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*Original Article*

# Similarity measure between Pythagorean fuzzy sets based on lower, upper and middle fuzzy sets with applications to pattern recognition and multicriteria decision making with PF-TODIM

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## Abstract

The choice of similarity measure (SM) plays an important role in distinguishing between objects. Similarity measure of Pythagorean fuzzy sets (PFSs) is very useful and effective in discriminating between different Pythagorean fuzzy sets. Therefore, in this paper, we suggest a new similarity measure for PFSs based on converting the PFSs into their lower, upper and middle fuzzy sets (FSs) to calculate their degree of similarity. We construct an axiomatic definition for a new SM of PFSs. Furthermore, we put forward a new way to express the similarity measure of PFSs to show its competency, reliability and applicability. For establishing reasonability and usefulness of the proposed methods, we present several practical examples related to pattern recognition and multicriteria decision making problems. Finally, we construct an algorithm for Portuguese of interactive and multiple attributes decision making (TODIM) method based on the proposed similarity measures, for handling complex multicriteria decision making problems related to day to day life. Our final results show that the suggested method is reasonable, reliable and useful in managing different complex decision making problems in the context of Pythagorean fuzzy sets as the domain.

**Keywords**: fuzzy set, Pythagorean fuzzy sets, similarity measures, pattern recognition TODIM, multicriteria decision making

## 1. Introduction

The concept of fuzzy sets was first proposed by Zadeh (1965). With widespread use in various fields, fuzzy sets not only provide us a broad opportunity to measure uncertainties in more powerful and logical way, but also are a meaningful way to represent vague concepts in natural language. It is known that most systems based on crisp set theory or two-valued logic are somehow difficult for handling imprecise and vague information. In this sense, fuzzy sets can be used and provide solutions to more real world problems. Furthermore, to treat more imprecise and vague information in daily life, various applications and extensions of fuzzy sets have been demonstrated, such as multi attribute decision making (MADM) which is a significant branch and plays an important role in human activities (Garg, 2017; Zeng *et al.*, 2018). Recently, many tools have been introduced for representing and communicating uncertainty. For example, Zadeh (1965) introduced the fuzzy set (FS). The theory of fuzzy sets has received lots of attention over many years, but the weakness of FS is that it just has a membership degree (MD) and ignores unsure data in real DM problems. To overcome this disadvantage of FS, Atanassov (1986) gave the amazing concept of intuitionistic fuzzy sets (IFSs), with the characterization of membership degree (MD) and non-membership degree (NMD) and the degree of hesitancy. Thus, IFSs are more powerful and capable than fuzzy sets. The condition of IFSs state that the sum of MD and NMD is always less than or equal to 1. However, there exist circumstances where the sum of MD and NMD is more than one. To overcome this disadvantage, the new generalization of IFSs called Pythagorean fuzzy sets (PFSs) was introduced by Yager (2013), Yager and Abbasov (2013), and Pythagorean membership grades in multicriteria decision making by Yager (2013). The range of PFSs is much wider than that of the IFSs, in which the sum of the square of MD and NMD is restricted

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to less than or equal to 1. For example, (0.6, 0.7) represents a situation in which both IFSs cannot evaluate the attribute value of (0.6, 0.7) because 0.6 + 0.7 > 1 and $(0.6)^2 + (0.7)^2 < 1$, respectively. Therefore, we can say that PFSs are more powerful and rigorous than IFSs in handling incomplete information related to daily life. Fuzzy sets have many applications in almost all fields, such as clustering, image processing, mathematical programming, fuzzy control, pattern recognition, water quality, engineering, medical diagnosis, business management, decision making, data mining, etc. Application of fuzzy sets in soil science, fuzzy logic, fuzzy measurements and fuzzy decisions is given by Mcbratney and Odeh (1997), and an application of fuzzy set theory to inventory control models is suggested by Gen *et al.* (1997). The possible and necessary inclusion of intuitionistic fuzzy sets is given by Grzegorzewski (2011). Image enhancement using fuzzy sets is proposed by Pal and King (1980). The similarity measures based on lower, upper and middle fuzzy sets corresponding to PFSs, provides us a new way to construct similarity measures between PFSs. On the basis of numerical analysis results, we find that our new construction can handle amicably different problems related to daily life, especially problems involving pattern recognition and multicriteria decision making processes. As a whole, our proposed construction method for similarity measures between PFSs can provide a more effective way for measuring similarity degrees between PFSs. Similarity measure is a significant instrument to determine the degree of SMs between two objects. Kaufman, and Rousseeuw (1990) introduced a few models to present traditional similarity measures with applications in various leveled group investigations. Different similarity measures between FSs have been introduced. Dengfeng, and Chuntian (2002) introduced a few SMs between IFSs utilized in design recognition. Liang, and Shi (2003) proposed similarity measures between IFSs and furthermore present the connections between these measures with applications to design recognition, as given by Mitchell (2003). Deciphered IFSs as ensembles of ordered FSs from the statistical perspective to exchange techniques was suggested by Dengfeng, and Chuntian (2002). Liang, and Shi (2019) utilized numerical comparisons to demonstrate that their suggested SMs are more reliable than those of Dengfeng, and Chuntian (2002). Hung, and Yang (2004) expressed a few SMs between IFSs dependent on Hausdorff distance, which are very much utilized with linguistic variables. Xu, and Chen (2008) gave comparisons of distance and SMs between IFSs. Pythagorean fuzzy sets are used in a variety of applications in almost all fields, including pattern recognition, multicriteria decision making, engineering, medical diagnosis, business management, clustering etc. A similarity measure for constrained Pythagorean fuzzy sets (CPFSs) is presented by Pan *et al.* (2021) who suggested a complex distance measure of PFS and applied it to pattern recognition. Yang, and Hussain (2019) suggested distance and SM between hesitant fuzzy sets and applied it to clustering. Li, and Lu (2019) suggested many new distance and SM between PFSs with applications and provided the concept of normalized Hamming and Hausdorff distances. Fuzzy entropy for Pythagorean fuzzy sets with application to multicriterion decision making was coined by Yang, and Hussain (2018). Many distance and similarity measure of PFSs with applications are discussed in this literature Hussain, and Yang (2019), Li, and Lu (2019), Peng, and Garg, (2019), Verma, and Merijo (2019) and Zhao, and Chen (2019). Similarity Measures for New Hybrid Models: mF Sets and mF Soft Sets were suggested by Akram, and Waseem (2019). A hybrid method for complex Pythagorean fuzzy decision making, Mathematical Problems in Engineering, was put forwarded by Akram *et al.* (2021). Minimum spanning tree hierarchical clustering algorithm: A new Pythagorean fuzzy similarity measure for the analysis of functional brain networks was given by Habib *et al.* (2022). A new outranking method for multicriteria decision making with complex Pythagorean fuzzy information has been reported (Akram *et al.*, 2022). Belief and plausibility measures on Pythagorean fuzzy sets and their applications with BPI-VIKOR were proposed by Hussain *et al.* (2022). An integrated ELECTRE-I approach for risk evaluation with hesitant Pythagorean fuzzy information was proposed by Akram *et al.* (2022). The similarity measure between two PFSs is very useful to indicate the degree of resemblance between two objects. Generally, information systems in PFSs are carried out by the lower, upper and middle fuzzy sets. In this manuscript, we put forward a new construction for similarity measures between PFSs based on lower, upper and middle fuzzy sets with the objective to develop useful and reasonable similarity measures between PFSs. We have utilized TODIM methods because the decision making outcome is determined by computing the degree of gain or loss of an alternative relative to the rest, to better reflect the behavioral preference of the decision makers.

The rest of the manuscript is arranged as follows. Section 2 consists of some basic preliminaries about PFSs. In Section 3, we introduce a new type of similarity measures between PFSs based on the lower, upper and center FSs. In Section 4, we show the performance of our proposed similarity measures using different examples. In section 5, we use the Pythagorean fuzzy TODIM method to manage a problem involving multicriteria decision making. At the end, conclusions are conveyed in Section 6.

## 2. Preliminaries

The basic concepts of IFS and PFS are respectively given in following section.

**Definition 1**. An intuitionistic fuzzy set (IFS) *S* in *X* is defined by (Atanassov, 1999), as an object of the following form as,

$$S = \left\{ \left( x_i, \mu_{\hat{S}}(x_i), \nu_{\hat{S}}(x_i) \right) : x_i \in X \right\},$$

where $\mu_{\hat{S}}(x_i): X \to [0,1]$, denotes the degree of membership of $x_i \in S$ and $\nu_{\hat{S}}(x_i): X \to [0,1]$, denotes the degree of non-membership of $x_i \in \widehat{S}$ and $0 \le \mu_{\hat{S}}(x_i) + \nu_{\hat{S}}(x_i) \le 1.$ The degree of non - determinacy of IFS $\hat{S}$ is symbolized by the following relation $\pi_{\hat{S}}(x_i) = 1 - \left( \mu_{\hat{S}}(x_i) + \nu_{\hat{S}}(x_i) \right).$

**Definition 2**. A Pythagorean fuzzy set PFS *G* in *X* is given by (Yager, & Abbasov, 2013) as

$$G=\left\{\left\langle x_i,\mu_G\left(x_i\right),\nu_G\left(x_i\right)\right\rangle : x_i\in X\right\},$$

where $\mu_G\left(x_i\right): X\to[0,1]$, represents the degree of membership and $\nu_G\left(x_i\right): X\to[0,1]$, represents the degree of membership. For every $x_i\in B$ with the condition that

$$0\le\mu_G^2\left(x_i\right)+\nu_G^2\left(x_i\right)\le 1.$$

**Definition 3**. (Atanassov, 1999). For all $P\in PFSs(X)$, the following expression is termed as Pythagorean index of the element $x_i\in G$

$$\pi_G\left(x_i\right)=\sqrt{1-\left\{\mu_G^2\left(x_i\right)+\nu_G^2\left(x_i\right)\right\}},\ \text{It is obvious that}\ 0\le\pi_G^2\left(x_i\right)\le 1,\ \forall\ x_i\in X.$$

**Definition 4**. Let $P_1$, $P_2$ and $P_3$ are three PFSs on *X*. A similarity SM*S*(*G*, *H*) is mapped as, $S:\ PFSs(X)\times PFSs(X)\to[0,1]$ have the following operations

$(S_1)$ $0\le S(G,H)\le 1;$

$(S_2)$ $S(G,H)=1\ if\ G=H;$

$(S_3)$ $S(G,H)=S(H,G);$

$(S_4)$ $S(G,H)\ge S(G,I)\ and\ S(H,I)\ge S(G,I)\ if\ G\subseteq H\subseteq I;$

$(S_5)$ $S(G,H)=0\ if\ G=X\ and\ H=\varnothing\ or\ G=\varnothing\ and\ H=X.$

**Definition 5**. (Peng *et al.*, 2017) If *G* and *H* be two PFSs on *X* the following operations can be defined as follows:

(1) $G^c=\left\{\left\langle x_i,\nu_G\left(x_i\right),\mu_G\left(x_i\right)\right\rangle : x_i\in X\right\};$

(2) $G\subseteq H\ iff\ \forall x_i\in X,\ \mu_G\left(x_i\right)\le\mu_H\left(x_i\right)\ and\ \nu_G\left(x_i\right)\ge\nu_H\left(x_i\right);$

(3) $G=H\ iff\ \forall x_i\in X,\ \mu_G\left(x_i\right)=\mu_H\left(x_i\right)\ and\ \nu_G\left(x_i\right)=\nu_H\left(x_i\right);$

(4) $G\cap H=\left\{x_i, min\left(\mu_G\left(x_i\right),\mu_H\left(x_i\right)\right), max\left(\nu_G\left(x_i\right),\nu_H\left(x_i\right)\right)\right\};$

(5) $G\cup H=\left\{x_i, max\left(\mu_G\left(x_i\right),\mu_H\left(x_i\right)\right), min\left(\nu_G\left(x_i\right),\nu_H\left(x_i\right)\right)\right\}.$

## 3. Construction of New Similarity Measures

In this section, we construct some new and useful similarity measures between two PFSs. We use similar ideas to Hwang, and Yang (2013) and define some similarity measures on PFSs based on lower, upper and middle Pythagorean fuzzy set. Let us take a PFS $G=\left\{\left\langle x_i,\mu_G\left(x_i\right),\nu_G\left(x_i\right)\right\rangle : x_i\in X\right\}$, we first define the lower, upper and middle Pythagorean fuzzy sets with reference to Gregorszewsk, (2011), Hwang, and Yang (2013). Assume that the lower, upper and middle Pythagorean fuzzy sets are denoted by $G^L$, $G^U$, and $G^M$ respectively as follows:

$$G^L=\left\{\left\langle x_i,\mu_G^L\left(x_i\right),\nu_G^L\left(x_i\right)\right\rangle : x_i\in X\right\},\mu_G^L\left(x_i\right)=\mu_G^2\left(x_i\right);$$

$$G^U=\left\{\left\langle x_i,\mu_G^U\left(x_i\right),\nu_G^U\left(x_i\right)\right\rangle : x_i\in X\right\},\mu_G^U\left(x_i\right)=\mu_G^2\left(x_i\right)+\pi_G^2\left(x_i\right)=1-\nu_G^2\left(x_i\right);$$

$$G^M=\left\{\left\langle x_i,\mu_G^M\left(x_i\right)\right\rangle : x_i\in X\right\},\mu_G^M\left(x_i\right)=\frac{\mu_G^2\left(x_i\right)+1-\nu_G^2\left(x_i\right)}{2}.$$

First, we extend the similarity measures between IFSs (Hwang & Yang, 2013) to the similarity measures between Pythagorean fuzzy sets as follows:

$$\tilde{S}_C(G,H)=1-\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\nu_G^2\left(x_i\right)\right|-\left|\mu_H^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\} \tag{1}$$

$$\tilde{S}_H(G,H)=1-\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|-\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\} \tag{2}$$

$$\begin{aligned}\tilde{S}_L(G,H)=1&-\frac{1}{4n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\nu_G^2\left(x_i\right)\right|-\left|\mu_H^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\}\\&-\frac{1}{4n}\sum_{i=1}^{n}\left\{\left|\mu_G^2\left(x_i\right)-\mu_H^2\left(x_i\right)\right|-\left|\nu_G^2\left(x_i\right)-\nu_H^2\left(x_i\right)\right|\right\}\end{aligned} \tag{3}$$

$$\tilde{S}_O(G,H)=1-\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^2\right\}} \tag{4}$$

$$\tilde{S}_{DC}(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{m}_G(i)-\tilde{m}_H(i)\right|^p} \tag{5}$$

where $\tilde{m}_G(i)=\left(\mu_G^2(x_i)+1-\nu_G^2(x_i)\right)/2$ and $\tilde{m}_H(i)=\left(\mu_H^2(x_i)+1-\nu_H^2(x_i)\right)/2\; 1\le P<\infty.$

$$\tilde{S}_{HF}(G,H)=\frac{1}{2}\left(\rho\tilde{\mu}^2(G,H)+\rho\tilde{\nu}^2(G,H)\right) \tag{6}$$

where $\rho\tilde{\mu}^2(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}$ and $\rho\tilde{\nu}^2(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}$

$$\tilde{S}_e^p(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\tilde{\phi}_{tGH}(i)+\tilde{\phi}_{fGH}(i)\right)^p} \tag{7}$$

where $\tilde{\phi}_{tGH}(i)=\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|/2$ and $\tilde{\phi}_{fGH}(i)=\left|\left(1-\nu_G^2(x_i)\right)-\left(1-\nu_H^2(x_i)\right)\right|/2\; 1\le P<\infty$

$$\tilde{S}_s^p(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\tilde{Q}_{s1}(x_i)+\tilde{Q}_{s2}(x_i)\right)} \tag{8}$$

$$\begin{aligned}&\tilde{Q}_{s1}(x_i)=\left|\tilde{m}_{G1}(x_i)-\tilde{m}_{H1}(x_i)\right|/2,\tilde{Q}_{s2}(x_i)=\left|\tilde{m}_{G2}(x_i)-\tilde{m}_{H2}(x_i)\right|/2,\ \tilde{m}_{G1}(i)=\left(\tilde{\mu}_G^2(x_i)+\tilde{m}_G(i)\right)/2,\\&\tilde{m}_{G2}(i)=\left(\tilde{m}_G(i)+1-\tilde{\nu}_G(i)\right)/2,\ \tilde{m}_{H1}(i)=\left(\tilde{\mu}_{\mathbf{B}}^2(i)+\tilde{m}_H(x_i)\right)/2,\ \tilde{m}_{H2}(i)=\left(\tilde{m}_B(i)+1-\tilde{\nu}_G^2(i)\right)/2\cdot\\&\tilde{S}_{HB}(G,H)=\frac{1}{2}\left(\tilde{\rho}_\mu^2(G,H)+\tilde{\rho}_\nu^2(G,H)\right)\end{aligned} \tag{9}$$

where $\tilde{\rho}_\mu^2(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{\mu}_G^2(x_i)-\tilde{\mu}_H^2(x_i)\right|^p}$ and $\tilde{\rho}_\nu^2(G,H)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\tilde{\nu}_G^2(x_i)-\tilde{\nu}_H^2(x_i)\right|^p}$

Next, we extend the above defined similarity measures Equations (1) - (9) between two PFSs *G* and *H* to the similarity measures between two PFSs *G* and *H* based on below, above and center (bac) fuzzy sets respectively as follows:

$$\begin{aligned}\tilde{S}_{lumc}(G,H)=1-\frac{1}{3n}\sum_{i=1}^{n}&\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|+\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|\\&+\frac{1}{2}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|\end{aligned} \tag{10}$$

$$\begin{aligned}\tilde{S}_{lumH}(G,H)=1-\frac{1}{3n}\sum_{i=1}^{n}&\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|+\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|\\&+\frac{1}{2}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|\end{aligned} \tag{11}$$

$$\begin{aligned}\tilde{S}_{lumL}(G,H)=1-\frac{1}{3n}\sum_{i=1}^{n}&\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|+\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|\\&\frac{1}{2}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|\end{aligned} \tag{12}$$

$$\begin{aligned}\tilde{S}_{lumO}(G,H)=1-\frac{1}{3}\Bigg(&\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^2}\\&+\sqrt{\frac{1}{4n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^2}\Bigg)\end{aligned} \tag{13}$$

$$\tilde{S}_{lumDC}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \quad (14)$$

$$\tilde{S}_{lumHB}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \quad (15)$$

$$\tilde{S}_{leum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \quad (16)$$

$$\tilde{S}_{lsum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{2^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}\right) \quad (17)$$

$$\tilde{S}_{lhum}^{p}(G,H)=1-\frac{1}{3}\left(\sqrt[p]{\frac{2^p}{3^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{2^p}{3^p n}\sum_{i=1}^{n}\left|\nu_G^2(x_i)-\nu_H^2(x_i)\right|^p}+\sqrt[p]{\frac{1}{3^p n}\sum_{i=1}^{n}\left|\mu_G^2(x_i)-\mu_H^2(x_i)+\nu_H^2(x_i)-\nu_G^2(x_i)\right|^p}\right) \quad (18)$$

To show the reasonability and usefulness of our proposed similarity measures in Equations (10) – (18), we put forward the following examples.

## 4. Demonstration of Results and Application

In the following section, we present some practical examples related to pattern recognition and multicriteria decision making to show reasonability and practicality of our proposed similarity measures in Equations (10) – (18) as follows:

**Example 1**. Let *G* and *H* be two PFSs as expressed in Table 1.

Table 1. Similarities of PFSs calculated using Equations (1) - (9).

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $G$ | (0.7, 0.7) | (0.6 ,0.5) | (0.2 ,0.5) | (0.6 ,0.6) | (0.6, 0.4) | (0.3,0.4) |
| $H$ | (0.8 ,0.4) | (0.8, 0.4) | (0.4 ,0.6) | (0.9 ,0.2) | (0.5 ,0.7) | (0.2,0.2) |
| $\tilde{S}_c$ | 0.7200 | 0.7050 | 0.7950 | 0.6150 | 0.9800 | 0.9650 |
| $\tilde{S}_L$ | 0.6000 | 0.8025 | 0.8475 | 0.4425 | 0.9150 | 0.9325 |
| $\tilde{S}_H$ | 0.8799 | 0.9527 | 0.9807 | 0.7964 | 0.8851 | 0.9843 |
| $\tilde{S}_O$ | 0.6534 | 0.7825 | 0.8611 | 0.5487 | 0.6610 | 0.8749 |
| $\tilde{S}_{DC}$ | 0.7600 | 0.8150 | 0.9950 | 0.6150 | 0.7800 | 0.9650 |
| $\tilde{S}_s^p$ | 0.7200 | 0.8200 | 0.9000 | 0.6200 | 0.7800 | 0.900 |
| $\tilde{S}_{HB}$ | 0.7300 | 0.8200 | 0.9000 | 0.6200 | 0.7800 | 0.9200 |
| $\tilde{S}_s^p$ | 0.7300 | 0.1960 | 0.7200 | 0.6200 | 0.8100 | 0.9700 |
| $\tilde{S}_h^p$ | 0.8800 | 0.9600 | 0.9600 | 0.9700 | 0.9600 | 0.6200 |

Table 1, reflects the calculation of suggested similarity measures using Equations (1) - (9). The numerical results show the reasonability of our proposed methods of calculating similarities between PFSs. Next, we utilized our newly constructed similarity measures in Equations (10) - (18) based on lower upper and middle fuzzy sets to calculate similarity measures between PFSs.

**Examples 2**. Let *G* and *H* be two PFSs as in the Table 2.

Table 2. Similarities of PFSs calculated using Equations (10) - (18).

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $G$ | $(0.7,\ 0.7)$ | $(0.6\ ,0.5)$ | $(0.2\ ,0.5)$ | $(0.6\ ,0.6)$ | $(0.6,\ 0.4)$ | $(0.3,0.4)$ |
| $H$ | $(0.8\ ,0.4)$ | $(0.8,\ 0.4)$ | $(0.4\ ,0.6)$ | $(0.9\ ,0.2)$ | $(0.5\ ,0.7)$ | $(0.2,0.2)$ |
| $\tilde{S}_{lumc}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}_{lumH}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}_{lumL}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}_{lumO}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}_{lumDC}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}_{lumHB}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}^{p}_{leum}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}^{p}_{lsum}$ | 0.73 | 0.82 | 0.92 | 0.62 | 0.78 | 0.92 |
| $\tilde{S}^{p}_{lhum}$ | 0.82 | 0.91 | 0.95 | 0.74 | 0.85 | 0.95 |

From the numerical simulations of Table 2, it is clear that the newly constructed similarity measures in Equations (10) - (18) based on lower, upper and middle fuzzy sets are reasonable and appropriate. In the following subsection, we apply our proposed similarity measures in Equations (10) – (18) to handling problems related to pattern recognition.

## 4.1 Application to pattern recognition

In this subsection, we particularly use our similarity measures in Equations (10) – (18) in an application to pattern recognition. We utilize the law of maximum similarity between two PFSs to measure the similarity between two PFSs.

**Example 3**. Let $G_1$ and $G_2$ be two given patterns in PFSs in finite universe of discourse *X*.

$$G_1 = \{(x,\ 0.70,\ 0.70)\}\ and\ G_2 = \{(x,\ 0.80,\ 0.30)\}.$$

We have a sample *Q* which is expressed by the PFSs

$$Q = \{(x,\ 0.70,\ 0.50)\}$$

The objective is to recognize the sample *Q* with one of the given patterns $G_1$ and $G_2$, using the principle of maximum degree of similarity between two PFSs the procedure of transmission *Q* to $G_m$ is

$$S_{PFS,m} = arg\max_{1\le i\le 2}\left(S_{PFS}\left(G_i,Q\right)\right)$$

Utilizing the proposed similarity measures in Equations (10) – (18) between PFSs, we have the following $G_i,\ i = 1,\ 2\ and\ Q$. The calculation of similarity measures in Equations (10) – (18) are given as follows:

$G_i,\ i = 1,\ 2\ and\ Q$. The calculation of similarity measures in Equations (1) - (18) are given as follows

$$\tilde{S}_C(G_1,Q)=0.88,\ \tilde{S}_C(G_2,Q)=0.896,\ \tilde{S}_{lumC}(G_1,Q)=0.64,\ \tilde{S}_{lumC}(G_2,Q)=0.86,$$
$$\tilde{S}_H(G_1,Q)=0.88,\ \tilde{S}_H(G_2,Q)=0.86,\ \tilde{S}_{lumH}(G_1,Q)=0.64,\ \tilde{S}_{lumH}(G_2,Q)=0.86,$$
$$\tilde{S}_L(G_1,Q)=0.88,\ \tilde{S}_L(G_2,Q)=0.86,\ \tilde{S}_{lumL}(G_1,Q)=0.88,\ \tilde{S}_{lumL}(G_2,Q)=0.86,$$
$$\tilde{S}_0(G_1,Q)=0.65,\ \tilde{S}_O(G_2,Q)=0.86,\ \tilde{S}_{lumO}(G_1,Q)=0.99,\ \tilde{S}_{lumO}(G_2,Q)=0.86,$$
$$\tilde{S}_{Dc}(G_1,Q)=0.88,\ \tilde{S}_{DC}(G_2,Q)=0.86,\ \tilde{S}_{lumDC}(G_1,Q)=0.88,\ \tilde{S}_{lumDC}(G_2,Q)=0.86,$$
$$\tilde{S}_{HB}(G_1,Q)=0.88,\ \tilde{S}_{HB}(G_2,Q)=0.86,\ \tilde{S}_{lumHB}(G_1,Q)=0.88,\ \tilde{S}_{lumHB}(G_2,Q)=0.86,$$
$$\tilde{S}^P_e(G_1,Q)=0.88,\ \tilde{S}^P_e(G_2,Q)=0.86,\ \tilde{S}^p_{lume}(G_1,Q)=0.88,\ \tilde{S}^P_{lume}(G_2,Q)=0.86,$$
$$\tilde{S}^p_s(G_1,Q)=0.79,\ \tilde{S}^P_S(G_2,Q)=0.86,\ \tilde{S}^p_{lumS}(G_1,Q)=0.88,\ \tilde{S}^P_{lumS}(G_2,Q)=0.86,$$
$$\tilde{S}^p_h(G_1,Q)=0.93,\ \tilde{S}^P_h(G_2,Q)=0.99,\ \tilde{S}^p_{lumh}(G_1,Q)=0.95,\ \tilde{S}^P_{lumh}(G_2,Q)=0.92.$$

Intuitively, we expect that $Q$ should belong to the pattern $G_1$. The above numerically computed results shows that the sample $Q$ belongs to the pattern $G_1$ according to the principle of maximum degree of similarity between PFSs. Most of the similarity measures agree expect for a few conflicts with negligible error. This might the result of poor approximation. So we can conclude that the sample $Q$ belongs to the pattern $G_1$ according to the principle of maximum degree of similarity between PFSs.

Next, we propose Pythagorean fuzzy TODIM to apply our proposed similarity measure in an application to daily life matters containing complex multicriteria decision making.

## 5. Pythagorean Fuzzy TODIM

Step 1. The Pythagorean fuzzy decision matrix with respect to alternatives $\tilde{H}_i, i=1,2,3,...,m$ to the criteria $\tilde{C}_j, j=1,2,3,...,n$ is given as follows:

$$\tilde{R}=\left(r_{ij}\right)_{m\times n}=\begin{array}{c|cccccc} \tilde{H}_i/\tilde{C}_j & \tilde{C}_1 & \tilde{C}_2 & . & . & . & \tilde{C}_n \\ \hline \tilde{H}_1 & r_{11} & r_{12} & . & . & . & r_{1j} \\ \tilde{H}_2 & r_{21} & r_{22} & . & . & . & r_{2j} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ \tilde{H}_m & r_{i1} & r_{i2} & . & . & . & r_{ij} \end{array}$$

Step 2. The transform of the decision matrix to the normalized Pythagorean decision matrix as follows:

$$\tilde{L}=\left(l_{ij}\right)_{m\times n}=\begin{cases} r_{ij} & \text{for beneficial attribute} \\ \left(r_{ij}\right)^c & \text{for cost attribute} \end{cases}$$

Step 3. Calculation of the relative weights of criteria by following formula

$$\tilde{w}_{jr}=\tilde{w}_j/\tilde{w}_r,\ 0\le w_{jr}\le 1$$

$$\tilde{w}_r=max\left[\tilde{w}_j : j=1,2,3,...,n\right] and\ 0\le \tilde{w}_{jr}\le 1$$

Step 4. Calculate the dominance degree of each alternative $\tilde{H}_i$ over each alternative $\tilde{H}_t$ with respect to the criterion by $\tilde{C}_j$ using,

$$\tilde{\phi}_j\left(\tilde{H}_i,\tilde{H}_t\right)=\begin{cases} \sqrt{\dfrac{\tilde{w}_{rj}\, d\left(\tilde{I}_{ij},\tilde{I}_{tj}\right)}{\sum_{j=1}^{n}\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}>\tilde{I}_{tj} \\ 0 & if\ \ \tilde{I}_{ij}=\tilde{I}_{tj} \\ -\dfrac{1}{\tilde{\theta}}\sqrt{\dfrac{\sum_{j=1}^{n}\tilde{w}_{rj} d\left(\tilde{I}_{ij},\tilde{I}_{tj}\right)}{\tilde{w}_{jr}}} & if\ \ \tilde{I}_{ij}<\tilde{I}_{tj} \end{cases}$$

Step 5. Calculate the overall dominance degree of $\tilde{H}_i$ over each alternative $\tilde{H}_t$ using

$$\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)=\sum_{j=1}^{n}\tilde{\phi}_j\left(\tilde{H}_i,\tilde{H}_t\right)$$

where $\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)$ denotes the measurement of dominance of alternative $\tilde{H}_i$ over alternative $\tilde{H}_t$

Step 6. Derive the overall value of each alternative $\tilde{H}_i$ by using

$$\varepsilon_i=\frac{\sum_{i=1}^{n}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)-\min_i\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right)}{\max_i\left\{\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right\}-\min_i\left(\sum_{i=1}^{m}\tilde{\delta}\left(\tilde{H}_i,\tilde{H}_t\right)\right)}$$

Clearly, $0\le\varepsilon_i\le 1$, and we select the greater value of $\varepsilon_i$ and consider this as better alternative $\tilde{H}_i$. Thus, one can choose the appropriate alternative, in accordance with a descending order of the overall values of all the alternatives.

Step 7. Find the best alternatives according to the given values.

**Example 4**. (Application in vehicle selection)
Consider a customer who wants to purchase a new car. Assume four types of cars as alternative candidates $A_j$ (j= 1, 2, 3, 4) available in the market. The customer considers the following four criteria before buying a new car:

$C_1$: *Economical to operate*, $C_2$ : *Price*, $C_3$ : *Reliable*, $C_4$ : *Capacious.*

We noticed that $C_2$ is a cost attribute while the other three are benefit attributes. The evaluated values of alternatives $A_i$ over criteria are given in the decision making Table 3.

Table 3. Pythagorean fuzzy decision matrix

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $(0.60, 0.70)$ | $(0.70, 0.30)$ | $(0.50, 0.40)$ | $(0.60, 0.60)$ |
| $A_2$ | $(0.50, 0.50)$ | $(0.60, 0.50)$ | $(0.40, 0.30)$ | $(0.40, 0.70)$ |
| $A_3$ | $(0.80, 0.20)$ | $(0.80, 0.30)$ | $(0.40, 0.60)$ | $(0.80, 0.40)$ |
| $A_4$ | $(0.70, 0.50)$ | $(0.70, 0.70)$ | $(0.70, 0.20)$ | $(0.50, 0.60)$ |

0

As $D_2$ is a cost attribute we have to find the complement of $C_2$ for normalizing the decision matrix, and it is given in Table 4.

Table 4. The normalized decision matrix

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $(0.60, 0.70)$ | $(0.30, 0.70)$ | $(0.50, 0.40)$ | $(0.60, 0.60)$ |
| $A_2$ | $(0.50, 0.50)$ | $(0.50, 0.60)$ | $(0.40, 0.30)$ | $(0.40, 0.70)$ |
| $A_3$ | $(0.80, 0.20)$ | $(0.30, 0.80)$ | $(0.40, 0.60)$ | $(0.80, 0.40)$ |
| $A_4$ | $(0.70, 0.50)$ | $(0.70, 0.70)$ | $(0.70, 0.20)$ | $(0.50, 0.60)$ |

Assume that the weights are known, so we denote these four criteria as $C_j$ where $j$=1, 2, 3, 4 and their respective weight vector is represented by $w$ = (0.40, 0.30, 020, 0.10).

Step 3. Since $w_1$ is maximum over all other weights, with $C_1$ the reference criterion and the weight $w_r$ = 0.4, the relative weights of all the given criteria $C_j$ (j=1, 2, 3, 4) are

$$w_r = max\ \{w_j,\ j = 1,\ 2,\ 3,\ 4\}$$

$$w_{1r} = 0.4/0.4 = 1, \quad w_{2r} = 0.2/0.4 = 0.5, \quad w_{3r} = 0.1/0.4 = 0.25, \quad w_{4r} 0.3/0.4 = 0.75$$

$$\sum_{i=1}^{n} w_{jr} = 1 + 0.75 + 0.5 + 0.25 = 2.5 = \theta$$

Step 4. Calculation of the degree of dominance by using

$$d(G, H) = \frac{1}{n}\sum_{i=1}^{n}\left|\mu_G^2(x_i) - \mu_H^2(x_i)| \nu | \nu_G^2(x_i) - \nu_H^2(x_i)\right|$$

$$\phi_1(A_x, A_t)\ (\mathrm{x, y} = 1, 2, 3, 4)$$

The calculated values are given in Tables 5, 6, 7, and 8 respectively.
The overall dominance degree of $A_i$ is summarized in Table 9.
The overall value of dominance degree of $A_i$ over each alternative $A_t$ can be found as

$$\varepsilon_i = \frac{\sum_{i=1}^{n}\delta(A_x, A_t) - min_i\left(\sum_{i=1}^{m}\delta(A_x, A_t)\right)}{max_i\left\{\sum_{i=1}^{m}\delta(A_x, A_t)\right\} - min_i\sum_{i=1}^{m}\delta(A_x, A_t))}$$

$$\varepsilon_1 = \frac{3.1506 - (-6.0942)}{7.528 - (-6.0942)} = 0.6787, \ \varepsilon_2 = \frac{-6.0942 - (-6.0942)}{7.528 - (-6.0942)} = 0$$

$$\varepsilon_3 = \frac{7.528 - (-6.0942)}{7.528 - (-6.0942)} = 1, \ \varepsilon_4 = \frac{5.9042 - (-6.0942)}{7.528 - (-6.0942)} = 0.88.$$

The final ranking is given in Table 10. From Table 10, we conclude that the final descending rank order is

$$A_3 \succ A_4 \succ A_1 \succ A_2$$

From the above ranking of alternatives, it is clear that the alternative $A_3$ is considered the best among all available four alternatives. The ranking of alternatives $A_i$ is in descending order based on the overall value $\varepsilon_i$ of each alternative $\tilde{H}_i$. The alternative $A_i$ having the highest overall value is selected as the best alternative. Hence, the alternative $A_3$ is considered the best alternative.

Table 5. The matrix for criteria $C_1$

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 0.0000 | 0.6000 | –0.4500 | –0.3098 |
| $A_2$ | –0.3098 | 0.0000 | –0.3900 | –0.3200 |
| $A_3$ | 1.0600 | 0.9800 | 0.0000 | 0.7200 |
| $A_4$ | 0.7700 | 0.7700 | 0.2890 | 0.0000 |

Table 6. The matrix for criteria $C_2$

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 0.0000 | 0.8940 | –0.3460 | –0.5656 |
| $A_2$ | –0.3577 | 0.0000 | –0.4730 | –1.3856 |
| $A_3$ | 0.7500 | 1.183 | 0.0000 | 1.4140 |
| $A_4$ | –0.6066 | 1.0900 | –1.4140 | 0.0000 |

Table 7. The matrix for criteria $C_3$

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 0.0000 | 0.9480 | 1.4140 | –0.1690 |
| $A_2$ | –0.3794 | 0.0000 | –0.6570 | –0.7260 |
| $A_3$ | –0.5656 | –0.6570 | 0.0000 | –0.7260 |
| $A_4$ | 1.4140 | 1.8160 | 1.8160 | 0.0000 |

Table 8. The matrix for criteria $C_4$

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 0.0000 | 0.8660 | –0.3860 | 0.6550 |
| $A_2$ | –32650 | 0.0000 | –0.5059 | –0.2633 |
| $A_3$ | 0.9660 | 1.2640 | 0.0000 | 1.1400 |
| $A_4$ | –0.2422 | 0.6580 | –0.4560 | 0.0000 |

Table 9. The overall dominance degree

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $\sum_{i=1}^{n}\phi_y(A_x, A_t)$ |
|---|---|---|---|---|---|
| $A_1$ | 0.0000 | 3.3080 | 0.2320 | –0.3894 | 3.1506 |
| $A_2$ | –1.3734 | 0.0000 | –2.0259 | –2.6949 | –6.0942 |
| $A_3$ | 2.2104 | 2.7700 | 0.0000 | 2.5480 | 7.5280 |
| $A_4$ | 1.3352 | 4.3340 | 0.235 | 0.0000 | 5.9042 |

## 6. Conclusions

Various information measures are suggested in the literature but still there is space to improve or modify them, and to create new ones. Adopting similarity measures between PFSs based on lower, middle and upper fuzzy sets, we provide a novel way of constructing similarity measures between PFSs. The new type of similarity measures between PFSs is based on converting the PFSs into their lower, upper and middle fuzzy sets (FSs) to calculate the degree of similarity very effectively and usefully. On the basis of numerical analysis results, we found that our novel construction provides very useful and reasonable results. Applications of our proposed methods related to pattern recognition and multicriteria decision making show the usefulness and practical applicability of the proposed methods. Finally, Pythagorean fuzzy TODIM method based on our proposed similarity measures was constructed to handle complex problems related to daily life. Holistically, our suggested measures between PFSs can provide a more reasonable and effective way of calculating degree of similarity between PFSs.

## 7. Future Direction

In the future, we will consider clustering objects in uncertain and ambiguous environments using our proposed new construction method for similarity measures between PFSs.

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