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Original Article

Mathematical analysis for classical Chua's circuit with two nonlinear resistors

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Abstract

We formulate a mathematical model for the classical Chua's circuit with two nonlinear resistors in terms of a system of nonlinear ordinary differential equations. The existence of two nonlinear resistors implies that the system has three equilibrium points. The behaviour of the trajectory in a neighbourhood of each equilibrium point depends on the eigenvalues of the system. The eigenvalues can be obtained from a cubic polynomial equation. It turns out that all possible solutions of the cubic equation lead to six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, saddle focus node. The chaotic behaviour of the circuit occurs when the equilibrium point is a stable focus node or a saddle focus node. The hidden attractor of our Chua's system is localized through a suitable initial point.

Keywords: chaos theory, circuit analysis, Chua's circuit, nonlinear resistors, hidden attractor

1. Introduction

In nonlinear dynamical systems, it is well known that there are two types of oscillations, namely, periodic oscillation and chaotic oscillation. Chaos system is thus a nonlinear dynamic system which has chaotic motion or random changing of waveform. It is sensitive to initial conditions and has the self-similarity property. Chaotic phenomenon has been received much attention for a few decades. Such behaviour has been successfully applied to signal transmission and cryptography (Kolumban, Kennedy, & Chua, 1998; Yang, Wu, & Chua, 1997; Dmitriev, Panas, & Starkov, 1995). Several types of oscillators have been studied and applied for generating chaos, e.g. Collpits, Wien bridge, Chua, Lorenz, etc. Among those, Chua's circuit is a famous one.

Chua's circuit (Chua, 1992; Sprott, 2000a) is a simple electronic circuit that exhibits classic chaos theory behaviour (Sprott, 2000b; Piper & Sprott, 2010; Chua & Lin, 1990; Sprott, 2011). It produces an oscillating waveform, which is different from usual electronic oscillators. The classical Chua's circuit, shown in Figure 1, consists of only resistors, capacitors, and a nonlinear resistor (Morgul, 1995; Aissi & Kazakos, 2008). The nonlinear resistor, also called Chua's diode, consists of many op-amps. In the literature, there are many ways to adjust the classical Chua's circuit to a more complicated one having chaotic behaviour. These include anti-monotonicity, and bubble formation (Kyprianidis, 2006; Stouboulos, Kyprianidis, & Papadopoulou, 2008). It is also possible to replace the piecewise linear characteristic of the Chua's diode with a smooth cubic function (Kyprianidis & Fotiadou, 2006). Applications of Chua's chaotic systems go to computer science, mathematical biology, communication system, weather forecast and other branches of sciences.

In the literature, an oscillation in a dynamical system (e.g. Chua's system) can be localized numerically if initial conditions from its neighbourhood lead to asymptotic behaviour that approaches the oscillation. Such an oscillation is called an attractor, and its attracting set is called the basin of attraction. There are two types of attractors classified by the basin of attractions (Bragin, Vagaitsev, Kuznetsov, & Leonov, 2011). A hidden attractor, discovered in (Kuznetsov, Leonov, & Vagaitsev, 2010; Bragin, Vagaitsev, Kuznetsov, & Leonov, 2011; Kuznetsov, Leonov, & Seledzhi, 2011), is an attractor whose basin of attractions does not intersect with small

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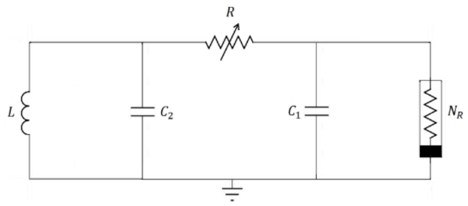

Figure 1. Classical Chua's circuit.

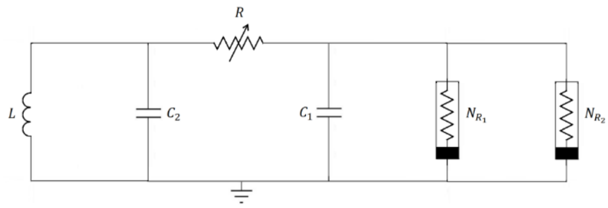

Figure 2. Classical Chua's circuit with two nonlinear resistors.

neighbourhoods of equilibrium points; otherwise an attractor is called a self-excited attractor.

The present paper investigates the classical Chua's circuit by adding a nonlinear resistor, so that the circuit has two nonlinear resistors as shown in Figure 2. The circuit of each nonlinear resistor is shown in Figure 3. We apply fundamental laws in electrical engineering to make a mathematical model of the circuit; see Section 2. Such a model is described in terms of a system of nonlinear differential equations. Then we shall find all equilibrium points of the system; see Section 3. To investigate the trajectory behaviour about a neighbourhood of each equilibrium point, we shall classify the type of each equilibrium point through the associated eigenvalues; see Section 4. In Section 5, we show that our system can be reduced to a simpler one via an invertible transformation. Our system has one hidden attractor, and its localization is discussed in Section 6. Our theory is then illustrated with a numerical simulation in Section 7. We finish the paper with the conclusion in Section 8.

2. Formulation of Classical Chua's Circuit to a System of ODEs

In this section, we formulate a mathematical model for the classical Chua's circuit (Figure 3) in terms of a system of nonlinear ordinary differential equations (ODEs).

We divide the circuit in Figure 3 into four parts as illustrated in Figures 4-5. Our analysis is based on fundamental theory of electrical circuit analysis such as Ohm's law, Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL).

To analyse the circuit parts in Figure 4, we use the following notations. Let i_L and i_{N_R} be the currents through the inductor L and the nonlinear resistor N_R. Let V_{C_1} and V_{C_2} be the voltages measured across the capacitors C_1 and C_2. Let R be the resistance of the variable resistor. Now, the circuit parts in Figure 4 can be described as

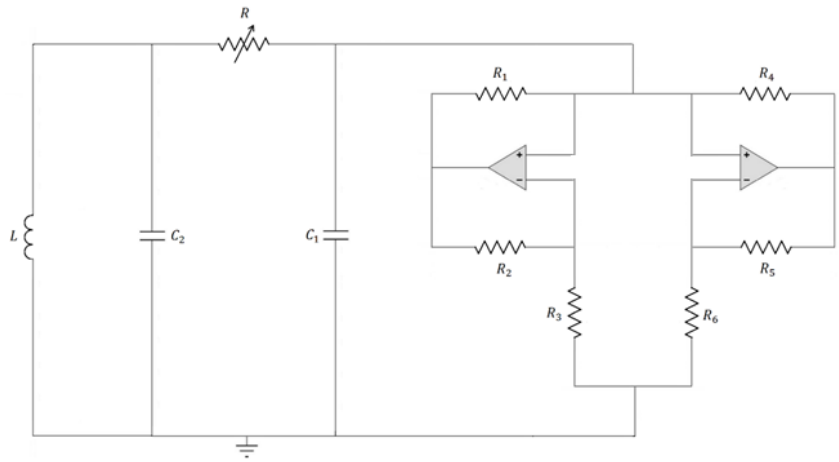

Figure 3. Fully classical Chua's circuit with two nonlinear resistors.

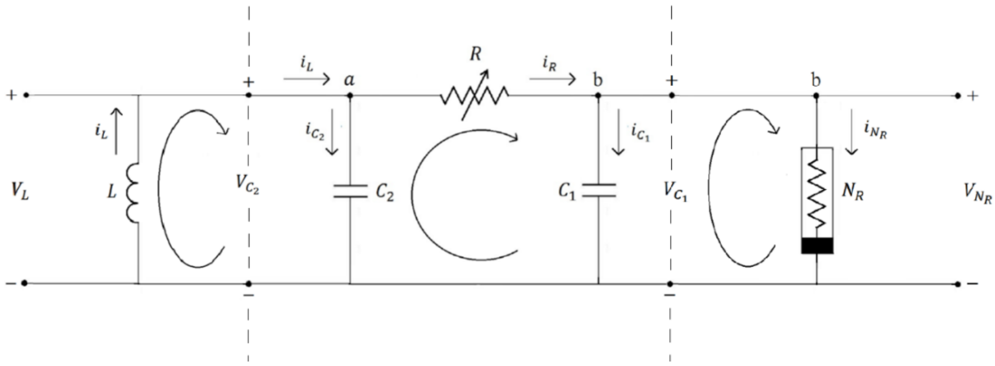

Figure 4. Chua's circuit analysis.

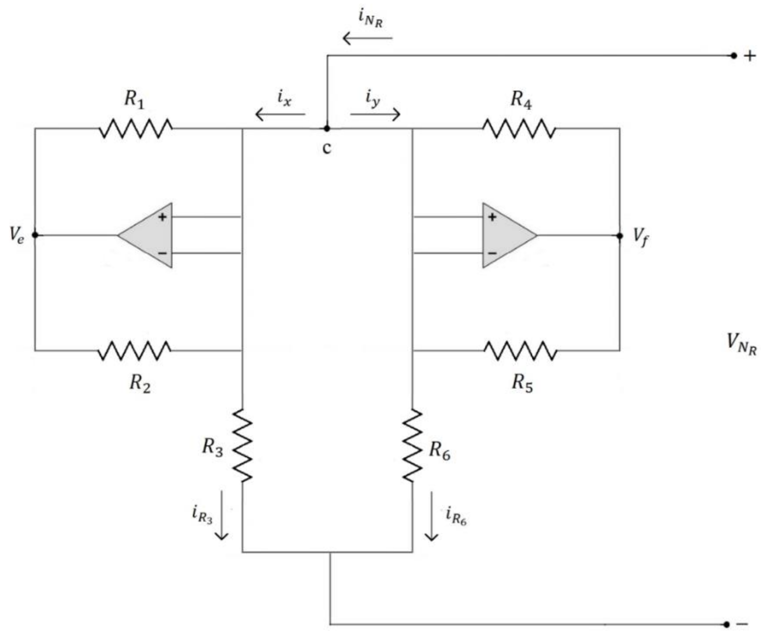

Figure 5. Two nonlinear resistors analysis in Chua's circuit.

$$\frac{di_L}{dt} = -\frac{V_{C_2}}{L}, \tag{1}$$

$$\frac{dV_{C_2}}{dt} = \frac{V_{C_1} - V_{C_2}}{RC_2} + \frac{i_L}{C_2}, \tag{2}$$

$$\frac{dV_{C_1}}{dt} = \frac{V_{C_2} - V_{C_1}}{RC_1} - \frac{i_{N_R}}{C_1}. \tag{3}$$

The circuit in Figure 5 is a more complicated one since it consists of two nonlinear resistors. For the nonlinear resistor on the left, using Ohm's law, we have $V_{N_R} = i_{R_3}R_3$, $V_e = (R_2 + R_3)i_{R_3}$ and $V_{N_R} - V_e = i_x R_1$, where V_e is the voltage of the op-amp on the left hand side. Combining these three equations to get $i_x = R_x V_{N_R}$ where

$$R_x = -\frac{R_2}{R_1 R_3}.$$

Similarly, for the nonlinear resistor on the right, we obtain that $i_y = R_y V_{N_R}$ where

$$R_y = -\frac{R_5}{R_4 R_6}.$$

Using KCL at node *c*, we have $i_{N_R} - i_x - i_y = 0$. Then the current i_{N_R} satisfies the relation

$$i_{N_R} = (R_x + R_y)V_{N_R}.$$

However, as pointed out in (Chua & Ying, 1982), the behavior of i_{N_R} depends on the voltage V_{C_1}. Indeed, when $V_e < V_f$, the graph of i_{N_R} with respect to V_{C_1} is as follows.

From Figure 6,

$$i_{N_R} = \left(R_x + \frac{1}{R_4}\right)V_{C_1} + \frac{1}{2}\left(R_y - \frac{1}{R_4}\right)$$
$$\left(\left|V_{C_1} + \frac{V_{f,max}}{V_f}V_{C_1}\right| - \left|V_{C_1} - \frac{V_{f,max}}{V_f}V_{C_1}\right|\right)$$

where $V_{f,max}$ is the maximum voltage at a node f. Introduce the following time-scale changing:

$$\tau = \frac{t}{RC_2}, \qquad x = \frac{V_f}{V_{f,max}},$$

$$y = \frac{V_f V_{C_2}}{V_{f,max}V_{C_1}}, \qquad z = \frac{V_f i_L R}{V_{f,max}V_{C_1}}.$$

Now, the equations (1), (2) and (3) become the following system of ODEs

$$\frac{dx}{d\tau} = \left(\frac{C_2}{C_1}\right)\left(-x + y - g(x)\right), \tag{4}$$

$$\frac{dy}{d\tau} = x - y + z, \tag{5}$$

$$\frac{dz}{d\tau} = -\left(\frac{R^2 C_2}{L}\right)y, \tag{6}$$

where

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)$$
$$(|x+1| - |x-1|). \tag{7}$$

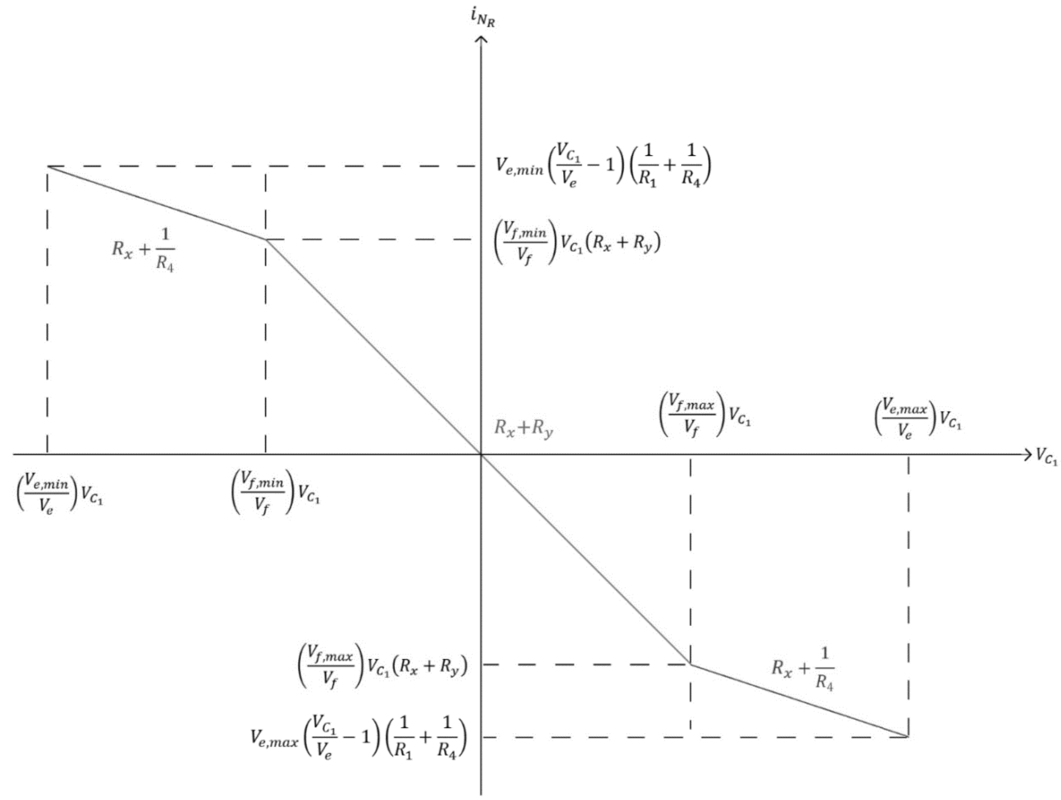

Figure 6. Reduced form of I-V Characteristic for nonlinear resistors ($V_e < V_f$).

3. Equilibrium Points of the Classical Chua's System

From the characteristic of nonlinear resistors in Figure 6 and the formula (7), we see that the behavior of the current i_{N_R} depends on the voltage V_{C_1}, which is considered into three cases, namely, Case 1: $-1 \leq x \leq 1$, Case 2: $-\frac{V_f}{V_e} \leq x \leq 1$, Case 3: $-1 \leq x \leq \frac{V_f}{V_e}$. Let E_i be the equilibrium point for Case i where $i = 1,2,3$. Denote $\zeta = \frac{C_2}{C_1}$ and $\eta = \frac{R^2 C_2}{L}$.

For Case 1, we have

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)$$

$$(x + 1 - 1 - x) = m_0 x,$$

where $m_0 = R(R_x + R_y)$. It follows that the equations (4) to (6) become

$$\zeta(y - x) - \zeta m_0 x = 0, \tag{8}$$

$$x - y + z = 0, \tag{9}$$

$$-\eta y = 0. \tag{10}$$

Thus, the equilibrium for Case 1 is given by $(x_1, y_1, z_1) = (0,0,0)$.

For Case 2, we have

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)$$

$$(-x - 1 - 1 + x), = m_1 x + m_0 - m_1,$$

where $m_1 = R\left(R_x + \frac{1}{R_4}\right)$. Now, the equation (4) reduces to

$$\zeta(y - x) - \zeta(m_1 x - m_0 - m_1 = 0. \tag{11}$$

From the system of equations (9) to (11), we obtain the equilibrium point to be

$$(x_2, y_2, z_2) = (\frac{m_1 - m_0}{m_1 + 1}, 0, \frac{m_0 - m_1}{m_1 + 1}).$$

Finally for Case 3, we can see that

$$g(x) = m_1 x + m_1 - m_0,$$

and thus the equilibrium point is determined by

$$(x_3, y_3, z_3) = (\frac{m_0 - m_1}{m_1 + 1}, 0, \frac{m_1 - m_0}{m_1 + 1}).$$

4. Eigenvalues and Trajectories of the System

In this section, we find the eigenvalues for the classical Chua's system and analyze the behaviour of trajectories of the system in a neighborhood of each equilibrium point.

4.1 Finding eigenvalues

We shall formulate our system into a vector differential equation. Let us denote

$$X(t)=\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix} \text{ and } \dot{X}(t)=\begin{bmatrix}\dot{x}(t)\\\dot{y}(t)\\\dot{z}(t)\end{bmatrix},$$

where $\dot{x}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$ are the derivatives of $x(t)$, $y(t)$ and $z(t)$ with respect to the time *t*, respectively.

From the equations (4)-(7), we again consider three cases. For Case 1 $(-1 \le x \le 1)$, we obtain the linear system

$$\dot{X}(t)=JX(t) \text{ where } J=\begin{bmatrix}-\zeta-\zeta m_0 & \zeta & 0\\ 1 & -1 & 1\\ 0 & -\eta & 0\end{bmatrix}.$$

Recall the following the result:

Theorem 1. (see *e.g.* (Goode, 2000)) The initial value problem

$$X(t)=A(t)X(t)+B(t), \qquad X(t_0)=X_0,$$

where $A(t)$ and $B(t)$ are continuous vector-valued functions on an interval *I*, has a unique solution $X(t)$ on *I*.

This theorem guarantees the existence and the uniqueness of the trajectories $x(t)$, $y(t)$ and $z(t)$, provided that initial values $x(0)$, $y(0)$ and $z(0)$ are given.

In order to get the solutions $x(t)$, $y(t)$ and $z(t)$ of the above system, we shall find the eigenvalues of the matrix J. Indeed, we have $\det(\lambda I_3 - J)=0$ and thus the characteristic equation of J is given by

$$\lambda^3+(\zeta+\zeta m_0+1)\lambda^2+(\zeta m_0+\eta)\lambda$$
$$+(\zeta\eta+\zeta\eta m_0)=0. \qquad (12)$$

For the second and the third cases $(-1 \le x \le 1)$, the coefficient matrix J is given by

$$J=\begin{bmatrix}-\zeta-\zeta m_1 & \zeta & 0\\ 1 & -1 & 1\\ 0 & -\eta & 0\end{bmatrix}.$$

Similarly, its characteristic equation is

$$\lambda^3+(\zeta+\zeta m_1+1)\lambda^2+(\zeta m_1+\eta)\lambda$$
$$+(\zeta\eta+\zeta\eta m_1)=0. \qquad (13)$$

We shall adopt a treatment on cubic equations (e.g. (Guilbeau, 1930)) to this situation. We shall find the solution of the cubic equation

$$\lambda^3+b\lambda^2+c\lambda+d=0 \qquad (14)$$

in which $b=\zeta+\zeta m+1, c=\zeta m+\eta,$ and $d=\zeta\eta+\zeta\eta m,$ where $m=m_0$ or $m=m_1$. The numbers of real and complex roots are determined by the discriminant of the cubic equation defined by

$$\Delta = 18bcd-4b^3d+b^2c^2-4c^3-27d^2.$$

The general solution of the cubic equation involves calculating:

$$\Delta_0=b^2-3c \text{ and } \Delta_1=2b^3-9bc+27d.$$

For $\Delta>0$, the equation has three distinct real roots. More precisely, substitutions $t-\frac{b}{3}$ into λ, we get $t^3+pt+q=0$, where $p=-\frac{\Delta_0}{3}$ and $q=\frac{\Delta_1}{27}$. The solution t will be in the form $t=u+v$, where

$$u=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}} \text{ and } v=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}}.$$

Let $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $A=|u|$ and $B=|v|$. Then, there are three possible values of u, namely $u_1=A$, $u_2=\omega A$ and $u_3=\omega^2 A$. Similarly, there are three positive values of v, namely, $v_1=B$, $v_2=\omega B$ and $v_3=\omega^2 B$. However, the pair (u,v) must satisfies the condition $uv=-\frac{p}{3}$. Hence, the solutions of (14) are given by

$$\lambda_1=A+B-\frac{b}{3}, \qquad \lambda_2=\omega A+\omega^2 B-\frac{b}{3},$$

$$\lambda_3=\omega^2 A+\omega B-\frac{b}{3}.$$

For $\Delta=0$, the equation has a multiple root and all of its roots are real. There are two subcases:

$\Delta_0=0$: it has a triple same root $\lambda_1=\lambda_2=\lambda_3=-\frac{b}{3}$.

$\Delta_0\neq 0$: it has a double same root $\lambda_1=\lambda_2=\frac{9d-bc}{\Delta_0}$ and a simple distinct root $\lambda_3=\frac{4bc-9d-b^3}{\Delta_0}$.

For $\Delta<0$, the equation has one real root and two non-real complex conjugate roots.

4.2 Analysis for Trajectories of the system in a neighborhood of equilibrium

The classical Chua's system can be classified in terms of the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ from the system $\dot{X}(t)=JX(t)$ as follows.

Case 1: λ_1,λ_2 and λ_3 are negative real numbers. In this case, the equilibrium point is called a stable node. Thus, the trajectories of $(x(t),y(t),z(t))$ will converge to the equilibrium point for any initial value $(x(0),y(0),z(0))$ (Figure 7).

Case 2: λ_1,λ_2 and λ_3 are positive real numbers. In this case, the equilibrium point is called an unstable node. Thus, the trajectories of $(x(t),y(t),z(t))$ will diverge to the equilibrium point for any initial value $(x(0),y(0),z(0))$ (Figure 8).

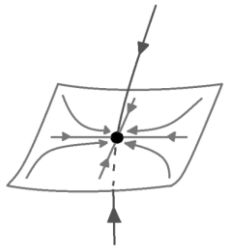

Figure 7. A trajectory of stable node.

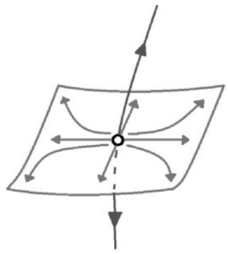

Figure 8. A trajectory of unstable node.

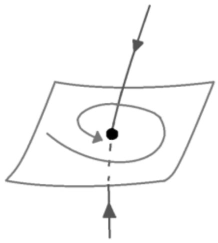

Figure 10. A trajectory of stable focus node.

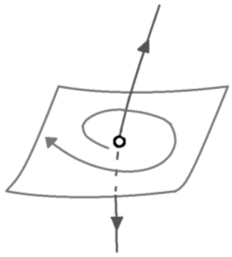

Figure 11. A trajectory of unstable focus node

Case 3: λ_1 is a positive real number, λ_2 and λ_3 are negative real numbers. In this case, the equilibrium point is called a saddle node. The trajectories lying on the x and y axes tend toward to equilibrium point, whereas the trajectories lying on the z axis tends away from equilibrium point (Figure 9 on the left hand side).

Case 4: λ_1 is a negative real number, λ_2 and λ_3 are positive real numbers. In this case, the equilibrium point is called a saddle node. The trajectories of $\left(x(t), y(t)\right)$ will diverge but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$ (Figure 9 on the right hand side).

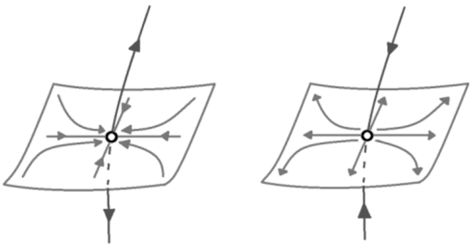

Figure 9. A trajectory of saddle node with $\lambda_1 > 0 > \lambda_{2,3}$ from left hand side and $\lambda_{2,3} > 0 > \lambda_1$ from right hand side.

Case 5: λ_1 is a negative real number, λ_2 and λ_3 are complex numbers having negative real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called a stable focus node. Thus, the trajectories of $\left(x(t), y(t)\right)$ will converge spiral form and $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$ (Figure 10).

Case 6: λ_1 is a positive real numbers, λ_2 and λ_3 are complex numbers having positive real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called an unstable focus node. The trajectories of $\left(x(t), y(t)\right)$ will diverge spiral form and $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$ (Figure 11).

Case 7: λ_1 is a negative real numbers but λ_2 and λ_3 are positive reals and pairs of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. Thus, the trajectories of $\left(x(t), y(t)\right)$ will diverge spiral form but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$ (Figure 12 on the left hand side).

Case 8: λ_1 is a positive real numbers but λ_2 and λ_3 are negative reals and pairs of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. Thus, the trajectories of $\left(x(t), y(t)\right)$ will converge spiral form but $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$ (Figure 12 on the right hand side).

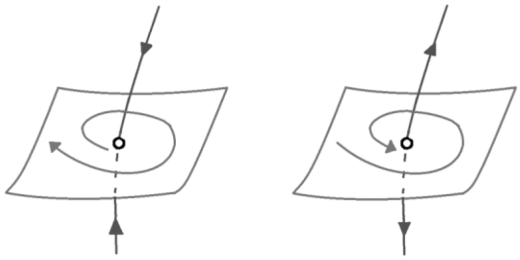

Figure 12. A trajectory of saddle focus node with $\mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3) > 0 > \mathrm{Re}(\lambda_1)$ from left hand side and $\mathrm{Re}(\lambda_1) > 0 > \mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3)$ from right hand side.

5. Reduction of the System

In this section, we show that our system can be reduced to a simpler system by introducing an invertible transformation.

From equations (4)-(7), we put them together in the following matrix form:

$$\frac{dM}{dt} = P_0 M + Q\mu(R^T M)\,, \tag{15}$$

where

$$P_0 = \begin{bmatrix} -\zeta(m_1 + 1 + k) & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix},\quad Q = \begin{bmatrix} -\zeta \\ 0 \\ 0 \end{bmatrix},\qquad R^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From (15), let us introduce an invertible transformation matrix S such that $M = SY$. We shall find the explicit formula of S later. Multiplying S^{-1} to both sides of (15) yields

$$\frac{dY}{dt} = S^{-1}P_0SY + S^{-1}Q\mu(R^TSY),$$

Let $A = S^{-1}P_0S, B = S^{-1}Q, C^T = R^TS$. Then we get

$$\frac{dY}{dt} = AY + B\mu(C^TY), \tag{16}$$

where

$$A = \begin{bmatrix} 0 & -\alpha_0 & 0 \\ \alpha_0 & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 0 \\ c_3 \end{bmatrix},$$

Write $S = [s_{ij}]$. From the conditions $A = S^{-1}P_0S, B = S^{-1}Q, C^T = R^TS$, it is straightforward to deduce that

$$s_{11} = 1, s_{12} = 0, s_{13} = c_3,$$

$$s_{21} = m_1 + 1 + k, s_{22} = -\frac{\alpha_0}{\zeta},$$

$$s_{23} = \frac{\beta}{\zeta} + c_3(m_1 + 1 + k),$$

$$s_{31} = m_1 + k - \frac{\alpha_0^2}{\zeta},$$

$$s_{32} = -\frac{\alpha_0}{\zeta} - \alpha_0(m_1 + 1 + k),$$

$$s_{33} = \frac{hd(1 + \zeta - d)}{\zeta} + c_3(m_1 + k)(1 - \beta).$$

Now, the transfer functions of the system (15) and (16) are

$$W_{P_0}(p) = R^T(P_0 - pI)^{-1}Q, \tag{17}$$

$$W_A(p) = C^T(A - pI)^{-1}B. \tag{18}$$

We obtain that

$$(A - pI)^1 = \frac{1}{\det(A - pI)}\operatorname{adj}(A - pI)$$

$$= \frac{1}{(-p-\beta)(p^2+\alpha_0^2)} \begin{bmatrix} p^2 + \beta p & -\alpha_0 p - \alpha_0\beta & 0 \\ \alpha_0 p + \alpha_0\beta & p^2 + \beta p & 0 \\ 0 & 0 & p^2 + \alpha_0^2 \end{bmatrix}.$$

It follows that

$$W_A(p) = [1 \quad 0 \quad c_3] \begin{bmatrix} \frac{p^2 + \beta p}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{-\alpha_0 p - \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ \frac{\alpha_0 p + \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{p^2 + \beta p}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ 0 & 0 & \frac{p^2 + \alpha_0^2}{(-p-\beta)(p^2+\alpha_0^2)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}$$

$$= \frac{b_1(p^2 + \beta p) + b_2(-\alpha_0 p - \alpha_0\beta) + c_3(p^2 + \alpha_0^2)}{(-p-\beta)(p^2+\alpha_0^2)}$$

$$= \frac{(b_1 + c_3)p^2 + (b_1\beta - b_2\alpha_0)p + (c_3\alpha_0^2 - b_2\alpha_0\beta)}{-p^3 - \beta p^2 - \alpha_0^2 p + \alpha_0^2\beta}. \tag{19}$$

Similarly,

$$W_{P_0}(p) = \frac{(-\zeta)p^2 + (-\zeta)p}{-p^3 + (-\zeta m_1 - \zeta - \zeta k - 1)p^2 + (-\zeta m_1 - \zeta - \zeta k - \eta)p + (-\zeta m_1\eta - \zeta\eta - \zeta k\eta)}. \tag{20}$$

Since the system (15) and the system (16) are the same system, we have $W_A(p) = W_{P_0}(p)$. Now, we compare the numerator and denominator coefficients of $W_A(p)$ and $W_{P_0}(p)$. We obtain that

$$c_3 = -\zeta - b_1,\ b_1\beta - b_2\alpha_0 = -\zeta, -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta$$

$$\beta = \zeta m_1 + \zeta + \zeta k + 1, b_1\beta - b_2\alpha_0 = -\zeta, -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta.$$

These imply the following relations

$$k = \frac{\alpha_0^2 - \eta}{\zeta} - m_1, \qquad \beta = \alpha_0^2 + \zeta - \eta + 1, \qquad b_1 = \frac{\zeta(\beta - \alpha_0^2 - \eta)}{\alpha_0^2 + \beta^2}, \tag{21}$$

$$b_2 = \zeta \frac{\eta - \beta + \beta^2}{\alpha_0(\alpha_0^2 + \beta^2)}, \quad c_3 = \frac{-\zeta(\eta - \beta + \beta^2)}{\alpha_0^2 + \beta^2}. \tag{22}$$

6. Localization of a Hidden Attractor for Classical Chua's Circuit

In this section, we will discuss the oscillation behaviour in the classical Chua's circuit with two nonlinear resistors, focused on hidden attractors.

In order to find a hidden attractor of the system, we will find a suitable initial point $(x(0), y(0), z(0))$ so that our system will have a chaos. The initial point depends on a parameter a_0, which is the solution of the equation $\Phi(a) = 0$ where Φ is the describing function (Bragin, Vagaitsev, Kuznetsov, & Leonov, 2011) defined by

$$\Phi(a) = \int_0^{2\pi/\omega_0} \varphi(a\cos(\omega_0 t))\cos(\omega_0 t)\mathrm{dt}.$$

Here, $\varphi(a) = g(x) - kx$, where k is a coefficient of harmonic linearization, $x = \omega_0 t$, $g(x)$ is the function of x in (7) and

$$g(\omega_0 t) = \frac{1}{2}R(m_0 - m_1)(|\omega_0 t + 1| - |\omega_0 t - 1|).$$

Then, $g(\omega_0 t)$ is equal to $(m_0 - m_1)\omega_0 t$ for all $x \in [0, 1]$ and it equals the constant $m_0 - m_1$ when $x \in (1, 2\pi]$. It follows that

$$\Phi(a) = \int_0^{\tau} a(m_0 - m_1)\cos^2 t\mathrm{dt} + \int_{\tau}^{2\tau} (m_0 - m_1)\cos t\mathrm{dt} - \int_0^{2\pi} ak\cos^2 t\mathrm{dt},$$

where $\tau = \arccos\left(\frac{1}{a}\right)$. Thus, we have

$$\begin{aligned}
\Phi(a) &= (m_0 - m_1)\left[a_0 \int_0^{\arccos\left(\frac{1}{a_0}\right)} \cos^2 t\,\mathrm{dt} + \int_{\arccos\left(\frac{1}{a_0}\right)}^{2\arccos\left(\frac{1}{a_0}\right)} \cos t\mathrm{dt}\right] - a_0 k \int_0^{2\pi} \cos^2 t\mathrm{dt}, \\
&= (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + \frac{(1 - 2a_0)}{2a_0}\sin\left(\arccos\left(\frac{1}{a_0}\right)\right)\right] - a_0 k\pi, \\
&= (m_0 - m_1)\left[\frac{a_0}{2} arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 k\pi.
\end{aligned} \tag{25}$$

From (Bragin, Vagaitsev, Kuznetsov, & Leonov, 2011), the first step of multistage localization for our Chua's system is

$$M(0) = SY(0) = S\begin{bmatrix} a_0 + \mathbf{O}(\mu) \\ 0 \\ \mathbf{O}_{\mathbf{n-2}}(\mu) \end{bmatrix},$$

where $\mathbf{O}(\mu)$ is the big-O notation of order μ and $\mathbf{O}_{n-2}(\mu)$ is the (n-2)-dimensional big-O notation so that all its coordinates are big-O notations of order μ. We can approximate $\mathbf{O}(\mu) \approx 0$ and $\mathbf{O}_{n-2}(\mu) \approx 0$. Thus,

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = S\begin{bmatrix} a_0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that

$$x(0) = a_0 s_{11} = a_0, \qquad y(0) = a_0 s_{21} = a_0(m_1 + 1 + k), \qquad z(0) = a_0 s_{31} = a_0\left(m_1 + k - \frac{\alpha_0^2}{\zeta}\right).$$

7. Numerical Simulation

Consider the classical Chua's circuit with two nonlinear resistors when the following parameters are given: $R = 1000\ \Omega$, $R_1 = 250\ \Omega$, $R_2 = 250\ \Omega$, $R_3 = 500\ \Omega$, $R_4 = 750\ \Omega$, $R_5 = 180\ \Omega$, $R_6 = 400\ \Omega$, $C_1 = 10\ \mu F$, $C_2 = 80\ \mu F$, $L = 70\ mH$.

We have the following parameters

$$\zeta = 8.4562, \qquad \eta = 12.0732, \qquad m_0 = -0.1768, \qquad m_1 = -1.1468.$$

From (21), we have $k = 0.2098$. To find a suitable indicial point of the system, we have to solve the equation $\Phi(a) = 0$, where Φ is given by the equation (21). Indeed, we have

$$0 = (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 k\pi$$

$$= (-0.97)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 0.2098\pi$$

An approximated solution a_0 via MATLAB is given by $a_0 = 9.4287$. It follows from the previous discussion that an initial point is given by $x(0) = 2.0392$, $y(0) = 0.5945$, $z(0) = -13.4705$. A numerical simulation for the equations (4)-(7) with the initial point via MATLAB is illustrated in the following figures.

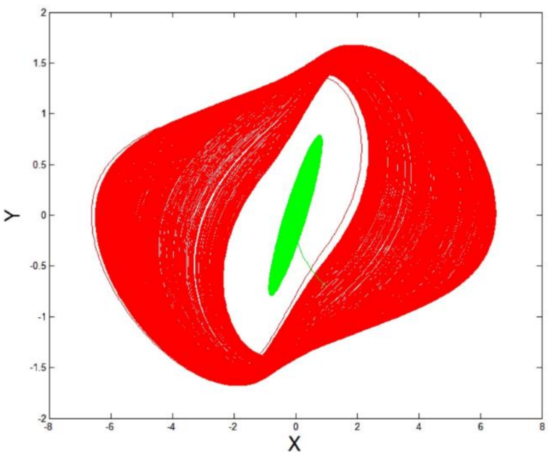

Figure 13. Attractors of the classical Chua's equations in two dimensions.

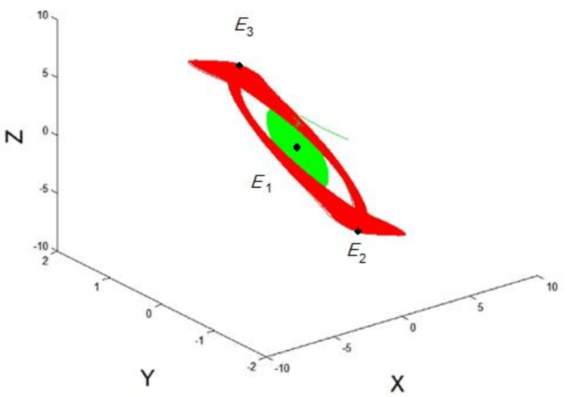

Figure 14. Attractors of the classical Chua's equations in three dimensions.

From the classical Chua's circuit, we formulate the following system

$$\frac{dx}{d\tau} = 8\big(-x + y - g(x)\big),$$

$$\frac{dy}{d\tau} = x - y + z,$$

$$\frac{dz}{d\tau} = -1142.8571y,$$

where $g(x) = -0.6667x - 0.9667(|x + 1| - |x - 1|)$. This system has three equilibrium points, namely,

$$E_1 = (0,0,0),$$

$$E_2 = \left(\frac{m_1 - m_0}{m_1 + 1}, 0, \frac{m_0 - m_1}{m_1 + 1}\right) = (6.6076, 0, -6.6076),$$

$$E_3 = \left(\frac{m_0 - m_1}{m_1 + 1}, 0, \frac{m_1 - m_0}{m_1 + 1}\right) = (-6.6076, 0, 6.6076).$$

The eigenvalues of the system corresponding to each equilibrium point are given by

$$E_1\colon \lambda_1 = -7.9587,\ \lambda_2 = -0.0038 + 3.2494i,\ \lambda_3 = -0.0038 - 3.2494i,$$

$$E_2\colon \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i,$$

$$E_3\colon \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i.$$

Thus, the equilibrium point E_1 is a stable focus node, that is, the trajectory of $\big(x(t), y(t)\big)$ converges spiral form and $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$. On the other hands, the equilibrium points E_2 and E_3 are saddle focus nodes, that is, the trajectory of $\big(x(t), y(t)\big)$ will diverge spiral form but $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$.

From Figures 13 and 14, we see that a chaotic behaviour occurs in our Chua's system has. The self-excited attractor of our system is appeared in the green lines, while the hidden attractor of our system is shown by the red lines in Figures 13 and 14.

8. Conclusions

We apply fundamental laws in electrical engineering to formulate a mathematical model for the classical Chua's circuit with two nonlinear resistors (Figure 3) in terms of the system of ordinary nonlinear differential equations (4)-(6). Each nonlinear resistor in the circuit plays a role like an op-amp. The existence of two nonlinear resistors implies that the system has three equilibrium points. The behavior of the trajectory in a neighborhood of each equilibrium point depends on the eigenvalues of the system. To obtain the eigenvalues for each equilibrium point, we must solve a cubic polynomial equation. It turns out that all possible solutions of the cubic equation lead to six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, saddle focus node. The chaotic behaviour of the circuit occurs when the equilibrium point is a stable focus node or a saddle focus node. The hidden attractor of our Chua's system is localized through a suitable initial point.

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