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Original Article

Application of continuous and discrete optimal control to feeding of farm animals

Sathaporn Puengpo[1], Elvin James Moore[1, 2*], and Chanakarn Kiataramkul[1, 2]

[1] *Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bang Sue, Bangkok, 10800 Thailand*

[2] *Centre of Excellence in Mathematics, Commission on Higher Education, Ratchathewi, Bangkok, 10400 Thailand*

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Abstract

An important problem in farm management is to feed animals in an optimal manner to achieve maximum food production subject to constraints. In this paper, we use optimal control to develop optimal feeding policies for sheep, swine, and shrimp. We derive optimal control policies by three different methods: 1) continuous optimal control, 2) hybrid continuous optimal control with discretized feeding policy, and 3) discrete optimal control with discretized feeding policy. For sheep, the aim is to optimally feed pregnant sheep from gestation to birth of lamb to achieve a desired weight of lamb at the time of birth of the lamb. For swine and shrimp the aim is to achieve a desired weight at the time of sale. Data for animal growth from the Thai government and Thai Agricultural Research Stations was used to develop the models. Numerical solutions from the three optimal control methods were obtained and compared.

Keywords: continuous optimal control, discrete optimal control, feeding of farm animals

1. Introduction

In this paper, we use optimal control models to find optimal methods for three animal feeding problems. In the first problem, we consider optimal methods for feeding sheep during pregnancy to produce new-born lambs of an optimal weight (Gatford *et al.*, 2008; Kiataramkul, Wake, Ben-Tal & Lenbury, 2011; Puengpo, Kiataramkul & Moore, 2016). In the second problem, we consider optimal control methods for feeding swine in their post-weaning period to attain the most profitable weight at time of sale (Kiataramkul & Matkhao, 2015; Puengpo *et al.*, 2016; Thailand Ministry of Agriculture and Cooperatives, 2014). In the third problem, we consider optimal control methods for feeding shrimp to attain the most profitable weight at time of sale which occurs after 120 days (Thailand Ministry of Agriculture and Cooperatives, 2013).

*Corresponding author
Email address: elvin.j@sci.kmutnb.ac.th

In this paper, we compared three different methods of formulating and solving the optimal control problem for the minimum cost of feeding sheep, swine, and shrimp. In the first formulation, we assumed that the feeding is a continuous process and the control is also a continuous variable. In this case, the standard continuous optimal control formalism, based on the Pontryagin maximum principle (Lenhart & Workman, 2007; Luenberger, 1979; Pontryagin, Boltyanskii, Gamkrelize, & Mishchenko, 1962), can be used to solve the optimal feeding problem. However, a continuous feeding policy is often not a practical policy for a farmer to follow. A farmer will usually feed a given constant amount of food in a given period of time, for example during a week or a month.

Therefore, in the second and third formulations, we assumed that the feeding policy is a discrete policy. In the second formulation, which we call a hybrid method, we assumed that the total feeding period is divided into subintervals and the control must have a constant value over each subinterval. However, we assumed that the state and co-state equations for the model are continuous ordinary differential equations. In this formulation, we can use the continuous optimal control formulation and the Pontryagin maximum principle. In the third formulation, we considered a fully discrete optimal control problem (Lenhart & Workman, 2007; Luenberger, 1979) in which the control is discrete and the state and co-state equations are difference equations.

The paper is organized as follows. In section 2, we summarize the theory for the continuous, hybrid, and discrete formulations of the optimal control problems and the methods of solution. In section 3, we describe the three optimal control models for the sheep feeding problem for the growth rate of pregnant sheep as a function of age and food intake modeled by the logistic equation (Giordano, Fox, & Horton, 2014) and the Michaelis-Menten relationship (Murray, 2007). In section 4, we repeat the discussion in section 3 for the swine feeding problem for growth rate as a function of age and food intake modeled first by the logistic equation and then by the Gompertz equation (Zeide, 1993) and a Michaelis-Menten relationship. In section 5, we repeat the discussion in section 3 for the shrimp feeding problem for growth rate as a function of age and food intake modeled by the Gompertz equation and the Michaelis-Menten relationship. In section 6, we show and compare the results of numerical calculations for the three optimal control models for the sheep, swine and shrimp feeding problems. For each animal, we use real data in the numerical calculations. Finally, in section 7, we give conclusions.

Results for the optimal feeding policy for the continuous model were published previously by one of the authors (C.K) and her collaborators for the logistic growth model for sheep (Kiataramkul *et al.,* 2011) and for the logistic and Gompertz growth models for swine (Kiataramkul & Matkhao, 2015). Preliminary results for the continuous and hybrid models for sheep and swine feeding were also reported previously by the current authors (Puengpo *et al.,* 2016). However, we briefly summarize these results in this paper as we wish to compare the results from the continuous, hybrid, and discrete models for feeding sheep, swine, and shrimp.

2. Optimal Control

2.1 Continuous optimal control, Pontryagin maximum principle (Lenhart & Workman, 2007; Luenberger, 1979; Pontryagin *et al.,* 1962)

We consider optimal control problems of the form:

Minimize $J[u]=\int_0^T g(t,x(t),u(t))\,dt,$

subject to $\dfrac{dx}{dt}= f(\mathrm{t}, x(t),u(t)),$ (1)

where T is a fixed time, $x=(x_1,x_2,...,x_n)^T$, and with boundary conditions of either $x_j(0)$ and $x_j(T)$ having given values or having free values. The Pontryagin maximum principle introduces Hamiltonian $H(t,x,\lambda,u)$ and costate variables $\lambda(t)$ defined as the solution of co-state differential equations given by

$$H(t,x,\lambda,u)=g(t,x,u)+\sum_{j=1}^{n}\lambda_j f_j(t,x,u),$$

$$\frac{d\lambda}{dt}=-\frac{\partial H}{\partial x_j}, \qquad j=1,2,... \tag{2}$$

with boundary conditions: $\lambda(T)$ is free if $x(\mathrm{T})=x_T$ is fixed and $\lambda(T)=0$ if $x(T)$ is free. In the applications in this paper, we assumed no upper or lower bounds on the values of the control variables and we are attempting to minimize the Hamiltonian with respect to the control variables. In this case, we can find the minimum of the Hamiltonian from the condition $\frac{\partial H}{\partial u}=0$ and check that the second derivative $\frac{\partial^2 H}{\partial u^2}>0$ at the optimal control $u^*(t)$. Using the Pontryagin maximum principle for the Hamiltonian to find a value for the control $u^*(t)$, we can numerically solve the boundary value problem for the state and co-state equations with the $bvp4c$ program in MATLAB.

2.2 Hybrid optimal control

Minimize $J[u]=\int_0^T g(t,x(t),u(t))\,dt,$

subject to $\dfrac{dx}{dt}=f(\text{t},\ x(t),u(t)),$ (3)

where the interval $[0,T]$ is assumed to be divided into N subintervals of length $h=\dfrac{T}{N}$ and $u(\text{t})$ is assumed to be constant over each subinterval. For this kind of control problem, the Pontryagin maximum principle can be used because the state and co-state equations are still ordinary differential equations, but with the extra boundary conditions that the state and co-state variables must be continuous at the ends of each subinterval.

As for the continuous case, we can numerically solve the boundary value problem for the state and co-state equations with the $bvp4c$ program in MATLAB.

2.3 Discrete optimal control

We consider discrete optimal control problems of the form (Lenhart & Workman, 2007; Luenberger, 1979):

Minimize $J[u]=\sum_{k=0}^{T-1} g(t,x(t),u(\text{t})),$

subject to $x(\text{t}+1)=f(t,x(t),u(t))$. (4)

The Pontryagin maximum principle for the discrete-time model is defined as follows (Lenhart & Workman, 2007; Luenberger, 1979):

We define the Hamiltonian as

$$H(t)=g(t,x(t),u(t))+\sum_{j=1}^{n}\lambda_j(t+1)f_j(t,x(t),u(t))$$

for $t=0,1,2,...,T-1,$ and then the necessary conditions on the state and co-state variables are the difference equations:

$$x(t+1)=f(t,x(t),u(\text{t})), \quad (5)$$

$$\lambda_j(t)=\frac{\partial H(t,x(t),\lambda(t+1),u(t))}{\partial x_j(t)},\quad k=1,2,...,t \quad (6)$$

for $u(t),\ k=0,1,2,...,T-1$, and with boundary conditions: $\lambda(T)$ is free if $x(\text{T})=x_T$ is fixed and $\lambda(T)=0$ if $x(T)$ is free. We can also check the second-derivative condition to distinguish between optimal controls $u^*(t)$ that maximize and those that minimize the objective functional since $\dfrac{\partial^2 H}{\partial u^2}<0$ at $u^*(t)$ implies maximization and $\dfrac{\partial^2 H}{\partial u^2}>0$ at $u^*(t)$ implies minimization.

3. Sheep Feeding

3.1 Continuous and hybrid model

The model of fetal growth is based on experimental data of sheep singleton pregnancies for 72 days after the differentiated fetus is formed (Gatford *et al.,* 2008; Kiataramkul *et al.,* 2011). As stated in Kiataramkul *et al.* (2011), the optimal control problem is:

Minimize $$J[u]=\int_0^T u(t)dt,$$

subject to
$$\frac{dx}{dt}=\frac{rxu}{u+L}\left(1-\frac{x}{K_0+\dfrac{\alpha y}{y+L_0}}\right), \tag{7}$$
$$\frac{dy}{dt}=u-\beta y.$$

The definitions of the variables and parameters in the model and for the parameter values used in the numerical calculations in section 6 are given in Table 1. The parameter values in Table 1 were obtained by Kiataramkul *et al.* (2011) as best fits for the data on the fetal growth rate of sheep published in Gatford *et al.* (2008). Kiataramkul *et al.* tested a straight-line function, a logistic growth function, a Gompertz growth function, and an exponential function as fits to the Gatford *et al.* data. They found that the logistic function with the parameter values in Table 1 gave the best fit of the four functions tested. Further details can be found in Kiataramkul *et al.* (2011).

Table 1. Sheep Feeding (data from Gatford *et al.,* 2008; Kiataramkul *et al.,* 2011).

Parameters	Description	Values
$x(t)$	Fetal weight of lamb at time t	kg
$u(t)$	Food given to pregnant sheep at time t	kg.day^{-1}
$y(t)$	Cumulative food intake at time t	kg
$x(0)$	Initial fetal weight of lamb	0.2 kg
T	Time of birth of lamb	72 days
x(T)	Weight of lamb at birth	5.5 kg
r > 0	Average growth rate	0.07 day^{-1}
β > 0 α	Discount factor for extent that cumulative food intake is influenced by the past	0.12 day^{-1}
α > 0	Factor indicating degree to which ultimate weight of lamb is determined by food given	0.1 kg
K_0	Capacity for weight of lamb	7 kg
L, L_0	Positive constants which are determined from data	0.09 kg. day^{-1}, 10 kg.

In the model in equation (7), it is assumed that the growth rate is given by the logistic-type equation $x(t)=\dfrac{x_0K}{x_0+(K-x_0)e^{-rt}}$ and the Michaelis-Menten relationship $\dfrac{u}{u+L}$ (Murray, 2001).

The Hamiltonian and co-state equations can be computed from equation (1) (Kiataramkul *et al.,* 2011). The boundary values on the state variables $x(t)$ and $y(t)$ and the two co-state variables $\lambda_1(t)$ and $\lambda_2(t)$ are $x(0)$=0.2 kg, $x(72)$ = 5.5 kg, $y(0)$=0, $y(72)$ free, $\lambda_1(72)$ free, and $\lambda_2(72)$=0.

For the hybrid model, the formulation is similar except that the control is discrete and the condition that the state and co-state variables are continuous functions of t must be imposed at each point of discontinuity in u.

3.2 Discrete model

The equivalent discrete optimal control problem to the continuous problem in (7) is as follows:

Minimize $J[u]=\sum_{t=0}^{T-1}u(t),$

subject to $$x(t+1)=x(t)+\frac{hrx(t)u(t)}{u(t)+L}\left(1-\frac{x(t)}{K_0+\frac{\alpha y(t)}{y(t)+L_0}}\right), \tag{8}$$

$$y(t+1)=y(t)+h(u(t)-\beta y(t)).$$

The Hamiltonian function and co-state equations are defined as:

$$H(t,x,\lambda,u)=\sum_{t=0}^{T-1}u(t)+\lambda_1(t+1)x(t+1)+\lambda_2(t+1)y(t+1),$$

$$\lambda_1(t)=\lambda_1(t+1)\left(1+\frac{hru(t)}{u(t)+L}\left(1-\frac{2x(t)}{K_0+\frac{\alpha y(t)}{y(t)+L_0}}\right)\right), \tag{9}$$

$$\lambda_2(t)=\lambda_1(t+1)\left(1+\frac{hr\alpha L_0u(t)x^2(t)}{\left(K_0+\frac{\alpha y(t)}{y(t)+L_0}\right)^2(u(t)+L)(y(t)+L_0)^2}\right)+h(1-\beta),$$

with boundary conditions $x(0)=0.2$ kg, $x(72)=0.2$ kg, $y(0)=0$ and with $y(72)$, $\lambda_1(0)$, $\lambda_1(72)$, $\lambda_2(0)$ free. We solve for optimal u^* from:

$$\frac{\partial H}{\partial u}=1+\lambda_1(t+1)\frac{hrLx(t)}{u(t)+L}\left(1-\frac{x(t)}{K_0+\frac{\alpha y(t)}{y(t)+L_0}}\right)+h\lambda_2(t+1)=0 \tag{10}$$

to obtain the optimal control as:

$$u^*(t)=-L+\sqrt{\frac{-\lambda_1 hrLx}{1+h\lambda_2}\left(1-\frac{x(t)}{K_0+\frac{\alpha y(t)}{y(t)+L_0}}\right)}. \tag{11}$$

As usual, we can check that $u^*(t)$ minimizes the Hamiltonian by checking that $\frac{\partial^2 H}{\partial u^2}>0.$ After substituting the optimal u^* into the state and co-state difference equations, we solve the system of difference equations by an iterative process until convergence occurs.

4. Swine Feeding

The definitions of the variables and parameters and for the values of parameters used in the numerical calculations in section 6 are given in Table 2. The parameter values in Table 2 were obtained by Kiataramkul & Matkhao (2015) as best fits the data on the growth rate of swine obtained from the Thailand Ministry of Agriculture and Cooperatives (2014). Kiataramkul & Matkhao tested a straight-line function, a logistic growth function, a Gompertz growth function, and an exponential function that fit the Ministry of Agriculture and Cooperatives data. They found that the Gompertz function with the parameter values in Table 2 gave the best fit of the four functions tested. However, they decided to test both the Gompertz and logistic growth functions to compare the optimal controls. Further details can be found in Kiataramkul & Matkhao (2015).

Table 2. Swine Feeding (data from Thailand Ministry of Agriculture and Cooperatives [2014]).

Parameters	Description	Values
$x(t)$	Weight of swine at time t	kg
$u(t)$	Food given to swine at time t	kg.day^{-1}
$x(0)$	Initial weight of swine at weaning	6.5 kg
T	Time for sale of swine after weaning	140 days
$x(T)$	Desired weight of swine at sale	100 kg
$r > 0$	Average growth rate	(Gompertz) 0.01191 day^{-1} (Logistic) 0.03275 day^{-1}
$\beta > 0$ α	Discount factor for extent that cumulative intake is influenced by the past	0.7129 kg^{-1}.day
$\alpha > 0$	Factor indicating degree to which ultimate weight is determined by food given	0.8497 kg^{-1}.day
K	Capacity of swine weight	(Gompertz) 188.6 kg. (Logistic) 113.4 kg.

4.1 Continuous and hybrid logistic growth models

Minimize $J[\mathrm{u}] = \int_0^T u\,dt,$

subject to $$\frac{dx}{dt} = \frac{rx\alpha u}{1+\beta u}\left(1-\frac{x}{K}\right), \tag{12}$$

In this model, it is assumed that the growth rate of the swine is given by the logistic growth function $rx\left(1-\frac{x}{K}\right)$ and the Michaelis-Menten relationship $\frac{\alpha u}{1+\beta u}$. For the swine model in (12), the Hamiltonian and co-state equations are:

$$\begin{aligned}
&H(t,x,\lambda,u) = u + \lambda rx\left(1-\frac{x}{K}\right)\frac{\alpha u}{1+\beta u}, \\
&\frac{d\lambda}{dt} = -\lambda r\left(1-\frac{2x}{K}\right)\frac{\alpha u}{1+\beta u}, \\
&\frac{\partial H}{\partial u} = 1 + \lambda rx\left(1-\frac{x}{K}\right)\frac{\alpha}{(1+\beta u)^2} = 0, \\
&u^* = \frac{\sqrt{-\lambda rx\alpha\left(1-\frac{x}{K}\right)}-1}{\beta},
\end{aligned} \tag{13}$$

The boundary conditions on the state equations are $x(0) = 6.5$ kg and $x(140) = 100$ kg, and the initial and final values for the co-state variable are free values. We checked that u^* gives minimum of H from the second derivative $\frac{\partial^2 H}{\partial u^2} > 0$.

For the hybrid model, the formulation is similar except that the control is discrete and the condition that the state and co-state variables are continuous functions of t must be imposed at each point of discontinuity in u.

4.2 Discrete logistic growth model

The discrete model and the corresponding Hamiltonian, co-state equation, boundary conditions, and optimal u for the optimal swine feeding problem in subsection 4.1 are as follows:

Minimize $J[u] = \sum_{t=0}^{139} u,$

subject to $x(t+1) = x(t) + hrx(t)\left(1 - \frac{x(t)}{K}\right)\frac{\alpha u(t)}{1+\beta u(t)},$

$$H(t,x,\lambda,u) = u(t) + \lambda hrx(t)\left(1 - \frac{x(t)}{K}\right)\frac{\alpha u(t)}{1+\beta u(t)},$$

$$\lambda(t) = \lambda(t+1)\left(1 + \frac{hr\alpha u(t)\left(-2x(t)+K\right)}{K\left(1+\beta u(t)\right)}\right),$$

$$\frac{\partial H}{\partial u} = 1 + \lambda(t+1)hrx(t)\left(1 - \frac{x(t)}{K}\right)\frac{\alpha}{\left(1+\beta u(t)\right)^2} = 0,$$

$$u^* = \frac{\sqrt{-\alpha\lambda hrx\left(1 - \frac{x(t)}{K}\right)} - 1}{\beta},$$

with boundary conditions $x(0) = 6.5$ kg, $x(140) = 100$ kg and with $\lambda(0)$, $\lambda(140)$ having free values.

4.3 Continuous Gompertz growth model

minimize $J[u] = \int_0^{140} u\,dt,$

subject to $\frac{dx}{dt} = r\log\left(\frac{K}{x}\right)x\frac{\alpha u}{1+\beta u},$

$$\frac{d\lambda}{dt} = -r\frac{\alpha u}{1+\beta u}\left(\log\left(\frac{k}{x}\right) - 1\right),$$

$$\frac{\partial H}{\partial u} = 1 + \lambda rx\log\left(\frac{K}{x}\right)\frac{\alpha}{\left(1+\beta u\right)^2} = 0,$$

$$u^*(t) = \frac{\sqrt{-\lambda rx(t)\alpha\log\frac{K}{x}} - 1}{\beta},$$

with boundary conditions $x(0) = 6.5$ kg, $x(140) = 100$ kg and with $\lambda(0)$, $\lambda(140)$ having free values.

4.4 Discrete Gompertz growth model

The discrete model and the corresponding Hamiltonian, co-state equation, boundary conditions, and optimal u are as follows:

$$\text{minimize } J[u] = \sum_{t=0}^{139} u,$$

$$\text{subject to } x(t+1) = x(t) + hrx(t)\log\left(\frac{K}{x(t)}\right)x\frac{\alpha u(t)}{1+\beta u(t)},$$

$$H(t,x,\lambda,u) = u(t) + \lambda(t+1)\left(x(t) + hrx(t)\log\left(\frac{K}{x(t)}\right)x(t)\frac{\alpha u(t)}{1+\beta u(t)}\right), \quad (14)$$

$$\lambda(t) = \lambda(t+1)\left(1 + hr\frac{\alpha u(t)}{1+\beta u(t)}\left(\log\frac{K}{x(t)} - 1\right)\right),$$

$$\frac{\partial H}{\partial u} = 1 + \lambda(t+1)hrx(t)\log\left(\frac{K}{x(t)}\right)x(t)\frac{\alpha}{(1+\beta u(t))^2} = 0,$$

$$u^*(t) = \frac{\sqrt{-\lambda rx(t)\alpha\log\frac{K}{x(t)}} - 1}{\beta}.$$

The boundary conditions are again $x(0) = 6.5$ kg and $x(140) = 100$ kg, and with $\lambda(0)$, $\lambda(140)$ having free values.

5. Shrimp Feeding

For the shrimp model, we assumed that growth rate, as a function of age and food intake, can be modeled by the Gompertz equation and the Michaelis-Menten relationship. The mathematical forms of the optimal control problems for the shrimp feeding are then the same as for the continuous and hybrid Gompertz growth models for swine feeding in section 4.1 and for the discrete Gompertz growth model for swine feeding in section 4.2. The definitions of the variables and parameters and the values of the parameters used in the numerical calculations in section 6 are given in Table 3. The parameter values were obtained by the present authors as the best fits to the data on the growth rate of Vannamei shrimp obtained from the Thailand Ministry of Agriculture and Cooperatives (2013). It was found that the Gompertz growth function with the parameter values in Table 3 gave the best fit.

Table 3. Shrimp Feeding (data from Thailand Ministry of Agriculture and Cooperatives [2013]).

Parameters	Description	Values
x(t)	Weight of shrimp at time t	g
u(t)	Food given at time t	g.day^{-1}
x(0)	Initial weight of shrimp	0.01 g
T	Time of sale of shrimp	120 days
x(T)	Desired weight of shrimp at sale	19 g
r > 0	Average growth rate	0.03117 day^{-1}
β > 0 α	Discount factor for extent that cumulative intake is influenced by the past	1.183 g^{-1}.day
α > 0	Factor indicating degree to which ultimate weight is determined by food given	0.6765 g^{-1}.day
K	Capacity of animal weight	21.13 g

6. Numerical Results

We give numerical results for the three optimal control methods for the sheep, swine, and shrimp feeding problems using real data for sheep obtained from Gatford *et al.* (2008) and Kiataramkul *et al.* (2011) and real data for swine and shrimp obtained from the Thailand Ministry of Agriculture and Cooperatives (2013, 2014). We used the Matlab program *bvp4c* to find the optimal control for the continuous and hybrid models and we wrote Matlab computer programs to solve the discrete control problems.

6.1 Sheep feeding

For pregnant sheep, our target was to use experimental data to find the optimal feeding policies to produce new-born lambs of an optimal weight. The data used to compute the optimal control policies are shown in Table 1. For pregnant sheep, we found that the total amounts of food required over 72 days were 127.1 kg for continuous, 127.03 kg for hybrid, and 131.23 kg for discrete models. These results showed that the optimal value for the total food would not appreciably change by replacing a continuous feeding policy by a discrete policy if the growth rates could be modeled by a differential equation. However, the optimal value for the total food estimated from a difference equation approximation to the differential equation growth equation was larger than from the differential equation.

For continuous control, the results showed that we should slightly increase the amount of food from 1.72 kg per day given on the first day to a maximum of 1.82 kg per day at day 65 and then decrease the food to approximately 1.71 kg per day on day 72 (Figure 1).

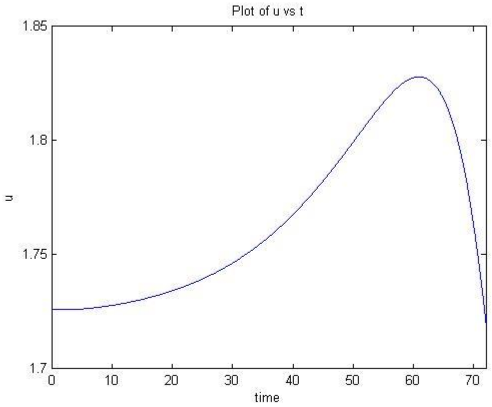

Figure 1. Continuous pregnant sheep feeding.

For the hybrid control (Figure 2), the total feeding time was divided into 9 intervals. In this strategy of feeding, the same amount of food was given for a total of 8 days and then changed for the next 8 days. A comparison of the continuous and hybrid feeding strategies showed that they follow a qualitatively similar pattern, but with differences in detail.

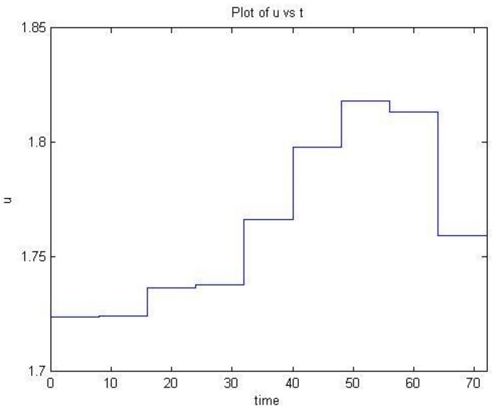

Figure 2. Hybrid pregnant sheep feeding.

For the discrete optimization results, the optimal feeding pattern was qualitatively similar to the continuous and hybrid patterns, but with a lower amount of food given on the first day of 1.1 kg per day and then rising to a maximum of 3 kg per day at day 68 (Figure 3). In Figure 3, the discrete feeding intervals were assumed to be one day.

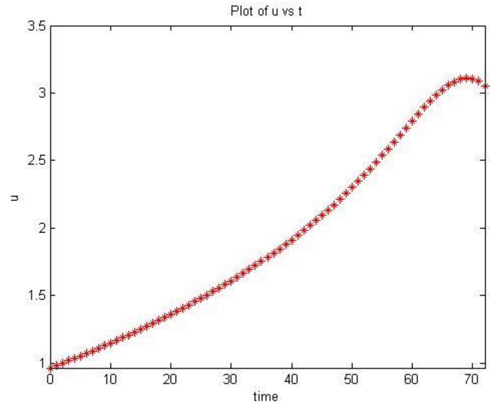

Figure 3. Discrete pregnant sheep feeding.

6.2 Swine feeding

For post-weaned pigs, the target was to minimize the feed to get a specified optimal weight at the day of sale. For swine, we considered two different growth rates: logistic and Gompertz. In each case, we used real data from the Thailand Ministry of Agriculture and Cooperatives (2014) (Table 1 of Kiataramkul & Matkhao 2015). The values of the parameters used are shown in Table 2. The results for the logistic and Gompertz models for the three different optimal control methods are shown in Figures 4, 5, 6, 7, 8, and 9. We found that, for each model, the continuous, hybrid, and discrete models gave similar amounts of total food required, but with some differences in detail, with the logistic model giving a total of 81.54 kg and the Gompertz model giving a total of 76.4 kg.

For continuous swine feeding in the Gompertz model, the results showed that we should increase the amount of food in a monotonic manner (Figure 4). However, in practice the value of the control is approximately constant during the feeding period.

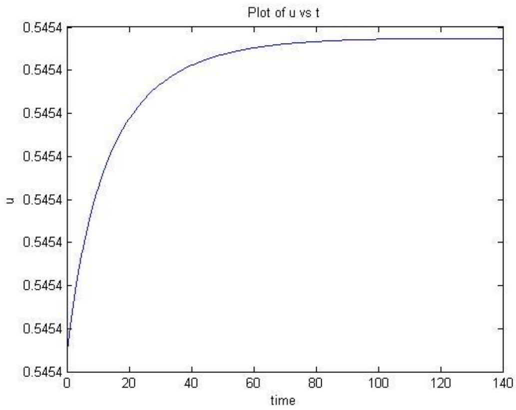

Figure 4. Continuous swine feeding – Gompertz.

For the hybrid control, the total feeding time was divided into four intervals (Figure 5). In this strategy, the same amount of food was given for 35 days and then changed for the next 35 days. A comparison of the continuous control and hybrid control shows that the hybrid control gives a step-wise reduction in optimal feeding in the second half of the feeding period, whereas the continuous control is a monotonically increasing function. However, in practice the value of the hybrid control is approximately constant during the feeding period.

For the discrete swine feeding results in the Gompertz model, the optimal feeding strategy is qualitatively similar to the continuous case (Figure 6). The results in Figure 6 were obtained by assuming that the discrete time steps were one day.

A comparison of the continuous results for the Gompertz model in Figure 4 with the continuous results for the logistic model in Figure 7 showed that the main difference is that the logistic feeding decreases at the end of the feeding period, whereas the Gompertz feeding is monotonically increasing.

Comparing the hybrid results for the Gompertz model in Figure 5 with the hybrid results for the logistic model in Figure 8, we see that the feeding patterns are qualitatively similar, but with some differences in detail.

Comparing the discrete results for the Gompertz model in Figure 6 with the results for the logistic model in Figure 9, we see that the feeding patterns are qualitatively similar, but with a higher feeding rate in the logistic model than in the Gompertz model. In Figure 9, the discrete feeding intervals were assumed to be one day.

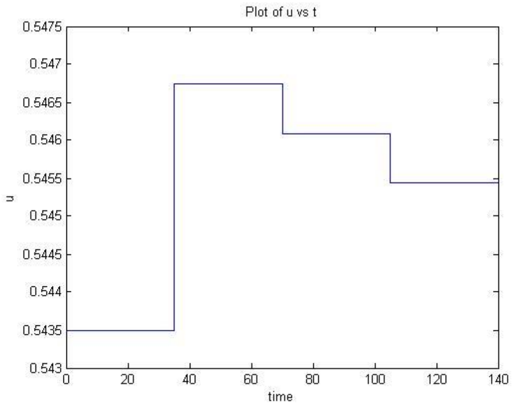

Figure 5. Hybrid swine feeding– Gompertz.

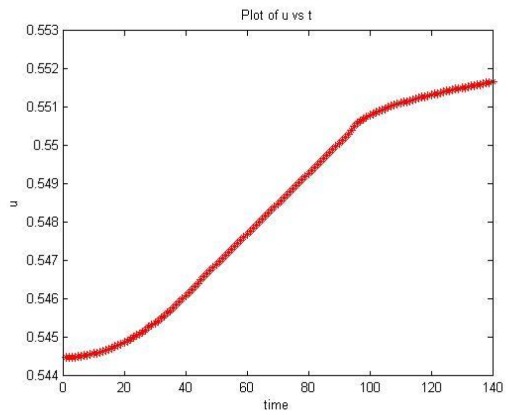

Figure 6. Discrete swine feeding– Gompertz.

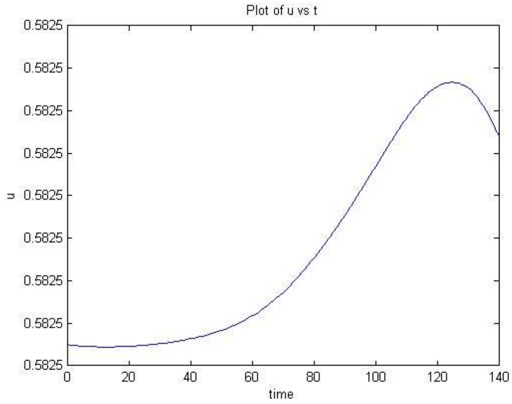

Figure 7. Continuous swine feeding– Logistic.

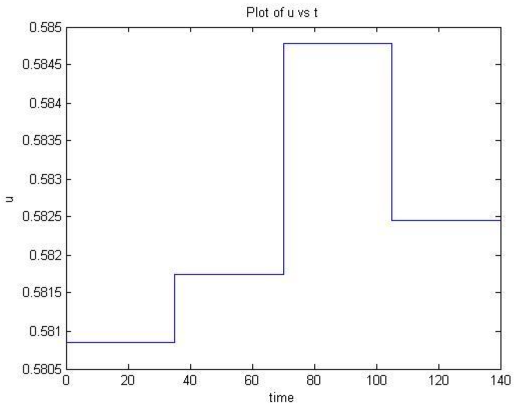

Figure 8. Hybrid swine feeding– Logistic.

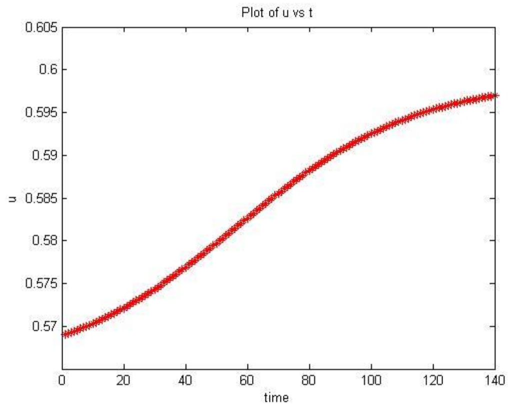

Figure 9. Discrete swine feeding– Logistic.

6.3 Shrimp feeding

For post-larvae shrimp, the target was to minimize the feed to get a specified optimal weight of shrimp on the day of sale. The values of the parameters used are shown in Table 3. For Vannamei-shrimp, we considered the three types of optimal control, i.e. continuous, hybrid, and discrete, and found that the total feeds for continuous, hybrid, and discrete were 26.7 g, 25.37 g, and 25.83 g, respectively. The optimal control policies for the continuous, hybrid, and discrete models are shown in Figures 10-12.

The results for the continuous feeding in Figure 10 show that, except for the slight dip at the start, there is a monotonic increase in the feeding rate in this Gompertz growth model. This pattern is qualitatively similar to the results in the Gompertz swine feeding model (Figure 4). However, there was a much bigger change in feeding rates for the shrimp model than the swine feeding model.

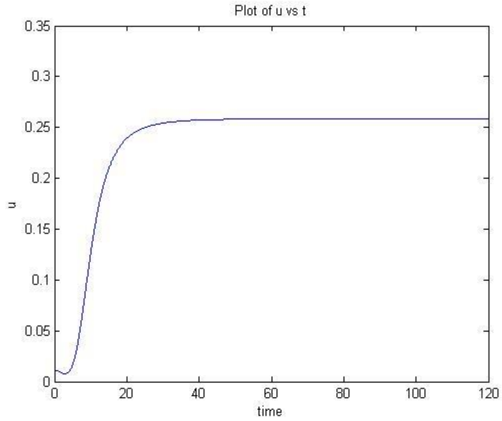

Figure 10. Continuous shrimp feeding.

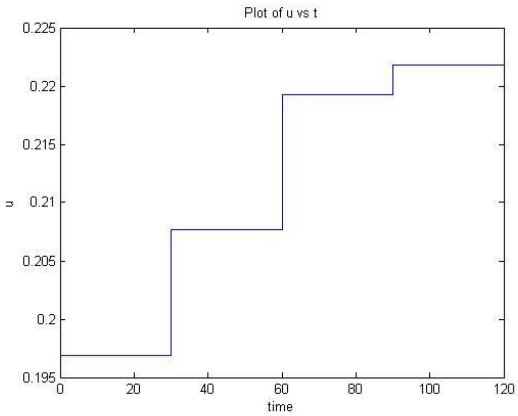

Figure 11. Hybrid shrimp feeding.

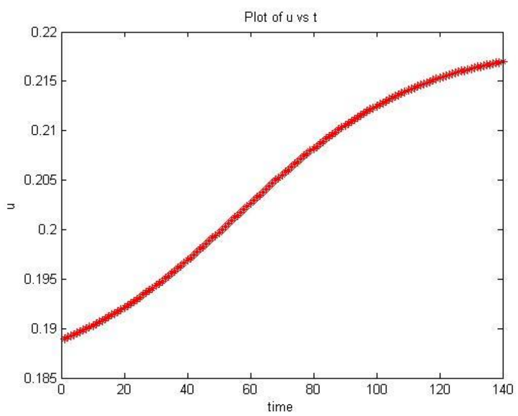

Figure 12. Discrete shrimp feeding.

The results for the hybrid and discrete shrimp feeding demonstrated that the feeding patterns were qualitatively similar to the continuous results with some differences in detail (Figures 11 and 12). In Figure 12, the discrete feeding intervals were assumed to be one day.

7. Discussion and Conclusions

We have used optimal control theory to find optimal feeding policies for pregnant sheep, post-weaned swine, and post-larvae period shrimp. Our objective for all animals was to minimize the total feed during a specified time. We have used three different optimal control methods: continuous, hybrid, and discrete. For each animal, we first considered a continuous feeding policy. However, a continuous feeding policy is not a practical policy for a farmer to follow. A farmer will usually feed a given constant amount in a period of time, for example during a week or a month. In the hybrid feeding strategy, we divided the feeding period into subintervals and assumed that the feed was constant over each subinterval. However, we assumed that the growth rate of an animal could be modeled by a continuous differential equation. For both the continuous and the hybrid models, the continuous form of the Pontryagin maximum principle could be used. However, for the hybrid case, extra boundary conditions of weight and co-state variable values that were continuous at the boundary of each sub-interval were required.

We developed computer programs in Matlab for each model. For both continuous and hybrid cases, we used the Matlab boundary value problem solver *bvp4c* to compute the optimal controls.

For the discrete-time optimal control case, we assumed that the feeding was constant over subintervals of the total period and also assumed difference equations for the growth rates. In this case, we used a discrete-time version of the Pontryagin maximum principle. We again wrote programs in Matlab to solve the systems of equations.

Our numerical results showed that the optimal feeding policies for the continuous and hybrid models were approximately the same and that the discrete results were also not appreciably different. Therefore, the results showed that the discrete feeding policies discussed in this paper can be used instead of a continuous feeding policy with very similar total feeding costs. We believe that optimal control can be a useful practical method to determine reasonable optimal feeding policies for animals.

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References

Gatford, K. L., Owens, J. A., Li, S., Moss, T. J. M., Newnham, J. P., Challis, J. R. G., & Sloboda, D. M. (2008). Repeated betamethasone treatment of pregnant sheep programs persistent reductions in circulating IGF-I and IGF-binding proteins in progeny. *American Journal of Physiology-Endocrinology and Metabolism*, *295*(1), E170-E178. doi: 10.1152/ajpendo.00047.2008.

Giordano, F. R., Fox, W. P., & Horton, S. B. (2014). *A first course in mathematical modeling* (5th ed.). Boston, MA: Cengage Learning.

Kiataramkul, C., Wake, G. C., Ben-Tal, A., & Lenbury, Y. (2011). Optimal nutritional intake for fetal growth. *Mathematical Biosciences and Engineering, 8*(3), 723-732. doi:10.3934/mbe.2011.8.723.

Kiataramkul, C., & Matkhao, P. (2015). Optimal control problem of food intake of swine during post weaning period. *Agriculture and Natural Resources-formerly Kasetsart Journal (Natural Science), 49*(6), 983-989.

Lenhart, S., & Workman, J. T. (2007). *Optimal control applied to biological models*. Boca Raton, FL: Chapman and Hall/CRC.

Luenberger, D. G. (1979). *Introduction to dynamic systems: Theory, models and applications*. New York, NY: John Wiley and Sons.

Thailand Ministry of Agriculture and Cooperatives (2013). *Culture of Vannamei Shrimp in recycle system compared with closed system by use of re-suspension of sediment (private communication)*. Bangkok, Thailand: Author.

Thailand Ministry of Agriculture and Cooperatives (2014). Retrieved from: http://guru.sanook.com/3950/

Murray, J. D. (2001). *Mathematical biology: I. An introduction* (3rd ed.). New York, NY: Springer.

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelize, R. V., & Mishchenko, E. F. (1962). *The mathematical theory of optimal processes*. New York, NY: John Wiley and Sons.

Puengpo, S., Kiataramkul, C., & Moore, E. J. (2016). Application of discrete optimal control to feeding of farm animals. *Proceeding of International Conference in Mathematics and Applications* (pp. 151–160).

Zeide, B. (1993). Analysis of growth equations. *Forest Science, 39*(3), 594–616.