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Original Article

Wave effects of the space–time fractional order (2+1)-dimensional breaking soliton equation via three consistent methods

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Abstract

The main goal of the present paper is to find exact traveling wave solutions of the space-time fractional order (2+1)-dimensional breaking soliton equation using the simple equation (SE) method with Bernoulli equation, the simple equation (SE) method with Riccati equation, and the Riccati sub-equation method. Using these techniques yields nineteen different solutions, which are in the form of exponential, trigonometric, and hyperbolic functions. This study investigated how varying the time affected the deeds of the solutions obtained for the given conditions. The predicted solutions, obtained under restricted conditions, were visualized through 2D, 3D, and contour plots using appropriate parameter values.

Keywords: SE with Bernoulli equation, SE with Riccati equation**,** Riccati sub-equation method**,** fractional partial differential equations, fractional order (2+1)-dimensional breaking soliton equation

1. Introduction

Fractional differential equations (FDEs) arise in numerous problems in many scientific fields, including control theory, engineering, biology, physics, mathematics, and chemistry. Therefore, exact solution methods for FDEs have become more important. Many researchers have used diverse methods to get exact solutions, such as the SE method (Chankaew, Phoosree, & Sanjun, 2023; Sanjun & Chankaew, 2022), the Kudryashov method (Thadee, Chankaew, & Phoosree, 2022), the modified Kudryshov method (Srivastava *et al*., 2020), the extended tanh-function method (Sadiya, Inc, Arefin, & Uddin 2022), the Tanh-coth method (Behera, Mohanty, & Virdi, 2023), the -expansion method (Sirisubtawee, Koonprasert, & Sungnul, 2019), the -expansion method (Djilali & Alı, 2023), the -expansion method (Behera, Aljahdaly, & Virdi, 2022), the Riccati sub-equation method (Phoosree, Khongnual, Sanjun, Kammanee, & Thadee, 2024; Thadee, Phookwanthong, Jitphusa, & Phoosree, 2023), the functional variable method (Rezazadeh, Vahidi, Zafar, & Bekir, 2020), the RKHS method (Abu Arqub & Rashaideh, 2018), and so on. There exists a great number of different

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definitions of fractional derivatives that may be found in the published research, such as Caputo (Abu Arqub, 2019), Atangana–Baleanu–Caputo fractional derivative (Momani, Abu Arqub, & Maayah, 2020), Riemann–Liouville (Seddek, Ebaid, El-Zahar, & Aljoufi, 2023), Beta-fractional derivative (Wang, 2023), and Jumarie's modified Riemann–Liouville (Zeng, Wang, Xiao, & Wang, 2023), etc.

The main aim of this study is to analyze the space–time fractional order (2+1)-dimensional breaking soliton equation (Ali *et al*., 2022),

$$D_t^\alpha\left(D_x^\alpha u\right)-4D_x^\alpha u D_x^\alpha\left(D_y^\alpha u\right)-2D_x^{2\alpha}u\left(D_y^\alpha u\right)+D_x^{3\alpha}u\left(D_y^\alpha u\right)=0,\quad 0<\alpha\le 1. \tag{1.1}$$

where $D_t^\alpha u$ denotes Jumarie's modified Riemann–Liouville derivatives of u, where $u = u(x, y, t)$.

Jumarie's modified Riemann-Liouville derivative and the properties of the modified Riemann-Liouville derivative (Sahoo & Ray, 2015) of order α are defined by the expression

$$f^{(\alpha)}(x)=\lim_{h\to 0}\left(\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f\left(x+(\alpha-k)h\right)}{h^\alpha}\right),\quad 0<\alpha\le 1, \tag{1.2}$$

which can be written as

$$D_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x(x-\xi)^{(-\alpha-1)}[f(\xi)-f(0)]d\xi & \text{if } \alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x(x-\xi)^{(-\alpha)}[f(\xi)-f(0)]d\xi & \text{if } 0<\alpha<1,\\ [f^{(\alpha-n)}(x)]^{(n)} & \text{if } n\le\alpha<n+1, n\ge 1,\end{cases} \tag{1.3}$$

and some properties of the modified Riemann-Liouville derivative are as follows:

$$\begin{aligned}&D_x^\alpha x^\gamma=\frac{\Gamma(\gamma+1)x^{\gamma-\alpha}}{\Gamma(\gamma+1-\alpha)},\quad \gamma>0,\quad x>0,\\&D_x^\alpha(f(x)g(x))=g(x)D_x^\alpha f(x)+f(x)D_x^\alpha g(x),\\&D_x^\alpha f(g(x))=f_g'[g(x)]D_x^\alpha g(x)=D_g^\alpha f[g(x)](g'(x))^\alpha.\end{aligned} \tag{1.4}$$

In this paper, we have solved the space-time fractional order (2+1)-dimensional breaking soliton equation by applying Jumarie's modified Riemann-Liouville derivative. We are also using three different approaches: the SE with Bernoulli equation, the SE with Riccati equation, and the Riccati sub-equation method. We have displayed the analytical solutions and the wave effects on a 2D graph, a 3D graph, and a contour graph.

2. General form of the Methods

In this part, we discuss the three methods for solving fractional PDEs. The general form of a fractional PDEs is

$$G\left(u,D_x^\alpha u,D_y^\alpha u,D_t^\alpha u,D_x^{2\alpha}u,D_y^\alpha D_x^\alpha u,D_t^\alpha D_x^\alpha u,\ldots\right)=0,\quad 0<\alpha\le 1, \tag{2.1}$$

where $u(x, y, t)$ is an unknown function and G is a polynomial of $u(x, y, t)$ and its derivatives. Start by considering combining the independent variables x, y, and t into one variable $\xi=\frac{kx^\alpha}{\Gamma(\alpha+1)}+\frac{ly^\alpha}{\Gamma(\alpha+1)}-\frac{\omega t^\alpha}{\Gamma(\alpha+1)}$.

We suppose the traveling wave solution of fractional PDEs is a solution that satisfies

$$u(x,y,t)=u(\xi),\ \xi=\frac{kx^\alpha}{\Gamma(\alpha+1)}+\frac{ly^\alpha}{\Gamma(\alpha+1)}-\frac{\omega t^\alpha}{\Gamma(\alpha+1)}, \tag{2.2}$$

where constants k and l are non-zero constants, and ω is the speed of the traveling wave. We call this a stationary wave when $\omega = 0$. For $\omega > 0$, the wave moves in the positive direction, and for $\omega < 0$, the wave moves in the negative direction (Phoosree, 2019). The traveling wave transformation Equation (2.2) permits us to reduce Equation (2.1) to the following ordinary differential equation (ODE):

$$Q\left(u,\frac{du}{d\xi},\frac{d^2u}{d\xi^2},\frac{d^3u}{d\xi^3},\ldots\right)=0, \tag{2.3}$$

where Q is a polynomial in $u(\xi)$ and its total derivatives $u'(\xi)=\frac{du}{d\xi}$, $u''(\xi)=\frac{d^2u}{d\xi^2}$, and so on.

2.1 Algorithm of the SE method with Bernoulli equation

We outline the fundamental steps of the SE method with Bernoulli equation (Chankaew *et al*., 2023; Phoosree & Thadee, 2022) as follows:

Step 1. Start by considering Equations (2.1) - (2.3).

Step 2. Suppose that the solution of Equation (2.3) is in the following form:

$$u(\xi)=\sum_{i=0}^{N}a_iZ^i(\xi). \tag{2.4}$$

in which a_i (i = 0, 1, 2, … , N) are constants that need to be determined such that $a_N \neq 0$ and $Z(\xi)$ conforms to the following Bernoulli equation:

$$Z'(\xi)=\beta Z(\xi)+\eta Z^2(\xi), \tag{2.5}$$

where β and η are non-zero constants. The two cases of solutions to Equation (2.5) are presented here.

Case 1: $\beta > 0$, $\eta < 0$,

$$Z(\xi)=\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}. \tag{2.6}$$

Case 2: $\beta < 0$, $\eta > 0$,

$$Z(\xi)=-\frac{\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}. \tag{2.7}$$

Step 3. The balance number N may be achieved by striking a balance between the derivative of the highest-order and the highest nonlinear terms that exist in Equation (2.4).

Step 4. For the terms that were all in the same power of Z, we added up all of the coefficients and set them to zero. We obtained β, η, ω, and a_i. As a result, the traveling wave solutions to Equation (2.1) are given.

2.2 Algorithm of the SE method with Riccati equation

The following five processes can be used in the simple equation method with Riccati equation (Nofal, 2016).

Step 1. Start by considering Equations (2.1) - (2.3).

Step 2. Suppose that the solution of Equation (2.3) is in the following form, Equation (2.4), and $Z(\xi)$ conforms to the following Riccati equation:

$$Z'(\xi) = cZ^2(\xi) + d, \tag{2.8}$$

where c and d are non-zero constants. The two cases of solutions to Equation (2.8) are presented here.

Case 1: $cd < 0$,

$$Z(\xi) = -\frac{\sqrt{cd}}{c}\tanh\left(\sqrt{-cd}\,\xi - \frac{\gamma\ln(\xi_0)}{2}\right),\ \xi_0 > 0,\ \ \gamma = \pm 1. \tag{2.9}$$

Case 2: $cd > 0$,

$$Z(\xi) = \frac{\sqrt{cd}}{c}\tan\left(\sqrt{cd}\,(\xi + \xi_0)\right), \tag{2.10}$$

where ξ_0 is a constant.

Step 3. The balance number N may be achieved by striking a balance between the derivative of the highest-order and the highest nonlinear terms that exist in Equation (2.4).

Step 4. For the terms that were all in the same power of Z, we added up all of the coefficients and set them to zero. We obtained c, d, ω, and a_i. As a result, the traveling wave solutions to Equation (2.1) are given.

2.3 Algorithm of the Riccati sub-equation method

This part presents the Riccati sub-equation method, which is a simple technique for finding traveling wave solutions. There are five processes to follow (Thadee *et al.*, 2023).

Step 1. Start by considering Equations (2.1) - (2.3).

Step 2. Suppose that the solution of Equation (2.3) is in the following form, Equation (2.4).

Step 3. The Riccati sub-equation method (Khoda dad, Nazari, Eslami, & Rezazadeh, 2017) is used to find Z as shown below:

$$\mathrm{Z}'(\xi) = \sigma + \mathrm{Z}^2(\xi), \tag{2.11}$$

where σ is an arbitrary constant. Here, the prime denotes the derivative with respect to ξ. By using the general solutions of Equation (2.11), we obtain the following expressions:

Case 1. When $\sigma < 0$,

$$Z_1(\xi) = -\sqrt{-\sigma}\tanh_{pq}\left(\sqrt{-\sigma}\xi\right), \tag{2.12}$$

$$Z_2(\xi) = -\sqrt{-\sigma}\coth_{pq}\left(\sqrt{-\sigma}\xi\right), \tag{2.13}$$

$$Z_3(\xi) = -\sqrt{-\sigma}\tanh_{pq}\left(2\sqrt{-\sigma}\xi\right) \pm i\sqrt{-\sigma}\,\mathrm{sech}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{2.14}$$

$$Z_4(\xi) = -\sqrt{-\sigma}\coth_{pq}\left(2\sqrt{-\sigma}\xi\right) \pm \sqrt{-\sigma}\,\mathrm{csch}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{2.15}$$

$$Z_5(\xi) = -\frac{1}{2}\left(\sqrt{-\sigma}\tanh_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right) + \sqrt{-\sigma}\coth_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right)\right), \tag{2.16}$$

$$Z_6(\xi) = \frac{\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{A\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right) + B}, \tag{2.17}$$

$$Z_7(\xi) = -\frac{\sqrt{-(B^2-A^2)\sigma} - A\sqrt{-\sigma}\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{A\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right) + B}, \tag{2.18}$$

Where A, B are two nonzero real constants and satisfy $B^2 - A^2 > 0$.

Case 2. When $\sigma > 0$,

$$Z_8(\xi) = \sqrt{\sigma}\tan_{pq}\left(\sqrt{\sigma}\xi\right), \tag{2.19}$$

$$Z_9(\xi) = -\sqrt{\sigma}\cot_{pq}\left(\sqrt{\sigma}\xi\right), \tag{2.20}$$

$$Z_{10}(\xi) = -\sqrt{\sigma}\tan_{pq}\left(2\sqrt{\sigma}\xi\right) \pm \sqrt{\sigma}\sec_{pq}\left(2\sqrt{\sigma}\right), \tag{2.21}$$

$$Z_{11}(\xi) = -\sqrt{\sigma}\cot_{pq}\left(2\sqrt{\sigma}\xi\right) \pm \sqrt{\sigma}\csc_{pq}\left(2\sqrt{\sigma}\xi\right), \tag{2.22}$$

$$Z_{12}(\xi) = \frac{1}{2}\left(\sqrt{\sigma}\tan_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right) - \sqrt{\sigma}\cot_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right)\right), \tag{2.23}$$

$$Z_{13}(\xi) = \frac{\pm\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}\cos_{pq}\left(2\sqrt{\sigma}\xi\right)}{A\sin_{pq}\left(2\sqrt{\sigma}\xi\right) + B}, \tag{2.24}$$

$$Z_{14}(\xi) = -\frac{\pm\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}\sin_{pq}\left(2\sqrt{\sigma}\xi\right)}{A\cos_{pq}\left(2\sqrt{\sigma}\xi\right) + B}, \tag{2.25}$$

where A, B are two nonzero real constants and satisfy $A^2 - B^2 > 0$.

Case 2. When $\sigma = 0$,

$$Z_{15}(\xi) = -\frac{1}{\xi + g}, \tag{2.26}$$

where g is a constant.

The different types of generalized hyperbolic functions are defined as follows (Thadee *et al.*, 2023), with p and q as arbitrary constants, $p > 0$, $q > 0$,

$$\sinh_{pq}(\xi)=\frac{pe^{\xi}-qe^{-\xi}}{2}, \tag{2.27}$$

$$\cosh_{pq}(\xi)=\frac{pe^{\xi}+qe^{-\xi}}{2}, \tag{2.28}$$

$$\tanh_{pq}(\xi)=\frac{pe^{\xi}-qe^{-\xi}}{pe^{\xi}+qe^{-\xi}}, \tag{2.29}$$

$$\coth_{pq}(\xi)=\frac{pe^{\xi}+qe^{-\xi}}{pe^{\xi}-qe^{-\xi}}, \tag{2.30}$$

$$\operatorname{sech}_{pq}(\xi)=\frac{2}{pe^{\xi}+qe^{-\xi}}, \tag{2.31}$$

$$\operatorname{csch}_{pq}(\xi)=\frac{2}{pe^{\xi}-qe^{-\xi}}, \tag{2.32}$$

where ζ is an independent variable.

The different types of generalized triangular functions are defined as follows (Thadee *et al.*, 2023), with p and q as arbitrary constants, $p > 0$, $q > 0$,

$$\sin_{pq}(\xi)=\frac{pe^{i\xi}-qe^{-i\xi}}{2i}, \tag{2.33}$$

$$\cos_{pq}(\xi)=\frac{pe^{i\xi}+qe^{-i\xi}}{2}, \tag{2.34}$$

$$\tan_{pq}(\xi)=-i\frac{pe^{i\xi}-qe^{-i\xi}}{pe^{i\xi}+qe^{-i\xi}}, \tag{2.35}$$

$$\cot_{pq}(\xi)=i\frac{pe^{i\xi}+qe^{-i\xi}}{pe^{i\xi}-qe^{-i\xi}}, \tag{2.36}$$

$$\sec_{pq}(\xi)=\frac{2}{pe^{i\xi}+qe^{-i\xi}}, \tag{2.37}$$

$$\csc_{pq}(\xi)=\frac{2i}{pe^{i\xi}-qe^{-i\xi}}, \tag{2.38}$$

where ζ is an independent variable.

Step 4. The balance number N may be achieved by striking a balance between the derivative of the highest-order and the highest nonlinear terms that exist in Equation (2.11).

Step 5. Substituting Equations (2.4) and (2.11) into Equation (2.3), the coefficients of all terms of the same order Z^i ($i = 0, 1, 2,\ldots$) are gathered, and the coefficients are then set to zero. We get an overdetermined system of algebraic equations with respect to a_i ($i = 0, 1, 2,\ldots, N$). When all the parameters in Equation (2.4) are substituted, the solutions to Equation (2.1) for the traveling wave are reached.

3. Mathematical analyses of the equation and their solutions

Next, we wish to apply the preceding methods in Sections 2.1–2.3 to solve both the space-time fractional order (2+1)-dimensional breaking soliton equation. We will reduce it to an ODE using $u(x, y, t) = u(\xi)$ and the traveling wave variable $\xi=\frac{kx^{\alpha}}{\Gamma(\alpha+1)}+\frac{ly^{\alpha}}{\Gamma(\alpha+1)}-\frac{\omega t^{\alpha}}{\Gamma(\alpha+1)}$. The substitution of the transformation into equation (1.1) yields

$$-k\omega u''-4k^2lu'u''-2k^2lu'u''+k^3lu^{(4)}=0. \tag{3.1}$$

Integrating Equation (3.1) with the zero constant, we get:

$$-k\omega u'-3k^2l\left(u'\right)^2+k^3lu'''=0. \tag{3.2}$$

The subsequent sections employ the planned methods to obtain the desired solutions.

3.1 Solutions with the SE method with Bernoulli equation

Next, we balanced the highest-order derivative terms u''' with the highest-power nonlinear terms $(u')^2$ of Equation (3.2). Then $N = 1$. We have the solution to Equation (3.2) as follows:

$$u(\xi)=a_0+a_1Z(\xi), \tag{3.3}$$

where z satisfies Equation (2.5). Therefore, the expressions for u', $(u')^2$, and u''' are expressed as:

$$u'=a_1\beta Z+a_1\eta Z^2, \tag{3.4}$$

$$\left(u'\right)^2=a_1{}^2\beta^2Z^2+2a_1{}^2\beta\eta Z^3+a_1{}^2\eta^2Z^4, \tag{3.5}$$

$$u'''=a_1\beta^3Z+7a_1\beta^2\eta Z^2+12a_1\beta\eta^2Z^3+6a_1\eta^3Z^4. \tag{3.6}$$

Substituting Equations (3.4)-(3.6) into Equation (3.2), the outcome is

$$\begin{aligned}&-k\omega\left(a_1\beta Z+a_1\eta Z^2\right)-3k^2l\left(a_1{}^2\beta^2Z^2+2a_1{}^2\beta\eta Z^3+a_1{}^2\eta^2Z^4\right)+\\&k^3l\left(a_1\beta^3Z+7a_1\beta^2\eta Z^2+12a_1\beta\eta^2Z^3+6a_1\eta^3Z^4\right)=0.\end{aligned} \tag{3.7}$$

Then we set each coefficient of Z^i to zero, where $i = 1, 2, 3, 4$ yields

$$Z^1;\ -k\omega\beta a_1+k^3l\beta^3a_1=0, \tag{3.8}$$

$$Z^2;\ -k\omega\eta a_1-3k^2l\beta^2a_1^2+7k^3l\beta^2\eta a_1=0, \tag{3.9}$$

$$Z^3;\ -6k^2l\beta\eta a_1^2+12k^3l\beta\eta^2a_1=0, \tag{3.10}$$

$$Z^4;\ -3k^2l\eta^2a_1^2+6k^3l\eta^3a_1=0. \tag{3.11}$$

Solving this system of algebraic equations, we obtain

$$a_1=2k\eta \text{ and } \omega=k^2l\beta^2. \tag{3.12}$$

By Equations (2.6), (2.7), (3.12), and

$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$ the exact solutions to the space-time fractional order (2+1)-dimensional breaking soliton equation are shown for two cases with an arbitrary constant ζ_0.

Case 1: $\beta > 0$, $\eta < 0$,

$$u_1(x,y,t) = a_0 + 2k\eta\left(\frac{\beta e^{\beta(\xi+\xi_0)}}{1-\eta e^{\beta(\xi+\xi_0)}}\right). \tag{3.13}$$

Where

$$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$$

and ζ_0 is a constant of the integration.

Case 2: $\beta < 0$, $\eta > 0$,

$$u_2(x,y,t) = a_0 + 2k\eta\left(\frac{-\beta e^{\beta(\xi+\xi_0)}}{1+\eta e^{\beta(\xi+\xi_0)}}\right). \tag{3.14}$$

Where

$$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$$

and ζ_0 is a constant of the integration.

3.2 Solutions with the SE method with Riccati equation

From $N = 1$. We have the solution to Equation (3.2), which is Equation (3.3). Here Z satisfies Equation (2.8), therefore, the expressions for u', $(u')^2$, and u''' are:

$$u' = ca_1Z^2 + da_1, \tag{3.15}$$

$$(u')^2 = c^2a_1^2Z^4 + 2cda_1^2Z^2 + d^2a_1^2, \tag{3.16}$$

$$u''' = 6c^3a_1Z^4 + 8c^2da_1Z^2 + 2cd^2a_1. \tag{3.17}$$

Substituting Equations (3.15)-(3.17) into Equation (3.2), the outcome is

$$\begin{aligned}&\left(-k\omega da_1 - 3k^2ld^2a_1^2 + 2k^3lcd^2a_1\right)\\&+\left(-k\omega ca_1 - 6k^2lcda_1^2 + 8k^3lc^2da_1\right)Z^2\\&+\left(-3k^2lc^2a_1^2 + 6k^3lc^3a_1\right)Z^4 = 0.\end{aligned} \tag{3.18}$$

Then we set each coefficient of Z^i to zero, where i = 0, 2, 4 yields

$$Z^0;\ -k\omega da_1 - 3k^2ld^2a_1^2 + 2k^3lcd^2a_1 = 0, \tag{3.19}$$

$$Z^2;\ -k\omega ca_1 - 6k^2lcda_1^2 + 8k^3lc^2da_1 = 0, \tag{3.20}$$

$$Z^4:\ -3k^2lc^2a_1^2 + 6k^3lc^3a_1 = 0. \tag{3.21}$$

Solving this system of algebraic equations, we obtain

$$a_1 = 2kc \text{ and } \omega = -4k^2lcd. \tag{3.22}$$

By Equations (2.9), (2.10), (3.22), and $\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$, the exact solutions of the space-time fractional order (2+1)-dimensional breaking soliton equation are shown for two cases with an arbitrary constant ζ_0.

Case 1: $cd < 0$,

$$u_3(x,y,t) = a_0 - 2k\sqrt{-cd}\,\tanh\left(\sqrt{-cd}\,\xi - \frac{\gamma\ln(\xi_0)}{2}\right);\quad \gamma = \pm 1. \tag{3.23}$$

where

$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$ and $\zeta_0 > 0$.

Case 2: $cd > 0$,

$$u_4(x,y,t) = a_0 + 2k\sqrt{cd}\,\tan\left(\sqrt{cd}\,(\xi+\xi_0)\right). \tag{3.24}$$

where

$$\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$$

and ζ_0 is a constant of the integration.

3.3 Solutions with the Riccati sub-ODE method

Using the balance between the derivative of the highest-order and the highest nonlinear terms in Equation (3.2), we obtain $N = 1$. For $N = 1$, Equation (2.4) has the form:

$$u(\xi) = a_0 + a_1Z, \tag{3.25}$$

where Z satisfies Equation (2.11). Therefore, the expressions for u', $(u')^2$, and u''' are expressed as:

$$u' = a_1\sigma + a_1Z^2, \tag{3.26}$$

$$(u')^2 = a_1^2\sigma^2 + 2a_1^2\sigma Z^2 + a_1^2Z^4, \tag{3.27}$$

$$u''' = 2a_1\sigma^2 + 8a_1\sigma Z^2 + 6a_1Z^4. \tag{3.28}$$

Substituting Equations (3.26)-(3.28) into Equation (3.2), the outcome is

$$\begin{aligned}&\left(-k\omega\sigma a_1 - 3k^2l\sigma^2a_1^2 + 2k^3l\sigma^2a_1\right)\\&+\left(-k\omega a_1 - 6k^2l\sigma a_1^2 + 8k^3l\sigma a_1\right)Z^2\\&+\left(-3k^2la_1^2 + 6k^3la_1\right)Z^4 = 0.\end{aligned} \tag{3.29}$$

Then we set each coefficient of Z^i to zero, where i = 0, 2, 4 yields

$$Z^0;\ -k\omega\sigma a_1 - 3k^2 l\sigma^2 a_1^2 + 2k^3 l\sigma^2 a_1 = 0, \tag{3.30}$$

$$Z^2;\ -k\omega a_1 - 6k^2 l\sigma a_1^2 + 8k^3 l\sigma a_1 = 0, \tag{3.31}$$

$$Z^4;\ -3k^2 la_1^2 + 6k^3 la_1 = 0. \tag{3.32}$$

Solving this system of algebraic equations, we obtain

$$a_1 = 2k \text{ and } \omega = -4k^2 l\sigma. \tag{3.33}$$

By Equations (2.12)-(2.26), (3.33), and $\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$, the exact solutions of the space-time fractional order (2+1)-dimensional breaking soliton equation are shown for three cases.

Case 1: $\sigma < 0$,

$$u_5(\xi) = a_0 - 2k\sqrt{-\sigma}\tanh_{pq}\sqrt{-\sigma}\xi, \tag{3.34}$$

$$u_6(\xi) = a_0 - 2k\sqrt{-\sigma}\coth_{pq}\sqrt{-\sigma}\xi, \tag{3.35}$$

$$u_7(\xi) = a_0 - 2k\sqrt{-\sigma}\tanh_{pq}\left(2\sqrt{-\sigma}\xi\right) \pm 2ki\sqrt{-\sigma}\,\mathrm{sech}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{3.36}$$

$$u_8(\xi) = a_0 - 2k\sqrt{-\sigma}\coth_{pq}\left(2\sqrt{-\sigma}\xi\right) \pm 2k\sqrt{-\sigma}\,\mathrm{csch}_{pq}\left(2\sqrt{-\sigma}\xi\right), \tag{3.37}$$

$$u_9(\xi) = a_0 - k\sqrt{-\sigma}\tanh_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right) - k\sqrt{-\sigma}\coth_{pq}\left(\frac{\sqrt{-\sigma}}{2}\xi\right), \tag{3.38}$$

$$u_{10}(\xi) = a_0 + 2k\left(\frac{\sqrt{-\left(A^2+B^2\right)\sigma} - A\sqrt{-\sigma}\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{A\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right)+B}\right), \tag{3.39}$$

$$u_{11}(\xi) = a_0 - 2k\left(\frac{\sqrt{-\left(B^2-A^2\right)\sigma} + A\sqrt{-\sigma}\sinh_{pq}\left(2\sqrt{-\sigma}\xi\right)}{A\cosh_{pq}\left(2\sqrt{-\sigma}\xi\right)+B}\right). \tag{3.40}$$

Where $\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$ and $B^2 - A^2 > 0$.

Case 2: $\sigma > 0$,

$$u_{12}(\xi) = a_0 + 2k\sqrt{\sigma}\tan_{pq}\left(\sqrt{\sigma}\xi\right), \tag{3.41}$$

$$u_{13}(\xi) = a_0 - 2k\sqrt{\sigma}\cot_{pq}\left(\sqrt{\sigma}\xi\right), \tag{3.42}$$

$$u_{14}(\xi) = a_0 - 2k\sqrt{\sigma}\tan_{pq}\left(2\sqrt{\sigma}\xi\right) \pm 2k\sqrt{\sigma}\sec_{pq}\left(2\sqrt{\sigma}\xi\right), \tag{3.43}$$

$$u_{15}(\xi) = a_0 - 2k\sqrt{\sigma}\cot_{pq}\left(2\sqrt{\sigma}\xi\right) \pm 2k\sqrt{\sigma}\csc_{pq}\left(2\sqrt{\sigma}\xi\right), \tag{3.44}$$

$$u_{16}(\xi) = a_0 + k\left(\sqrt{\sigma}\tan_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right) - \sqrt{\sigma}\cot_{pq}\left(\frac{\sqrt{\sigma}}{2}\xi\right)\right), \tag{3.45}$$

$$u_{17}(\xi) = a_0 + 2k\left(\frac{\pm\sqrt{\left(A^2-B^2\right)\sigma} - A\sqrt{\sigma}\cos_{pq}\left(2\sqrt{\sigma}\xi\right)}{A\sin_{pq}\left(2\sqrt{\sigma}\xi\right)+B}\right), \tag{3.46}$$

$$u_{18}(\xi) = a_0 - 2k\left(\frac{\pm\sqrt{\left(A^2-B^2\right)\sigma} - A\sqrt{\sigma}\sin_{pq}\left(2\sqrt{\sigma}\xi\right)}{A\cos_{pq}\left(2\sqrt{\sigma}\xi\right)+B}\right). \tag{3.47}$$

Where $\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$ and $A^2 - B^2 > 0$.

Case 3: $\sigma = 0$,

$$u_{19}(\xi) = a_0 - \left(\frac{2k}{\xi+g}\right). \tag{3.48}$$

Where $\xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ly^\alpha}{\Gamma(\alpha+1)}$.

4. Discussion and Results

The analytical solutions of the space-time fractional order (2+1)-dimensional breaking soliton equation of the three methods are in the form of exponentials, trigonometric functions, and hyperbolic functions. We substitute the parameters shown in Table 1, Table 2, and Table 3. The graph effects of Equations (3.13) and (3.14) by the SE method with Bernoulli, which portrays the wave behaviors as kink waves, are presented in Table 1. Table 2 shows the resulting graph effects of the SE method with Riccati in Equations (3.23), which present the wave behaviors as kink waves, and Equations (3.24), which present the wave behaviors as periodic waves. Table 3 by using the Riccati sub-equation method, Equations (3.34), (3.40), and (3.48), which show the wave behaviors as kink waves.

The graphs of Equations (3.35)-(3.39) and (3.41)-(3.47) exhibit periodicity.

Table 1. Parameters values of Equations (3.13) and (3.14)

Equations	Parameters	Figure	Wave effects
(3.13)	$a_0 = 0,\ a_1 = -2,\ k = 1,\ l = 1,\ \beta = \sqrt{2},\ \eta = -1,$ $\omega = 2,\ \alpha = 0.5,\ \xi_0 = 0,\ 1 \le x, y \le 30,\ t = 50, 100$	1	kink
(3.14)	$a_0 = 0, a_1 = 2,\ k = 1,\ l = 1,\ \beta = -\sqrt{2},\ \eta = 1,$ $\omega = 2,\ \alpha = 0.5,\ \xi_0 = 0,\ 1 \le x, y \le 30,\ t = 50, 100$	-	kink

Table 2. Parameters values of Equations (3.23) and (3.24)

Equations	Parameters	Figure	Wave effects
(3.23)	$a_0 = 0,\ a_1 = -1,\ k = 1,\ l = 1,\ c = -0.5,\ d = 1,\ \gamma = -1,$ $\omega = 2,\ \alpha = 0.5,\ \xi_0 = 10,\ 1 \le x, y \le 30,\ t = 50, 100$	-	kink
(3.24)	$a_0 = 0,\ a_1 = 1,\ k = 1,\ l = -1,\ c = 0.5,\ d = 1,\ \gamma = -1,$ $\omega = 2,\ \alpha = 0.5,\ \xi_0 = 0,\ 1 \le x, y \le 30,\ t = 50, 100$	2	periodic

Table 3. Parameters values of Equations (3.34)-(3.35) and (3.37)-(3.48)

Equations	Parameters	Figure	Wave effects
(3.34)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = -0.15,$ $\omega = 0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	kink
(3.35)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = -0.15,$ $\omega = 0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.37)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = -0.15,$ $\omega = 0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	3	periodic
(3.38)	$a_0 = 0, a_1 = 2,\ k = 1,\ l = 1,\ \sigma = -0.15,$ $\omega = 0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.39)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = -0.15, A = 1,$ $\omega = 0.6, B = 2,\ \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	4	periodic
(3.40)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = -0.15, A = 1,$ $\omega = 0.6, B = 2,\ \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	kink
(3.41)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,$ $\omega = -0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.42)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,$ $\omega = -0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	5	periodic
(3.43)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,$ $\omega = -0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.44)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,$ $\omega = -0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.45)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,$ $\omega = -0.6, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.46)	$a_0 = 0,\ a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,\ A = 2,$ $\omega = -0.6, B = 1,\ \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	periodic
(3.47)	$a_0 = 0,\ a_1 = 2, k = 1,\ l = 1,\ \sigma = 0.15,\ A = 2,$ $\omega = -0.6, B = 1,\ \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	6	periodic
(3.48)	$a_0 = 0, a_1 = 2, k = 1,\ l = 1,\ \sigma = 0,\ g = 10,$ $\omega = 0, \alpha = 0.5,\ 1 \le x, y \le 30,\ t = 50, 100$	-	kink

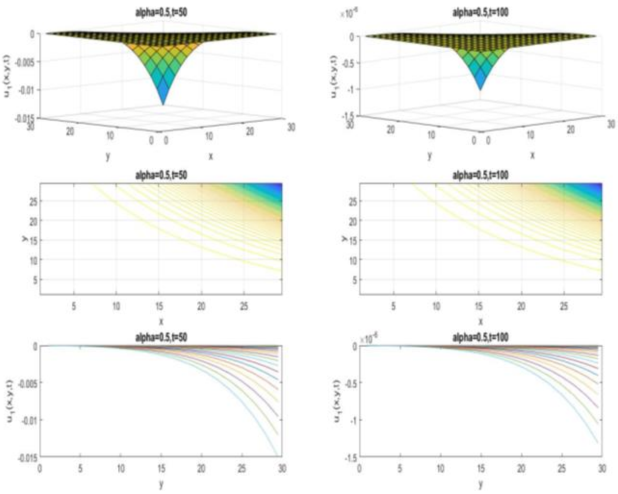

Figure 1. The kink effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.13)

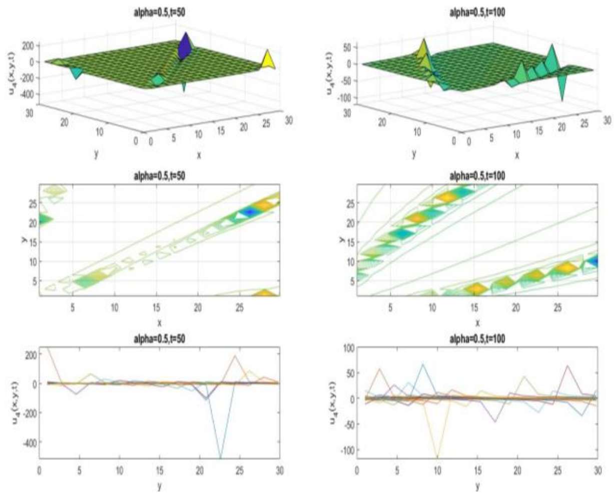

Figure 2. The periodic effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.24)

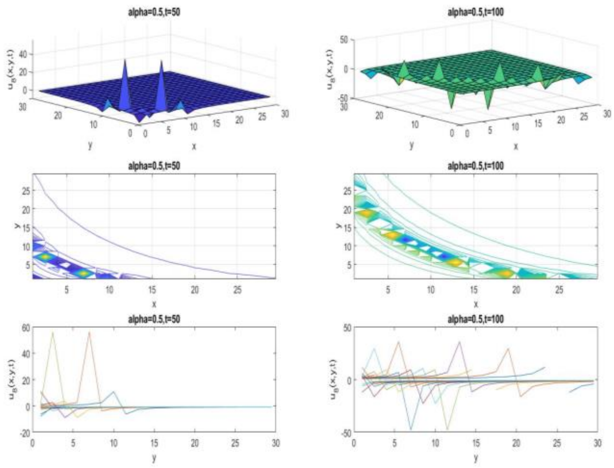

Figure 3. The periodic effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.37)

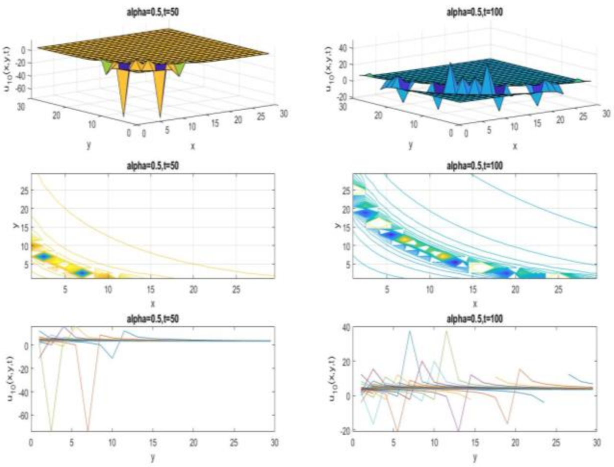

Figure 4. The periodic effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.39)

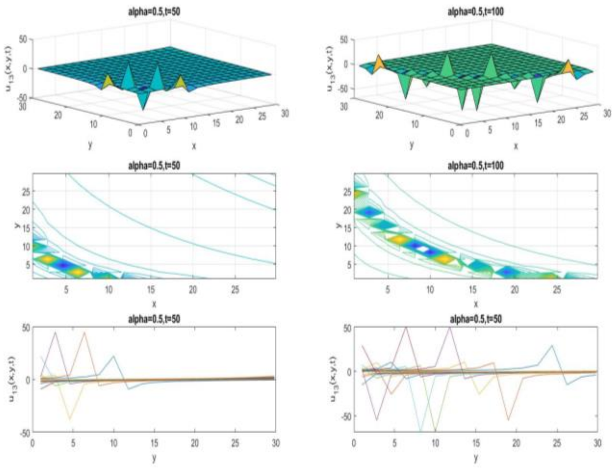

Figure 5. The periodic effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.42)

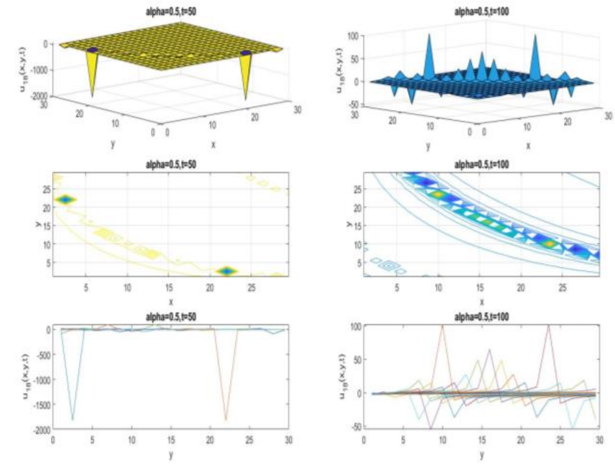

Figure 6. The kink effect in 3D, contour, and 2D plot for t=50, 100 in equation (3.47)

5. Conclusions

In this work, we determined the traveling wave solution for the space-time fractional order (2+1)-dimensional breaking soliton equation using the SE method with Bernoulli, the SE method with the Riccati equation, and the Riccati sub-equation method. Nineteen distinct solutions are produced by the results: two using the SE method with Bernoulli equation, two using the SE method with Riccati equation, and fifteen using the Riccati sub-equation method. Consequently, we came up with a lot of different kinds of exact traveling wave solutions for the model, exponentials, trigonometric functions, and hyperbolic functions.

The SE method with Bernoulli, the SE method with Riccati equation, and the Riccati sub-equation method are straightforward to comprehend and can be applied to other nonlinear fractional differential equations (FDEs). This study shows that the suggested method is suitable and effective for accurately solving the breaking soliton equation in space-time fractional order (2+1) dimensions. Three approaches proved to be dependable and effective in providing accurate solutions for solitary waves.

We also presented 3D, 2D, and contour plots for a few of the space-time fractional order (2+1)-dimensional breaking soliton equation solutions, which we demonstrated in Figures 1–6, where all graphs are kink and periodic waves.

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