_{k=0}^{\beta-1}f^{(k)}(0)\frac{D^{\alpha}(t^k)}{k!}. \tag{14}$$

Applying (13) and (14) in (1), we get

$$C^T\overline{B}(t,\beta-\alpha)+\sum_{k=0}^{\beta-1}f^{(k)}(0)\frac{D^{\alpha}(t^k)}{k!}=$$
$$y(t)+\int_0^t k(t,s)\left(C^T\overline{B}(s,\beta)+\sum_{k=0}^{\beta-1}f^{(k)}(0)\frac{t^k}{k!}\right)ds. \tag{15}$$

We obtain an $N\times M$ system of equations in $N\times M$ unknown constants c_{nm}. The system of equations is then solved by substituting (15) with $t_i=\frac{i+1}{2NM}$, where

$i = 0,1,\ldots,2NM-1$. The gained values of initially unknown constants are substituted in (13) in order to get the required approximate solution.

5. Numerical Examples

Example 1. Consider the following fractional integro-differential equation

$$D^{0.5}\left(f(t)\right)=\frac{\left(8/3\right)t^{3/2}-2t^{1/2}}{\sqrt{\pi}}+\frac{t}{12}+\int_0^1 tsf(s)ds \tag{16}$$

subject to $f(0)=0$, with the known exact solution $f(t)=t^2-t$.

We solve this problem by using the hybrid functions with $N=2$ and $M=2$. We let

$$D\,f(t)=C^T B(t), \tag{17}$$

where

$C^T=[\ c_{10}\ \ c_{11}\ \ c_{20}\ \ c_{21}\ ]$ and

$B(t)=[\ b_{10}(t)\ \ b_{11}(t)\ \ b_{20}(t)\ \ b_{21}(t)\ ]^T$.

From (12) and (13), we have

$$f(t)=C^T\overline{B}(t,1)+f(0) \tag{18}$$

where

$$\overline{B}(t,1)=[\ Ib_{10}(t)\ \ Ib_{11}(t)\ \ Ib_{20}(t)\ \ Ib_{21}(t)\ ]^T$$
$$=[\ t\ \ t^2\ \ t\ \ (t^2-t)\ ]^T.$$

From (18), we get

$$D^{0.5}\left(f(t)\right)=C^T\overline{B}(t,0.5)\,. \tag{19}$$

Next, applying (18) and (19) in (16) we have

$$C^T\overline{B}(t,0.5)=\frac{\left(8/3\right)t^{3/2}-2t^{1/2}}{\sqrt{\pi}}+\frac{t}{12}+\int_0^1 ts\left(C^T\overline{B}(s,1)\right)ds,\quad t\in[0,1]\,. \tag{20}$$

Calculating $C^T\overline{B}(t,0.5)$ by using (2), we have

$$\frac{c_{10}}{\Gamma(0.5)}\int_0^t\frac{1}{(t-s)^{0.5}}ds+\frac{c_{11}}{\Gamma(0.5)}\int_0^t\frac{2s}{(t-s)^{0.5}}ds+$$

$$\frac{c_{20}}{\Gamma(0.5)}\int_0^t\frac{1}{(t-s)^{0.5}}ds+\frac{c_{21}}{\Gamma(0.5)}\int_0^t\frac{2s-1}{(t-s)^{0.5}}ds$$

$$=\frac{\left(8/3\right)t^{3/2}-2t^{1/2}}{\sqrt{\pi}}+\frac{t}{12}+\int_0^1 ts\left[c_{10}s+c_{11}s^2+c_{20}s+c_{21}(s^2-s)\right]ds. \tag{21}$$

We substitute in (21) $t_0=\frac{1}{8}$, $t_1=\frac{2}{8}$, $t_2=\frac{3}{8}$ and $t_3=\frac{4}{8}$, and we get c_{20} and c_{21} are in terms of c_{10} and c_{11} where $c_{10},c_{11}\in R$. If $c_{10}=c_{11}=1$ then $c_{20}=-2$ and $c_{21}=1.57083\times10^{-16}$. Hence we get $f(t)=-t+t^2-1.57083\times10^{-16}(-t+t^2)$. Figure 1 shows the exact solution and the approximate solution and Figure 2 shows the absolute error in $[0,1]$.

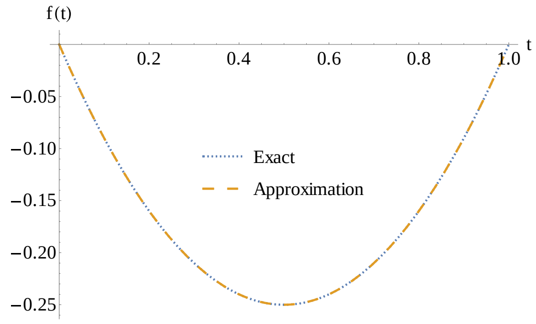

Figure 1. Numerical results of Example 1.

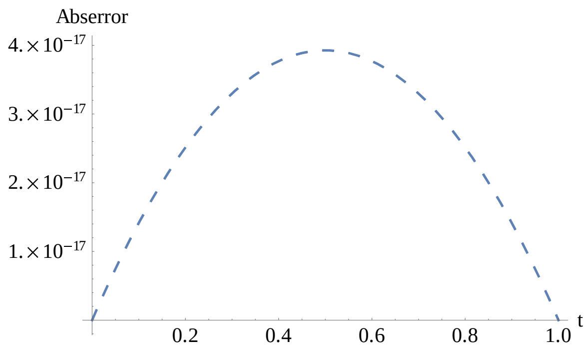

Figure 2. Absolute error of Example 1.

Example 2. Consider the following fractional integro-differential equation

$$D^{5/6}\left(f(t)\right)=-\frac{3}{91}\frac{t^{\frac{1}{6}}\Gamma(5/6)(-91+216t^2)}{\pi}+(5-2e)t+\int_0^1 te^s f(s)ds,\quad t\in[0,1] \tag{22}$$

subject to $f(0)=0$, with the known exact solution $f(t)=t-t^3$.

Here, we solve this problem by using the hybrid functions with $N=1$ and $M=3$. We let

$$D\,f(t)=C^T B(t), \tag{23}$$

where

$C^T=[\ c_{10}\ \ c_{11}\ \ c_{12}\ ]$ and $B(t)=[\ b_{10}(t)\ \ b_{11}(t)\ \ b_{11}(t)\ ]^T$.

From (23) with using (13), we have

$$f(t)=C^T\overline{B}(t,1)+f(0) \tag{24}$$

where

$$\bar{B}(t,1) = [\; Ib_{10}(t) \quad Ib_{11}(t) \quad Ib_{12}(t) \;]^T = \left[\; t \quad \frac{t^2}{2} \quad \frac{t^3}{3} \;\right]^T.$$

From (24), we get

$$D^{\frac{5}{6}}\big(f(t)\big) = C^T\bar{B}(t, \tfrac{1}{6}). \tag{25}$$

Next, we substitute (24) and (25) in (22), then

$$C^T\bar{B}(t,\tfrac{1}{6}) = -\frac{3}{91}\frac{t^{\frac{1}{6}}\Gamma(5/6)(-91+216t^2)}{\pi} + (5-2e)t + \int_0^1 te^s C^T\bar{B}(s,1)ds. \tag{26}$$

Calculating $C^T\bar{B}(t,0.5)$ by using (2), we have

$$c_{10}\left[\frac{1}{\Gamma\left(\frac{1}{6}\right)}\int_0^t \frac{1}{(t-s)^{\frac{5}{6}}}ds\right] + c_{11}\left[\frac{1}{\Gamma\left(\frac{1}{6}\right)}\int_0^t \frac{s}{(t-s)^{\frac{5}{6}}}ds\right] + c_{12}\left[\frac{1}{\Gamma\left(\frac{1}{6}\right)}\int_0^t \frac{s^2}{(t-s)^{\frac{5}{6}}}ds\right] = -\frac{3}{91}\frac{t^{\frac{1}{6}}\Gamma(5/6)(-91+216t^2)}{\pi} + (5-2e)t + \int_0^1 te^s\left[c_{10}s + \frac{c_{11}s^2}{2} + \frac{c_{12}s^3}{3}\right]ds. \tag{27}$$

We substitute in (27) $t_0 = \frac{1}{6}$, $t_1 = \frac{1}{3}$ and $t_2 = \frac{1}{2}$. The constants obtained are $c_{10} = 1$, $c_{11} = 5.13185\times10^{-15}$ and $c_{12} = -3$. Hence we get $f(t) = t + 2.56593\times10^{-15}t^2 - t^3$. Figure 3 shows the exact solution and the approximate solution in $[0,1]$, Figure 4 shows the absolute error in $[0,1]$, and Table 1 shows the absolute error of Example 2.

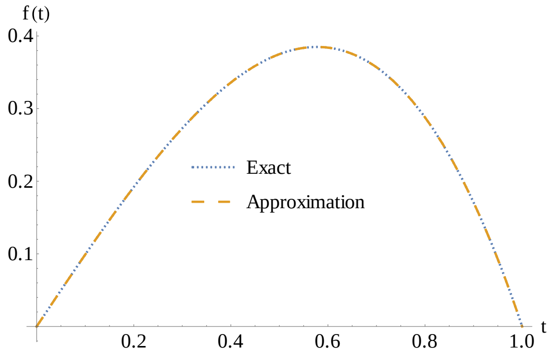

Figure 3. Numerical results of Example 2.

Example 3. Consider the following fractional integro-differential equation

$$D^{5/3}\big(f(t)\big) = \frac{3\sqrt{3}\Gamma(2/3)t^{\frac{1}{2}}}{\pi} - \frac{1}{5}t^2 - \frac{1}{4}t + \int_0^1 (st + s^2t^2) f(s)ds, \quad t\in[0,1]. \tag{28}$$

subject to $f(0) = f'(0) = 0$, with the known exact solution $f(t) = t^2$.

Here, we solve this problem by using the hybrid functions with $N = 1$ and $M = 1$. We let

$$D^2 f(t) = C^T B(t), \tag{29}$$

where

$C^T = [\; c_{10} \;]$ and $B(t) = [\; b_{10}(t) \;]^T$.

From (29) on using (13), we have

$$f(t) = C^T\bar{B}(t,2) + f(0) + f'(0) \tag{30}$$

where

$$\bar{B}(t,2) = [\; I^2 b_{10}(t) \;]^T = \left[\; \frac{t^2}{2} \;\right]^T.$$

Table 1. Absolute error of Example 2.

t	Absolute error of standard least squares method (SLM)	Absolute error of perturbed least squares method (PLM)	Absolute error of the present method
0.0	3.478×10^{-5}	1.0800×10^{-4}	0
0.1	1.6795×10^{-5}	9.1044×10^{-5}	2.14142×10^{-17}
0.2	2.7842×10^{-6}	7.8855×10^{-5}	7.49707×10^{-18}
0.3	7.2646×10^{-6}	7.1358×10^{-5}	2.57642×10^{-17}
0.4	1.3243×10^{-5}	6.8475×10^{-5}	6.23823×10^{-17}
0.5	1.5190×10^{-5}	7.0128×10^{-5}	8.63701×10^{-17}
0.6	1.3106×10^{-5}	7.6237×10^{-5}	8.17404×10^{-17}
0.7	6.9911×10^{-6}	8.6724×10^{-5}	3.25059×10^{-17}
0.8	3.1530×10^{-6}	1.0151×10^{-4}	7.73205×10^{-17}
0.9	1.7325×10^{-5}	1.2052×10^{-4}	2.63726×10^{-16}
1.0	3.5224×10^{-5}	1.4367×10^{-4}	5.42698×10^{-16}

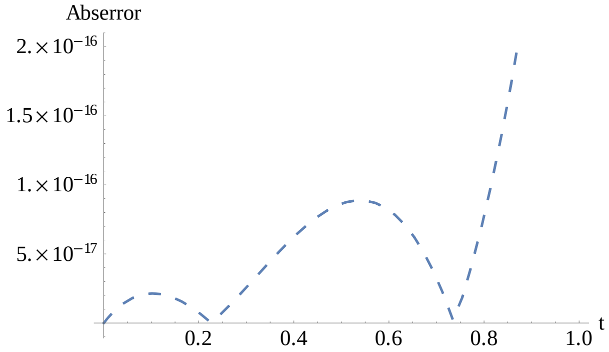

Figure 4. Absolute error of Example 2.

From (30), we get

$$D^{\frac{5}{3}}\left(f(t)\right)=C^T\bar{B}(t,\frac{1}{3}). \tag{31}$$

Next, we substitute (30) and (31) in (28), then

$$C^T\bar{B}(t,\frac{1}{3})=\frac{3\sqrt{3}\Gamma(2/3)t^{\frac{1}{2}}}{\pi}-\frac{1}{5}t^2-\frac{1}{4}t+\int_0^1(st+s^2t^2)C^T\bar{B}(s,2)ds. \tag{32}$$

We substitute in (32) $t_0=\frac{1}{2}$. The constant obtained is $c_{10}=2$. Hence we get $f(t)=t^2$. Figure 5 shows the exact solution and the approximate solution in $[0,1]$.

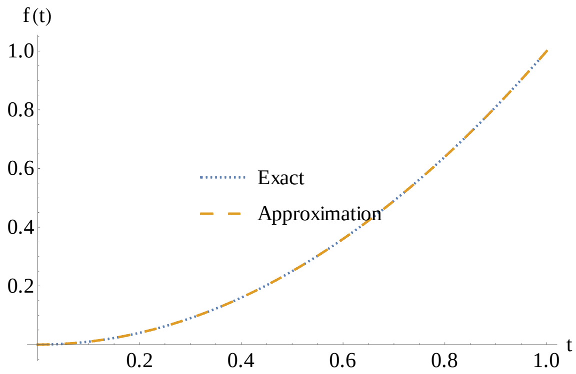

Figure 5. Numerical results of Example 3.

Example 4. Consider the following fractional integro-differential equation

$$D^{0.5}\left(f(t)\right)=2\sqrt{\frac{t}{\pi}}+\frac{3t\sqrt{\pi}}{4}-\frac{9}{10}+\int_0^1 f(s)ds. \tag{33}$$

subject to $f(0)=0$, with the known exact solution $f(t)=t+t^{3/2}$.

Here, we solve this problem by using the hybrid functions with $N=2$ and $M=3$. We let

$$Df(t)=C^TB(t), \tag{34}$$

where
$C^T=[\ c_{10}\ \ c_{11}\ \ c_{12}\ \ c_{20}\ \ c_{21}\ \ c_{22}\ ]$ and
$B(t)=\left[\ b_{10}(t)\ \ b_{11}(t)\ \ b_{12}(t)\ \ b_{20}(t)\ \ b_{21}(t)\ \ b_{22}(t)\right]^T$.

From (34) with use of (13), we have

$$f(t)=C^T\bar{B}(t,1)+f(0) \tag{35}$$

where

$$\bar{B}(t,1)=\left[\ Ib_{10}(t)\ \ Ib_{11}(t)\ \ Ib_{12}(t)\ \ Ib_{20}(t)\ \ Ib_{21}(t)\ \ Ib_{22}(t)\right]^T$$

$$=\left[\ t\ \ t^2\ \ \frac{4t^3}{3}\ \ t\ \ (t^2-t)\ \ (\frac{4t^3}{3}-2t^2+t)\ \right]^T.$$

From (35), we get

$$D^{0.5}\left(f(t)\right)=C^T\bar{B}(t,0.5). \tag{36}$$

Next, we substitute (35) and (36) in (33), then

$$C^T\bar{B}(t,0.5)=2\sqrt{\frac{t}{\pi}}+\frac{3t\sqrt{\pi}}{4}-\frac{9}{10}+\int_0^1 C^T\bar{B}(s,1)ds. \tag{37}$$

We get c_{20}, c_{21} and c_{22} in terms of c_{10}, c_{11} and c_{12} where $c_{10}, c_{11}, c_{12}\in R$. On selecting $c_{10}=c_{11}=c_{12}=0.5$, then $c_{20}=0.616239$, $c_{21}=0.976958$ and $c_{21}=0.861823$. Hence, we get the approximate solution as $f(t)=1.23137\mathrm{t}+1.24669\mathrm{t}^2-0.48243\mathrm{t}^3$. Figure 6 shows the exact solution and the approximate solution in $[0,1]$ and Table 2 shows a comparison of exact solution with approximate solution.

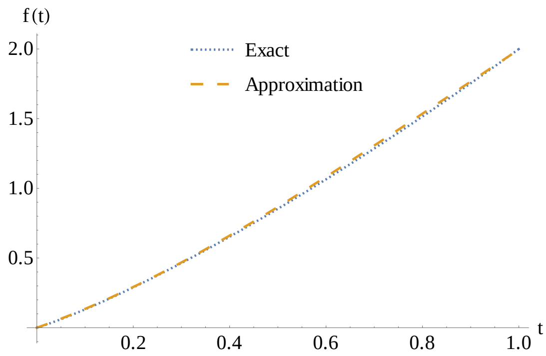

Figure 6. Numerical results of Example 4

6. Conclusions

In this work, we presented numerical solutions to four examples by using a hybrid of block-pulse function and Taylor polynomials. We compared the approximate results with the exact solutions, and it was seen that our method provides better approximate solutions than SLM and PLM.

Table 2. Comparison of exact solution with approximate solution

t	Exact solution	Chebyshev wavelet solution with $k=4, M=2$	Approximate solution of the present method
0.1	0.1316	0.1360	0.1351
0.2	0.2894	0.2955	0.2922
0.3	0.4643	0.4705	0.4685
0.4	0.6530	0.6587	0.6611
0.5	0.8536	0.8584	0.8670
0.6	1.0648	1.0713	1.0834
0.7	1.2857	1.2931	1.3073
0.8	1.5155	1.5233	1.5359
0.9	1.7538	1.7614	1.7663
1.0	2.0000	2.0070	1.9956

References

Altan, S. (2017). A numerical method for solution of integro-differential equations of fractional order. *Sakarya Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, *21*(2), 82-89. doi:10.16984/saufenbilder.296796

Huang, L., Xian-Fang, L., Zhao, Y., & Xiang-Yang, L. (2011). Approximate solution of fractional integro-differential equations by Taylor expansion method. *Computers and Mathematics with Applications*, *62*, 1127–1134. Retrieved from https://core.ac.uk/download/pdf/82805221.pdf

Mittal, R. C., & Nigam, R. (2008). Solution of fractional integro-differential equations by Adomian decomposition method. *International. Journal of Applied Mathematics and Mechanics*, *4*(2), 87-94. Retrieved from https://pdfs.semanticscholar.org/efe1/6ed60eb8d2b9b334434952ef9f21abe74873.pdf

Mohammed, Sh. D. (2014). Numerical solution of fractional Integro-differential equations by least squares method and shifted Chebyshev polynomial. *Mathematical Problems in Engineering*. doi:10.1155/2014/431965

Oyedepo, T., Taiwo, O., Abubakar, J., & Ogunwobi, Z. (2016). Numerical studies for solving fractional Integro-differential equations by using least squares method and Bernstein polynomials. *Fluid Mechanics: Open Access*, *3*, 142. doi:10.4172/2476-2296.1000142

Setia, A., Liu, Y. & A. S. Vatsala. (2014). Solution of linear fractional Fredholm integro- differential equation by using second kind Chebyshev wavelet. *International Conference on Information Technology: New Generations*. 465-469. doi: 10.1109/ITNG.2014.69