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Original Article

Fuzzy ANCOVA and its application in drug yield analysis

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Abstract

Analysis of Covariance (ANCOVA) is a method of data analysis used in experimental design to compare treatments with high precision by reducing experimental error. The uncertainty involved during data collection cannot be ignored when planning and executing experiments. Consequently, it is important to design a one-way analysis of covariance for data collected under uncertainty. This paper introduces a new methodology for the one-way ANCOVA model in fuzzy environments using a fuzzy completely randomized layout. The fuzzy ANCOVA model is de-fuzzified into two crisp models: the lower-level model and the upper-level model. Analysis is performed separately for each model, and fuzzy logic is employed to draw overall conclusions. The research focuses on identifying and mitigating the effects of fuzzy covariates that do not contribute to the prediction of fuzzy response variables. The method is applied and elucidated through a numerical illustration in the context of pharmaceutical laboratories, where it is used to assess drug yield variations across different laboratories with imprecise data. The application demonstrates the effectiveness of the proposed methodology in handling data uncertainty and provides insights into drug yield optimization in a practical setting.

Keywords: design of experiments, fuzzy environment, one-way analysis of covariance, fuzzy logic, drug yield analysis

1. Introduction

Most of the experimental research has been carried out by considering the influence of a set of independent variables on a response (called dependent) variable. Many experiments are designed to look at the influence of a single independent variable (factor or group) while holding other factors constant. These experiments are called single-factor or single-group experiments and are analysed with the one-way analysis of variance (ANOVA). Analysis of covariance (ANCOVA) is an extension of ANOVA and is useful to improve precision by removing extraneous sources of variation from experiments by including a covariate. ANCOVA was first introduced by Fisher (1932). Since then, many authors have applied, developed and expanded ANCOVA in various fields such as medicine, agriculture, industry, biology and engineering. Cochran (1957) provided an overview of ANCOVA models along with assumptions needed to test for significance. Yang and Juskiw (2011) highlighted traditional ANCOVA as well as developed ANCOVA for mixed models. Porter and Raudenbush (1987) described the statistical model ANCOVA and talked about its applications in psychological research. They concluded that ANCOVA works best in randomized trials and warned against its use in non-randomized trials. Egbewale, Lewis, and Sim (2014) demonstrated that ANCOVA continues to be highly efficient in terms of reducing bias, improving precision, and enhancing statistical power when analyzing continuous outcomes in randomized controlled trials.

In modern times, scientific and technological processes are often complex and imprecise, for which complete or exact information is not always available. A key limitation of ANCOVA is that it requires precise measurements for the available data. Consequently, applying traditional ANCOVA within classical statistics becomes unfeasible when

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observations are uncertain. Therefore, we need an extended version of ANCOVA to deal with such vague data. To make this possible, Zadeh (1965) introduced fuzzy set theory and extended it in many areas like decision making, to deal with ambiguous and unclear data. A fuzzy set represents the linguistic variable expressed by the membership functions of fuzzy numbers like triangular, trapezoidal, normal, pentagonal etc. Dubois and Prade (1978) discussed the algebraic operations by applying the fuzzification rule on fuzzy numbers. Gonzàlez de Garibay (1986) analyzed two – way classification of the linear models for one fuzzy dummy variable and solved its normal equations to derive the estimable function. He also (1986) analyzed two – way classification of the linear models for two fuzzy dummy variables and solved its normal equations to derive the estimable function concerning the model. Arnold (1998) defined the fuzzy hypothesis testing for crisp data and proposed one and two-sided hypothesis testing under fuzzy environment on Type I and Type II errors. The fuzzy test for testing hypothesis under vague circumstances was proposed by Gajivaradhan (2000). The author also developed the fuzzy decision showing a scale of acceptability for the binary hypothesis. Grzgorzewski and Mrowka (2005) formulated the operators to approximate the fuzzy numbers to trapezoidal expected intervals. Buckley (2006) compares the fuzzy means of 'm' populations where each population was normally distributed with same but unknown variance. Buckley (2006) introduced the fuzzy concept of two-way analysis of variance assuming both the variables are independent of each other. Wu (2009) discussed the hypothesis testing for Fuzzy ANOVA using fuzzy data with the concept of optimistic degree and pessimistic degree for h-level set and solved optimization problem. Lubiano and Trutsching (2010) introduced an R-package "Statistical Analysis of Fuzzy data (SAFD)" for Fuzzy ANOVA using the de-fuzzification process for Fuzzy Random Variables. Dutta *et al*. (2011) compared the algebraic operation on fuzzy numbers using alpha cut and without using alpha cut methods of de-fuzzification. They found that the alpha cut method of de-fuzzification is simpler than the other de-fuzzification methods. De Andrés-Sánchez (2012) described the fuzzy least square regression and combined it with two-way ANOVA to expose a claim reserving method. Kalpanapriya and Pandian (2012) initiated decision-making-criteria for testing the vague hypothesis in Fuzzy ANOVA considering triangular uncertain observations. Jiryaei *et al*. (2013) presented the least-square method for Fuzzy ANOVA based on fuzzy random variables. Gajivaradhan and Parthibhan (2015) discussed the Fuzzy ANOVA through de-fuzzification by α – cut intervals for trapezoidal fuzzy numbers. Lin *et al*. (2016) discussed the application of one-way ANOVA with fuzzy data in the field of consumer demand. Parchami *et al.* (2017) extended the concept of one-way ANOVA for imprecise observations. Kalpanapriya and Unnissa (2018) elaborated the ANOVA for intuitionistic fuzzy set and its efficiency in career development. Mariappan and Pachamuthu (2020) generalized the Randomized Block Design for Triangular Fuzzy Numbers for testing the crisp hypothesis based on observed fuzzy test statistic. Traneva *et al*. (2022) introduced a command-line utility for the calculation of intuitionistic Fuzzy ANOVA results. They also demonstrated the comparative analysis of the result obtained using intuitionistic Fuzzy ANOVA with the classical ANOVA. Rahul *et al*. (2022) discussed the problem of two-way analysis of variance with replication under fuzzy data by using the alpha cut interval for defuzzification. AlAita and Aslam (2023) developed the Neutrosophic ANCOVA and proved that this approach is adaptable and efficient in handling indeterminacy compared to the existing methods. While planning and executing any experiment, we cannot ignore the uncertainty involved in the collection of the data. Hence, it becomes important to design one-way analysis of covariance for the data collected in uncertain situations. After reviewing the existing literature and to the best of our understanding, there has been no prior research on ANCOVA within the context of fuzzy statistics. In this paper, a new dimension of methodology for the one-way ANCOVA model under fuzzy environment with fuzzy completely randomised layout will be introduced. We anticipate that the proposed Fuzzy ANCOVA model will offer greater flexibility, effectiveness and insight compared to the traditional ANCOVA model within classical statistics.

This paper is organized as follows. In section 2, some definitions related to the paper are described. Section 3 describes the generalization of the Fuzzy ANCOVA which is de-fuzzified into the two crisp models, i.e. a lower-level model and an upper-level model, respectively. In section 4, a numerical example has been considered for illustrating the findings. Finally, some concluding remarks are presented in section 5.

2. Preliminaries

In this section, we recall some basic definitions of fuzzy numbers, triangular fuzzy numbers, and alpha cut intervals, which are useful for the analysis.

Definition 2.1 Generalized fuzzy numbers

A generalized fuzzy number $\widetilde{A} = (p, q, r, s{:}\,\omega)$ is the fuzzy subset defined on real line $\Re$ whose membership function $\mu_A(x)$ satisfies the following conditions:

a) $\mu_{\widetilde{A}}(x)$ is continuous mapping on $\Re$ to the closed interval [0, ω].
b) $\mu_{\widetilde{A}}(x) = 0$ for all $\in (-\infty, p]$.
c) $\mu_L(x) = l_{\widetilde{A}}(x)$ is strictly increasing on [p,q].
d) $\mu_{\widetilde{A}}(x) = \omega$ for all [q,r].
e) $\mu_R(x) = r_{\widetilde{A}}(x)$ is strictly decreasing on [r,s].
f) $\mu_{\widetilde{A}}(x) = 0$ for all $x \in$ [s, ∞).

where $0 \leq \omega \leq 1$; p,q, r and s are real numbers such that p < q ≤ r < s. $\widetilde{A}$ is fuzzy number and $\mu_{\widetilde{A}}(x) \in \mathbb{R}$. $l_{\widetilde{A}}(x) and\ r_{\widetilde{A}}(x)$ are left and right side of a fuzzy number $\widetilde{A}$ respectively.

Definition 2.2 Triangular fuzzy numbers (*TrFNs*)

The TrFNs is a set of three real numbers (p,q,r) such that p <q <r where p, q and r are the lower, central and upper value of the TrFNs. The membership function of triangular fuzzy number can be expressed as

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & ; \quad x < p \\ \dfrac{x-p}{q-p} & ; p \leq x \leq q \\ \dfrac{r-x}{r-q} & ; q \leq x \leq r \\ 0 & ; x > r \end{cases}$$

Definition 2.3 α -cut

A most certain tool used for handling the fuzzy numbers is through their α –cuts. For any fuzzy number $\tilde{\mathcal{A}}$, the α-cut can be defined as

$$\mathcal{A}_\alpha = \{x \in \mathcal{X};\ \mu_{\tilde{\mathcal{A}}}(x) \geq \alpha\}$$

Where $\mathcal{A}_\alpha$ is a non-fuzzy set and $\alpha \in [0,1]$. It can be seen from the definition of fuzzy number that α-cut interval of fuzzy number is a closed interval. So, for a fuzzy number $\tilde{\mathcal{A}}$, we have

$$\mathcal{A}_\alpha = [\mathcal{A}_l^\alpha , \mathcal{A}_u^\alpha]$$

where,

$$\mathcal{A}_l^\alpha = \inf\{x \in \mathcal{X} ;\mu_{\tilde{\mathcal{A}}}(\mathcal{X}) \geq \alpha \ for\ all\ \alpha \in [0,1]\}$$

i.e. left side of fuzzy number $\tilde{\mathcal{A}}$.
and

$$\mathcal{A}_u^\alpha = \sup\{x \in \mathcal{X} ;\mu_{\tilde{\mathcal{A}}}(\mathcal{X}) \geq \alpha \ for\ all\ \alpha \in [0,1]\}$$

i.e. right side of fuzzy number $\tilde{\mathcal{A}}$.

3. Proposed One-Way Fuzzy Analysis of Covariance

The Fuzzy ANCOVA is a newly introduced method to enhance experimental precision under uncertain, linguistic, or fuzzy conditions. The features of Fuzzy ANOVA Modal and Fuzzy Multiple Regression Modal are used in the Fuzzy ANCOVA. Let us consider ‘k’ treatments, i.e. $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_k$, with the í[th] treatment replicated $r_í$ times (í = 1,2…v) so that $N = \sum_í^v r_í$ is the total number of experimental units. Here, the experiment is conducted under Complete Block Design (CRD) layout. Suppose that there are two Fuzzy variables under study that are Fuzzy Response variable ‘d’ and Fuzzy concomitant variable. Assuming a linear relationship between these two variables, the Fuzzy ANCOVA model for CRD with one Fuzzy concomitant variable is given by

$$\widetilde{d_{íȷ}} = \tilde{\mu} + \widetilde{t_í} + \tilde{\beta}\left(\widetilde{c_{íȷ}} - \tilde{\bar{c}}_{..}\right) + \widetilde{e_{íȷ}} \quad ; \quad (í = 1,2 \dots v ; j = 1,2, \dots, r_í) \qquad (1)$$

where $\tilde{\mu}$ is the general mean effect, $\widetilde{t_í}$ is the additional effect due to the íth treatment, $\widetilde{e_{íȷ}}$ is the random error effect, $\tilde{\beta}$ is the coefficient of Fuzzy regression of ‘d’ on ‘c’, and $\widetilde{c_{íȷ}}$ is the value of concomitant variable corresponding to the response variable $\widetilde{d_{íȷ}}$ such that $\sum_í^v \widetilde{t_í}$=0. Here, the observations are considered in terms of triangular fuzzy numbers. Using the α-cut interval method, we transform the Fuzzy ANCOVA model to the lower level crisp ANCOVA model and the upper-level crisp ANCOVA model. For the lower-level model, we have the observations $p_{íj} + \alpha(q_{íj} - p_{íj})$ for all ,(í = 1,2 … v ; j = 1,2, …, $r_í$) and for the upper-level model, we have the observations $r_{íj} - \alpha(r_{íj} - q_{íj})$ for all ,(í = 1,2 … v ; j = 1,2, …, $r_í$).

3.1 Lower-level model

Let $\widetilde{d_{íȷ}}^l$ represent the j[th] Lower fuzzy unit receiving the íth treatment. The linear model for Fuzzy ANCOVA is given by

$$\widetilde{d_{íȷ}}^l_\alpha = \tilde{\mu}^l_\alpha + \widetilde{t_í}^l_\alpha + \tilde{\beta}^l_\alpha\left(\widetilde{c_{íȷ}}^l_\alpha - \tilde{\bar{c}}^l_{..\alpha}\right) + \widetilde{e_{íȷ}}^l_\alpha \quad ; \quad (í = 1,2 \dots v ; j = 1,2, \dots, r_í) \qquad (2)$$

where $\tilde{\mu}^l_\alpha$ is the general mean effect for Lower-Level Model, $\widetilde{t_í}^l_\alpha$ is the additional effect due to the íth treatment in lower level model , $\widetilde{e_{íȷ}}^l_\alpha$ is the random error effect for Lower Level model, $\tilde{\beta}^l_\alpha$ is the coefficient of Fuzzy regression of ‘d’ on ‘c’ for lower level model, and $\widetilde{c_{íȷ}}^l_\alpha$ is the value of concomitant variable corresponding to the response variable $\widetilde{d_{íȷ}}^l_\alpha$ for lower level model such that $\sum_í^v \widetilde{t_í}^l_\alpha$=0. Table 1 presents the ANOVA Lower Level Fuzzy ANCOVA table in which

$$\widetilde{\varphi^l_{cc\alpha}} = \sum_{í=1}^{v} r_í\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{í.\alpha}}\right)^2 , \widetilde{\varphi^l_{dd\alpha}} = \sum_{í=1}^{v} r_í\left(\widetilde{d^l_{íȷ\alpha}} - \widetilde{\bar{d}^l_{í.\alpha}}\right)^2,$$

$$\widetilde{\varphi^l_{cd\alpha}} = \sum_{í=1}^{v} r_í\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{í.\alpha}}\right)\left(\widetilde{d^l_{íȷ\alpha}} - \widetilde{\bar{d}^l_{í.\alpha}}\right)$$

$$\widetilde{\vartheta^l_{dc\alpha}} = \sum_í\sum_j(\widetilde{d^l_{íȷ\alpha}} - \tilde{\bar{d}}^l_{..\alpha})\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{..\alpha}}\right), \widetilde{\vartheta^l_{cc\alpha}} = \sum_í\sum_j\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{..\alpha}}\right)^2,$$

$$\widetilde{\vartheta^l_{dd\alpha}} = \sum_í\sum_j(\widetilde{d^l_{íȷ\alpha}} - \tilde{\bar{d}}^l_{..\alpha})^2$$

$$\widetilde{\mho^l_{dc\alpha}} = \sum_í\sum_j(\widetilde{d^l_{íȷ\alpha}} - \widetilde{\bar{d}^l_{í.\alpha}})\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{í.\alpha}}\right), \widetilde{\mho^l_{cc\alpha}} = \sum_í\sum_j\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l_{í.\alpha}}\right)^2 \text{ and}$$

$$\widetilde{\mho^l_{dd\alpha}} = \sum_í\sum_j(\widetilde{d^l_{íȷ\alpha}} - \widetilde{\bar{d}^l_{í.\alpha}})^2$$

$$\widetilde{SSE^l_\alpha} = \widetilde{\mho^l_{dd\alpha}} - \frac{\widetilde{\mho^l_{cd\alpha}}}{\widetilde{\mho^l_{cc\alpha}}}$$

where $\widetilde{\mho^l_{dd\alpha}}$ is the error sum of squares for C.R.D.
Under the null hypothesis, all the treatments are equally effective.
The Lower-Level model reduces to

$$\widetilde{d_{íȷ}}^l_\alpha = \tilde{\mu}^l_\alpha + \tilde{\beta}^{l\,0}_\alpha\left(\widetilde{c^l_{íȷ\alpha}} - \widetilde{\bar{c}^l}_{..\alpha}\right) + \widetilde{e^l_{íȷ\alpha}}^{\,0} \quad ; \quad (í = 1,2 \dots v ; j = 1,2, \dots, r_í)$$

$$\widetilde{SSE^l_\alpha}^{\,0} = \widetilde{\vartheta^l_{dd\alpha}} - \frac{\widetilde{\vartheta^l_{dc\alpha}}}{\widetilde{\vartheta^l_{cc\alpha}}}$$

The Fuzzy sum of squares of treatment for lower-level model is given by $\widetilde{SST^l_\alpha} = \widetilde{SSE^l_\alpha}^{\,0} - \widetilde{SSE^l_\alpha}$

The Fuzzy Mean Sum of squares due to treatments for Lower-Level Model is $\widetilde{MST^l_\alpha} = \frac{\widetilde{SST^l_\alpha}}{v-1}$

The Fuzzy Mean Sum of squares due to Error for Lower-Level Model is $\widetilde{MSE^l_\alpha} = \frac{\widetilde{SSE^l_\alpha}}{n-v-1}$

The adjusted Fuzzy test statistic for Lower-Level Model is

$$\widetilde{F_{adj}}^l_{\cdot\alpha} = \frac{(\widetilde{SSE^l_\alpha}^{\,0} - \widetilde{SSE^l_\alpha})/(v-1)}{\widetilde{SSE^l_\alpha}/(n-v-1)}$$

3.2 Upper-level model

Let $\widetilde{d_{ij}}^{u}$ represent the u^{th} Lower fuzzy unit receiving the íth treatment. The linear model for Upper-Fuzzy ANCOVA is given by

$$\widetilde{d_{ij}}^{u}_{\alpha} = \tilde{\mu}^{u}_{\alpha} + \tilde{t}^{u}_{i\alpha} + \tilde{\beta}^{u}_{\alpha}\left(\widetilde{c_{ij}}^{u}_{\alpha} - \tilde{\tilde{c}}^{u}_{..\alpha}\right) + \widetilde{e_{ij}}^{u}_{\alpha} \quad ; \quad (i = 1,2 \ldots v\, ; j = 1,2,\ldots,r_i)$$

where $\tilde{\mu}^{u}_{\alpha}$ is the general mean effect for Upper-Level Model, $\tilde{t}^{u}_{i\alpha}$ is the additional effect due to the íth treatment in Upper level model, $\widetilde{e_{ij}}^{u}_{\alpha}$ is the random error effect for Upper Level model, $\tilde{\beta}^{u}_{\alpha}$ is the coefficient of Fuzzy regression of 'd' on 'c' for lower level model, and $\widetilde{c_{ij}}^{u}_{\alpha}$ is the value of concomitant variable corresponding to the response variable $\widetilde{d_{ij}}^{u}_{\alpha}$ for upper level model such that $\sum_{i}^{v}\tilde{t}^{u}_{i\alpha}=0$. Table 2 is for Fuzzy ANCOVA table for Lower-level Model where

$$\sum_i\sum_j(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{i.\alpha})(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{i.\alpha}) = \widetilde{\mho^{u}_{dc\alpha}},\ \sum_i\sum_j(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{i.\alpha})^2 = \widetilde{\mho^{u}_{cc\alpha}},\ \sum_i\sum_j(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{i.\alpha})^2 = \widetilde{\mho^{u}_{dd\alpha}}$$

$$\widetilde{\varphi^{u}_{cc\alpha}} = \sum_{i=1}^{v} r_i\left(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{i.\alpha}\right)^2,\ \widetilde{\varphi^{u}_{dd\alpha}} = \sum_{i=1}^{v} r_i\left(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{i.\alpha}\right)^2,$$

$$\widetilde{\varphi^{u}_{cd\alpha}} = \sum_{i=1}^{v} r_i\left(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{i.\alpha}\right)\left(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{i.\alpha}\right)$$

$$\sum_i\sum_j(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{..\alpha})(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{..\alpha}) = \widetilde{\vartheta^{u}_{dc\alpha}},\ \sum_i\sum_j(\widetilde{c^{u}_{ij_\alpha}} - \tilde{\tilde{c}}^{u}_{..\alpha})^2 = \widetilde{\vartheta^{u}_{cc\alpha}},\ \sum_i\sum_j(\widetilde{d^{u}_{ij_\alpha}} - \tilde{\tilde{d}}^{u}_{..\alpha})^2 = \widetilde{\vartheta^{u}_{dd\alpha}}$$

$$\widetilde{SSE^{u}_{\alpha}} = \widetilde{\mho^{u}_{dd\alpha}} - \frac{\widetilde{\mho^{u}_{cd\alpha}}^{2}}{\widetilde{\mho^{u}_{cc\alpha}}}$$

where $\widetilde{\mho^{u}_{dd\alpha}}$ is the error sum of squares for C.R.D.

Under the null hypothesis, all the treatments are equally effective.

The Upper-Level model reduces to

$$\widetilde{d_{ij}}^{u}_{\alpha} = \tilde{\mu}^{u}_{\alpha} + \tilde{\beta}^{u\,0}_{\alpha}\left(\widetilde{c_{ij}}^{u}_{\alpha} - \tilde{\tilde{c}}^{u}_{..\alpha}\right) + \widetilde{e_{ij}}^{u\,0}_{\alpha} \quad ; \quad (i = 1,2 \ldots v\, ; j = 1,2,\ldots,r_i)$$

$$\widetilde{SSE^{u}_{\alpha}}^{0} = \widetilde{\vartheta^{u}_{dd\alpha}} - \frac{\widetilde{\vartheta^{u}_{dc\alpha}}^{2}}{\widetilde{\vartheta^{u}_{cc\alpha}}}$$

The Fuzzy Sum of Squares of Treatment for Upper-Level Model is given by $\widetilde{SST^{u}_{\alpha}} = \widetilde{SSE^{u}_{\alpha}}^{0} - \widetilde{SSE^{u}_{\alpha}}$

The Fuzzy Mean Sum of squares due to treatments for Upper-Level Model is $\widetilde{MST^{u}_{\alpha}} = \frac{\widetilde{SST^{u}_{\alpha}}}{v-1}$

The Fuzzy Mean Sum of squares due to Error for Upper-Level Model is $\widetilde{MSE^{u}_{\alpha}} = \frac{\widetilde{SSE^{u}_{\alpha}}}{n-v-1}$

The Fuzzy test statistic for testing the null hypothesis is

$$\widetilde{F_{adj}}^{u}_{\alpha} = \frac{(\widetilde{SSE^{u}_{\alpha}}^{0} - \widetilde{SSE^{u}_{\alpha}})/(v-1)}{\widetilde{SSE^{u}_{\alpha}}/(n - v - 1)}$$

3.3 Fuzzy hypotheses and decision rule

In Fuzzy completely randomized design, the null and alternative hypothesis under the fuzzy statistics could be expressed as

$$\widetilde{H_o}: \tilde{t}_1 = \tilde{t}_2 = \cdots = \tilde{t}_v = 0 \ vs\ \widetilde{H_A}: at\ least\ one\ \tilde{t}_i \neq 0$$

Table 1. Fuzzy ANCOVA table for lower-level model

Source	Degree of freedom	Sum of Squares			Estimate of $\tilde{\beta}$	Adjusted $\widetilde{\vartheta^{l}_{dd\alpha}}$	Adjusted degree of freedom
Treatments	v − 1	$\widetilde{\varphi^{l}_{cc\alpha}}$	$\widetilde{\varphi^{l}_{dd\alpha}}$	$\widetilde{\varphi^{l}_{cd\alpha}}$			
Error	n-v − 1	$\widetilde{\mho^{l}_{cc\alpha}}$	$\widetilde{\mho^{l}_{dd\alpha}}$	$\widetilde{\mho^{l}_{cd\alpha}}$	$\frac{\widetilde{\mho^{l}_{cd\alpha}}}{\widetilde{\mho^{l}_{cc\alpha}}}$	$\widetilde{SSE^{l}_{\alpha}} = \widetilde{\mho^{l}_{dd\alpha}} - \frac{\widetilde{\mho^{l}_{cd\alpha}}}{\widetilde{\mho^{l}_{cc\alpha}}}$	n-v − 1
Total	n-1	$\widetilde{\vartheta^{l}_{cc\alpha}}$	$\widetilde{\vartheta^{l}_{dd\alpha}}$	$\widetilde{\vartheta^{l}_{cd\alpha}}$	$\frac{\widetilde{\vartheta^{l}_{cd\alpha}}}{\widetilde{\vartheta^{l}_{cc\alpha}}}$	$\widetilde{SSE^{l}_{\alpha}}^{0} = \widetilde{\vartheta^{l}_{dd\alpha}} - \frac{\widetilde{\vartheta^{l}_{dc\alpha}}}{\widetilde{\vartheta^{l}_{cc\alpha}}}$	n-2
Difference						$\widetilde{SSE^{l}_{\alpha}}^{0} - \widetilde{SSE^{l}_{\alpha}}$	v − 1

Table 2. Fuzzy ANCOVA table for upper-level model

Source	Degree of freedom	Sum of Squares			Estimate of $\tilde{\beta}$	Adjusted $\widetilde{\vartheta^{l}_{dd\alpha}}$	Adjusted degree of freedom
Treatments	v − 1	$\widetilde{\varphi^{u}_{cc\alpha}}$	$\widetilde{\varphi^{u}_{dd\alpha}}$	$\widetilde{\varphi^{u}_{cd\alpha}}$			
Error	n-v − 1	$\widetilde{\mho^{u}_{cc\alpha}}$	$\widetilde{\mho^{u}_{dd\alpha}}$	$\widetilde{\mho^{u}_{cd\alpha}}$	$\frac{\widetilde{\mho^{u}_{cd\alpha}}}{\widetilde{\mho^{u}_{cc\alpha}}}$	$\widetilde{SSE^{u}_{\alpha}} = \widetilde{\mho^{u}_{dd\alpha}} - \frac{\widetilde{\mho^{u}_{cd\alpha}}}{\widetilde{\mho^{u}_{cc\alpha}}}$	n-v − 1
Total	n-1	$\widetilde{\vartheta^{u}_{cc\alpha}}$	$\widetilde{\vartheta^{u}_{dd\alpha}}$	$\widetilde{\vartheta^{u}_{cd\alpha}}$	$\frac{\widetilde{\vartheta^{u}_{cd\alpha}}}{\widetilde{\vartheta^{u}_{cc\alpha}}}$	$\widetilde{SSE^{u}_{\alpha}}^{0} = \widetilde{\vartheta^{u}_{dd\alpha}} - \frac{\widetilde{\vartheta^{u}_{dc\alpha}}}{\widetilde{\vartheta^{u}_{cc\alpha}}}$	n-2
Difference						$\widetilde{SSE^{u}_{\alpha}}^{0} - \widetilde{SSE^{u}_{\alpha}}$	v − 1

The decision rule is given in Figure 1. If the calculated value of F statistic for lower and upper levels for a particular value of $\alpha - cut$ is greater than the Critical value of F statistic at $\dot{\alpha}$ significance level, then reject the null hypothesis; otherwise accept the null hypothesis.

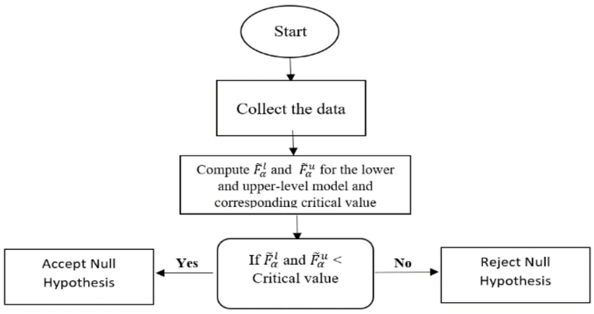

Figure 1. Test procedure for fuzzy ANCOVA

4. Drug Yield Analysis in Pharmaceutical Laboratories using FANCOVA

A pharmaceutical company operates across three different laboratories, each with varying levels of equipment precision and technician expertise. The company wants to assess whether there are significant differences in the yield of a specific drug compound between these laboratories, taking into account the different quantities of raw chemicals used. The company collects data on the quantities of raw chemicals and the corresponding drug yields from multiple production batches within each laboratory. However, due to factors such as measurement error or variability in equipment calibration, the data collected are not precise and are represented in fuzzy triangular number form. For instance, in Laboratory A, production batches use approximately 500 grams of a key chemical (recorded as (495, 500, 505)), resulting in a drug yield of about 200 milligrams (recorded as (190, 200, 210)). Similarly, other data are collected for Laboratories B and C, as shown in Table 3. Recognizing the positive relationship between raw material input and production output, the company aims to assess whether differences in production outputs between factories remain significant after accounting for variations in raw material quantities. To achieve this, they utilize Fuzzy ANCOVA, a statistical technique that enables them to control for the effect of raw material usage while testing for differences in outputs between factories. By employing Fuzzy ANCOVA, the company can make more informed decisions about resource allocation and manufacturing strategies across different factories, ultimately optimizing production efficiency and maximizing profitability in their operations. The alpha cut values of Table 3 is given in Table 4.

For Lower Level: The Fuzzy Sum of Squares for Lower-Level Model is given in Table 5. Now,

$$Adjusted\ \widetilde{SSE}_\alpha^{l\ 0} = \widetilde{\vartheta^l_{dd\alpha}} - \frac{\widetilde{\vartheta^l_{cd\alpha}}^2}{\widetilde{\vartheta^l_{cc\alpha}}}$$

$$Adjusted\ \widetilde{SSE}_\alpha^{l\ 0} = ((32.25\alpha^2 - 146.5\alpha + 976.9167) * (20.415\alpha^2 + 17.5033\alpha + 406.4164) - (1.5875\alpha^2 + 8.433\alpha + 533.7667)^2)/((32.25\alpha^2 - 146.5\alpha + 976.9167))$$

$$\text{Adjusted } \widetilde{SSE}_\alpha^l = \widetilde{\upsilon^l_{dd\alpha}} - \frac{\widetilde{\upsilon^l_{cd\alpha}}^2}{\widetilde{\upsilon^l_{cc\alpha}}}$$

$$\text{Adjusted } \widetilde{SSE}_\alpha^l = ((19.75\alpha^2 - 31\alpha + 506.75) * (20.255\alpha^2 + 24.4873\alpha + 323.5793) - (0.475\alpha^2 + 46.8225\alpha + 337.0457)^2)/((19.75\alpha^2 - 31\alpha + 506.75))$$

The Fuzzy ANCOVA table for lower-level model is given in Table 6.

At 5% significance level, the tabulated value of F is 4.459. Hence each value of $\widetilde{F_{adj}}{}^l_{\cdot\alpha}$ is less than the tabulated value for each value of α. Thus, we can conclude that there is no significant difference in the average drug yields between the laboratories.

Table 3. Data for fuzzy completely randomized design

Laboratory A		Laboratory B		Laboratory C	
Raw chemicals	Drug yield	Raw chemicals	Drug yield	Raw chemicals	Drug yield
(120,125,127)	(34.58,39.52,42.46)	(117,119,127)	(24.4,28.5,29.6)	(103,105,110)	(23.5,25.7,27.4)
(117,120,122)	(27.5,29.4,33.2)	(128,130,135)	(32.5,37.45,42.4)	(109,115,117)	(25.7,29.7,30.2)
(107,114,116)	(22.4,26.5,29.3)	(125,127,130)	(30.4,32.4,36)	(117,120,124)	(24.7,29.4,32.7)
(100,105,109)	(19.3,22.4,25.5)	(129,133,137)	(40.4,42.5,46.4)	(119,123,129)	(33.4,38.7,43.5)

Table 4. α −cut values of fuzzy completely randomized design

Laboratory A		Laboratory B		Laboratory C	
Drug yield	Raw chemicals	Drug yield	Drug yield	Raw chemicals	Drug yield
(120+5α, 127-2 α)	(34.58+4.94α, 42.46-2.94α)	(117+2α, 127-8α)	(24.4+4.1α, 29.6-1.1α)	(103+2α, 110-5α)	(23.5+2.2α, 27.4-1.7α)
(117+3α, 122-2α)	(27.5+1.9α, 33.2-3.8α)	(128+2α, 135-5α)	(32.5+4.95α, 42.4-4.95α)	(109+6α, 117-2α)	(25.7+2α, 30.2-0.5α)
(107+7α, 116-2α)	(22.4+4.1α, 29.3-2.8α)	(125+2α, 130-3α)	(30.4+2α, 36-3.6α)	(117+3α, 124-4α)	(24.7+4.7α, 32.7-3.3α)
(100+5α, 109-4α)	(19.3+3.1α, 25.5-3.1α)	(129+4α, 137-4α)	(40.4+2.1α, 46.4-3.9α)	(119+4α, 129-6α)	(33.4+5.3α, 43.5-4.8α)

For Upper Value: The Fuzzy Sum of Squares for Upper-Level Model is given in Table 7. Now,

$$Adjusted\ \widetilde{SSE_\alpha^u}^0 = \widetilde{\vartheta_{dd_\alpha}^u} - \frac{\widetilde{\vartheta_{cd_\alpha}^u}^2}{\widetilde{\vartheta_{cc_\alpha}^u}}$$

$$Adjusted\ \widetilde{SSE_\alpha^u}^0 =$$

$$((20.52\alpha^2 - 139.63\alpha + 551.6115) * (38.917\alpha^2 - 81.167\alpha + 904.92) - 〚(0.011\alpha^2 - 79.267\alpha + 616.503)〛^2)/((38.917\alpha^2 - 81.167\alpha + 904.92))$$

$$\text{Adjusted } \widetilde{SSE_\alpha^u} = \widetilde{\mho_{dd_\alpha}^u} - \frac{\widetilde{\mho_{cd_\alpha}^u}^2}{\widetilde{\mho_{cc_\alpha}^u}}$$

$$\text{Adjusted } \widetilde{SSE_\alpha^u} =$$

$$((19.12\alpha^2 - 125.935\alpha + 465.3162) * (25.75\alpha^2 + 42\alpha + 449.75) - 〚(-0.195\alpha^2 - 35.6225\alpha + 418.335)〛^2)/((25.75\alpha^2 + 42\alpha + 449.75))$$

The Fuzzy ANCOVA table for Upper-Level Model is given in Table 8. The range of α-cut values from 0.1 to 1.0 reflects different levels of confidence in the fuzzy observations. The α-cut method allows the transformation of fuzzy numbers into crisp intervals, where each α level corresponds to a specific degree of membership. Lower α values capture wider uncertainty (more conservative estimates), whereas higher α values represent stronger belief in the data (narrower intervals).

This systematic variation helps analyze the sensitivity of the results under varying degrees of uncertainty, making the model more robust and adaptable for real-world imprecise data.

At 5% significance level, the tabulated value of F is 4.459. Hence each value of $\widetilde{F_{adj}\cdot_\alpha^u}$ is less than the tabulated value for each value of α. Thus, there is no strong evidence to suggest that the average drug yields across the three laboratories are significantly different when accounting for the varying quantities of raw chemicals used. This indicates that, after considering the differences in raw chemical usage, the drug yield efficiency across the laboratories is relatively consistent, and no laboratory significantly outperforms the others in terms of yield.

5. Conclusions

The paper presents an enhanced approach to covariance analysis within fuzzy environments by introducing an alpha cut interval method for one-way analysis of covariance when data on observations are uncertain, such as imprecise measurements of raw chemical quantities and drug yields. The method involves transforming the initial fuzzy model into lower and upper-level crisp models through de-fuzzification using the alpha cut interval approach. Fuzzy test statistics are computed for both models based on various alpha values, and decisions are made by comparing these statistics with tabulated F statistic values. In the illustrative application, the results show no strong evidence to reject the fuzzy null hypothesis for either the lower or upper-level models across different alpha values. This implies that, within the pharmaceutical context, the differences in drug yields between laboratories are not statistically significant when accounting for the uncertainty in raw chemical usage. For a specific alpha value, the Classical One-way ANCOVA is shown to be a special case of the proposed One-way Fuzzy ANCOVA, highlighting the method's versatility and robustness. This approach could be further applied to other experimental designs in fuzzy environments, offering a valuable tool for future research in areas where data uncertainty is common.

Table 5. Fuzzy sum of squares for lower-level model

Source	Degree of freedom	Sum of Squares for $\widetilde{c_\alpha^l}$	Sum of Squares for $\widetilde{d_\alpha^l}$	Sum of Squares for $\widetilde{cd_\alpha^l}$
Treatments	2	$(12.5\alpha^2-177.5\alpha-189.6)$	$(0.16\alpha^2-6.984\alpha+82.837)$	$(1.1125\alpha^2-38.3892\alpha+196.7217)$
Error	9	$(19.75\alpha^2-31\alpha+7987.32)$	$(20.255\alpha^2+24.4873\alpha+323.5793)$	$(0.475\alpha^2+46.8225\alpha+337.045)$
Total	11	$(32.25\alpha^2-146.5\alpha+976.9167)$	$(20.415\alpha^2+17.50337\alpha+406.4164)$	$(1.5875\alpha^2+8.4333\alpha+533.7667)$

Table 6. Fuzzy ANOVA table for lower-level model

Source	Degree of freedom	Adjusted sum of squares	Mean square	F-ratio adjusted
Treatments	2	$Adjusted\ \widetilde{SSE_\alpha^l}^0$ − Adjusted $\widetilde{SSE_\alpha^l}$	$\frac{Adjusted\ \widetilde{SSE_\alpha^l}^0 - \text{Adjusted } \widetilde{SSE_\alpha^l}}{2}$	$\frac{\widetilde{MS_{Treatments_\alpha}^l}}{\widetilde{MS_{Error_\alpha}^l}}$
Error	8	$Adjusted\ \widetilde{SSE_\alpha^l}$	$\frac{Adjusted\ \widetilde{SSE_\alpha^l}}{8}$	
Total	10	Adjusted $\widetilde{SSE_\alpha^l}^0$		

Table 7. Fuzzy sum of squares for upper-level model

Source	Degree of freedom	Sum of Squares for $\widetilde{c_\alpha^u}$	Sum of Squares for $\widetilde{d_\alpha^u}$	Sum of Squares for $\widetilde{cd_\alpha^u}$
Treatments	2	$(13.167\alpha^2-123.167\alpha+455.167)$	$(1.41\alpha^2-13.695\alpha+86.295)$	$(0.206\alpha^2-43.645\alpha+198.173)$
Error	9	$(25.75\alpha^2+42\alpha+449.75)$	$(19.12\alpha^2-125.935\alpha+465.3162)$	$(-0.195\alpha^2-35.6225\alpha+418.335)$
Total	11	$(38.917\alpha^2-81.167\alpha+904.92)$	$(20.52609\alpha^2-139.6297\alpha+551.6115)$	$(0.011\alpha^2-79.267\alpha+616.503)$

Table 8. Fuzzy ANCOVA table for upper-level model

Source	Degree of freedom	Adjusted sum of squares	Mean square	F-ratio adjusted
Treatments	2	$Adjusted\ \widetilde{SSE_\alpha^u}^0$ − Adjusted $\widetilde{SSE_\alpha^u}$	$\frac{Adjusted\ \widetilde{SSE_\alpha^u}^0 - \text{Adjusted } \widetilde{SSE_\alpha^u}}{2}$	$\frac{\widetilde{MS^u_{Treatments_\alpha}}}{\widetilde{MS^u_{Error_\alpha}}}$
Error	8	$Adjusted\ \widetilde{SSE_\alpha^u}$	$\frac{Adjusted\ \widetilde{SSE_\alpha^u}}{8}$	
Total	10	Adjusted $\widetilde{SSE_\alpha^u}^0$		

Author Contributions

Rahul Thakur: Conceptualization, Data curation, Formal analysis, Methodology, Writing - Original draft. S.C. Malik: Supervision, Validation, Writing - Review and editing. Masum Raj: Validation, Investigation, Writing - Review and editing.

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