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Review Article

Numerical simulation of non-linear integro-differential equation

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Abstract

This study introduces a novel numerical method for solving first-order Volterra-Fredholm Integro-Differential Equations (V-FIDEs). The proposed approach reformulates V-FIDEs as integral equations, which are then approximated using power series polynomials. The resulting problem is converted into a system of algebraic equations and solved using the standard collocation method. After establishing the uniqueness and convergence of the approach, numerical examples were employed to evaluate its effectiveness. The results indicate that the method performs competitively compared to existing techniques.

Keywords: numerical method, first-order Volterra-Fredholm Integro-differential equations (V-FIDEs), power series polynomials, collocation method, competitive performance

1. Introduction

Integro-differential equations (IDEs) are powerful mathematical tools that combine integral and derivative operators, making them essential for modeling complex phenomena in engineering, physics, biology, and economics (Rahman, 2007). These equations are classified into Fredholm, Volterra, and Fredholm-Volterra types, each with distinct applications ranging from fluid dynamics and heat transfer to population growth models. For instance, Vito Volterra pioneered the use of IDEs in 1926 to study hereditary effects in population dynamics (Rahman, 2007).

The analytical solution of IDEs is often intractable, necessitating robust numerical methods. Recent advances include the Bernstein polynomial method for linear Volterra-Fredholm equations (Shahooth, Jameel, & Ameen, 2020), collocation techniques for first-order and fractional-order equations (Ajileye & Aminu, 2022) and (Ajileye, James, Ayinde, & Oyedepo, 2022), and spectral methods employing Chebyshev-wavelets and Rational Chebyshev functions. Hybrid approaches, such as the Adomian decomposition method (Bakodah & Darwish 2013) and the simplified reproducing kernel method combined with homotopy perturbation (Hou, Niu, Xu, & Ngolo, 2021), have also proven effective for nonlinear Volterra-Fredholm IDEs. (Amiraliyev, Durmaz & Kudu 2018) have recently constructed an exponential-difference scheme with an accuracy of $O\ N^{-1}$ for the first-order linear singularly perturbed Fredholm integro-differential equation (SPFIDE) on a uniform grid. (Amiraliyev, Durmaz & Kudu 2016). Delay forms of SPVIDEs were discretized.

This work proposes a novel collocation approach for numerically solving first-order V-FIDEs of the form

$$z^{'}(\tau)+h(\tau)z(\tau)+\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{H}_1(z(t))dt + \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{H}_2(z(t))dt=\eta(\tau) \quad (1)$$

with initial condition

$$z(\sigma)=\upsilon \quad (2)$$

Here $\rho_1(\tau,t)$ and $\rho_2(\tau,t)$ represent the Volterra and Fredholm integral kernel functions, respectively,

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μ_1, μ_1, σ, v *and* h are predefined constants, $\eta(\tau)$ is the given function, and $z(\tau)$ represents the unknown function to be solved.

2. Basic Definitions and Terms

In this section, we provide certain definitions and fundamental concepts for problem formulation.

Definition 1. (Ajileye & Amoo, 2023) Let $(a_m), m \geq 0$ represent a sequence of real numbers. The power series in k, with coefficients a_m, is expressed as:

$$y(w) = \sum_{m=0}^{M} a_m w^m = \psi(w)A \tag{3}$$

where $\psi(w) = [1\ w\ w^2 \ldots w^M]$, $A = [a_0\ a_1 \ldots a_M]^T$

Definition 2. The method determines the desired collocation points within the interval, i.e. $[a, b]$, which are given by:

$$l_u = a + \frac{(b-a)u}{M}, u = 1,2,3, \ldots M \tag{4}$$

(Ajileye, James, Ayinde, & Oyedepo, 2022)

Definition 3. Let $\omega(s)$ be an integrable function, then

$${}_0I_\tau^\alpha(\omega(s)) = \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau - s)^{\alpha-1}\omega(s)ds \tag{5}$$

(Ajileye, James, Ayinde, & Oyedepo, 2022)

Definition 4. Let $z(x)$ be a continuous function, then

$${}_0I_\tau^\beta(z^{(\beta)}z(\tau)) = z(\tau) - \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}\tau^k \tag{6}$$

where $M - 1 < \beta < 1$. (Ajileye, James, Ayinde, & Oyedepo, 2022)

Definition 5. A metric on a set M is a function $d: M \times M \longrightarrow \mathbb{R}$ that satisfies the following properties for all $x, y \in M$

(a) $d(x, y) \geq 0$;
(b) $d(x, y) = 0 \Leftrightarrow x = y$
(c) $d(x, y) = d(y, x)$
(d) $d(x, y) \leq d(x, z) + d(x, y)$

If $d$is a metric on M, the pair (M, d) is referred to as a metric space. (Ajileye, James, Ayinde, & Oyedepo, 2022)

Definition 6. Berinde, V. (2007). (Strict Contraction) Let (X, d) be a metric space. A mapping $T: X \longrightarrow X$ is called a strict contraction if T is Lipschitz continuous with $\alpha \in [0,1]$.

$$d(T(x), T(y)) \leq Ld(x, y)\ \forall x, y \in X$$

Definition 7. The inner product is jointly continuous in both arguments. That is, if $x_n \to x$ and $y_n \to y$, then: $\langle x_n, y_n \rangle \to \langle x, y \rangle$. (Boyd & Vandenberghe, 2004).

Definition 8. (Quarteroni, Sacco & Saleri, 2020). An error norm is a mathematical measure used to quantify the difference between a numerical solution and the exact (or reference) solution in computational mathematics. It provides a way to assess the accuracy and convergence of numerical methods, such as finite difference, finite element, or spectral methods.

2.1 Relative error

$$Relative\ error = \frac{\|u_{exact} - u_{numerical}\|}{\|u_{exact}\|}$$

useful when the exact solution has large magnitude variations, (Quarteroni, Sacco & Saleri 2007).

Theorem 1. (Berinde, 2007). (Banach fixed point theorem) Let (X, d) be a complete metric space, and let $T: X \longrightarrow X$ be a contraction mapping, then T has a unique fixed point $\tau \in X$ such that$T(\tau) = \tau$, Furthermore, the iterative scheme

$$\tau_n = T\tau_{n-1}$$

Converges to this unique solution from any arbitrary starting point.

3. Materials and Methods

In this section, we demonstrate the uniqueness of the solution and apply the collocation approach to obtain the numerical solution of Volterra-Fredholm integro-differential equations.

Lemma 1. (Integral form): Let $y \in \mathbb{C}((0,1), \mathbb{R})$ be the solution to equation (1), then it is equivalent to

$$z(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau - s)^{\alpha-1}(h(\tau)z(\tau))ds + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau - s)^{\alpha-1}\begin{pmatrix} \mu_1 \int_0^\tau \rho_1(\tau, t)\mathrm{H}_1(z(t))dt + \\ \mu_2 \int_0^1 \rho_2(\tau, t)\mathrm{H}_2(z(t))dt \end{pmatrix} ds = \Omega(\tau) \tag{7}$$

where

$$\Omega(\tau) = \sum_{k=0}^{N} \frac{z^{(k)}(0)}{k!}\tau^k + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau - s)^{\alpha-1}\eta(s)ds$$

Proof. Multiply equation (1) by ${}_0I_\tau^\alpha$ (:) gives

$${}_0I_\tau^\alpha(z'(\tau)) + {}_0I_\tau^\alpha(h(\tau)z(\tau)) + {}_0I_\tau^\alpha\begin{pmatrix} \mu_1 \int_0^\tau \rho_1(\tau, t)\mathrm{H}_1(z(t))dt + \\ \mu_2 \int_0^1 \rho_2(\tau, t)\mathrm{H}_2(z(t))dt \end{pmatrix} = {}_0I_\tau^\alpha(\eta(\tau))$$

Using equation (5) gives

$$z(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,(h(\tau)z(\tau))+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{H}_1(z(t))dt+\\ \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{H}_2(z(t))dt\end{pmatrix}ds=\sum_{k=0}^{N}\frac{z^{(k)}(0)}{k!}\tau^k+\frac{1}{\Gamma(\alpha)}\int_0^{x}(\tau-s)^{\alpha-1}\eta(s)ds$$

$$z(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,(h(x)z(\tau))+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(x,t)\mathrm{H}_1(z(t))dt+\\ \mu_2\int_0^{1}\rho_2(x,t)\mathrm{H}_2(z(t))dt\end{pmatrix}ds=\Omega(\tau)$$

$$z(\tau)=\Omega(\tau)-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,(h(\tau)z(\tau))-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{H}_1(z(t))dt+\\ \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{H}_2(z(t))dt\end{pmatrix}ds \tag{8}$$

where

$$\Omega(\tau)=\sum_{k=0}^{N}\frac{z^{(k)}(0)}{k!}\tau^k+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}h(s)ds$$

3.1 Uniqueness of the method

In order to establish the method’s uniqueness, we provide the following hypotheses:

*U*1: There exist a constant, G_1, G_2 such that for any $z_1, z_2 \in \mathbb{C}\,((0,1),\mathbb{R})$ then

$$|\mathrm{G}_1(\mathrm{z}_1)-\mathrm{G}_1(\mathrm{z}_2)|\le \mathrm{G}_1|\mathrm{z}_1-\mathrm{z}_2|$$

and

$$|\mathrm{G}_2(\mathrm{z}_1)-\mathrm{G}_2(\mathrm{z}_2)|\le \mathrm{G}_2|\mathrm{z}_1-\mathrm{z}_2|$$

*U*2: There exist two functions $g_1^* and\ g_2^* \in \mathbb{C}([0,1]\times[0,1],\mathbb{R})$the set of all positive functions such that

$$g_1^*=\max_{\tau\in[0,1]}|\mu_1|\int_0^{\tau}|\rho(\tau,t)|dt<\infty$$

$$g_2^*=\max_{\tau\in[0,1]}|\mu_2|\int_0^{1}|\rho(\tau,t)|dt<\infty$$

*U*3: The function $g\in\mathbb{R}$ is continuous and satisfies:

$$h^*=\max_{x\in[0,1]}|h(\tau)|$$

Theorem 2. Let $T: X\longrightarrow X$ be a mapping defined by (8) then T is a strict contraction in $C([0,1])$ if

$$\left(\frac{h^*+g_1^*+g_2^*}{\Gamma(\alpha+1)}\right)<1$$

A function $f: R^n\to R^m$ is called Lipschitz continuous if there exists a constant $L>0$ such that for all $x,y\in R^n$

$$\|f(x)-f(y)\|<L\|x-y\|$$

The smallest such constant L is called the Lipschitz constant of f (Goodfellow, Bengio & Courville 2016).

Proof. Let $z(\tau), z(\tau)\in X$ applying Banach fixed point to equation (8)

$$(Tz_1)(\tau)=\Omega(\tau)-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,(h(\tau)z_1(\tau))-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{G}_1(z_1(t))dt+\\ \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{G}_2(z_1(t))dt\end{pmatrix}ds \tag{9}$$

and

$$(Tz_2)(\tau)=\Omega(\tau)-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,(h(x)z_2(\tau))-\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{G}_1(z_2(t))dt+\\ \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{G}_2(z_2(t))dt\end{pmatrix}ds \tag{10}$$

Subtracting equation (10) from equation (9) gives

$$(Tz_1)(\tau)-(Tz_2)(\tau)=\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,[h(\tau)(z_2(\tau)-z_1(\tau))]ds+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau}\rho_1(\tau,t)\mathrm{G}_1(z_2(t)-z_1(t))dt+\\ \mu_2\int_0^{1}\rho_2(\tau,t)\mathrm{G}_2(z_2(t)-z_1(t))dt\end{pmatrix}ds \tag{11}$$

Taking the absolute value of both sides gives

$$|(Tz_1)(\tau)-(Tz_2)(\tau)|\le\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\,|[h(\tau)(z_2(\tau)-z_1(\tau))]|ds+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\begin{pmatrix}|\mu_1|\int_0^{\tau}|\rho_1(\tau,t)||\mathrm{G}_1(z_2(t)-z_1(t))|dt+\\ |\mu_2|\int_0^{1}|\rho_2(\tau,t)||\mathrm{G}_2(z_2(t)-z_1(t))|dt\end{pmatrix}ds$$

Taking maximum of both sides and applying *U*1 - *U*3 *gives*

$$d(Tz_1,Tz_2)\le\left(\frac{h^*+g_1^*+g_2^*}{\Gamma(\alpha+1)}\right)d(z_1,z_2)$$

By the Banach contraction principle, T is a strict contraction mapping.

Theorem 3. (*Continuity*) Let (X,d) be a metric space $C([0,1])$ and $T:X\longrightarrow X$ be a mapping in C, let $z_n(\tau),z(\tau)\in X$ and the $\lim_{\tau\in[0,1]} z_n(\tau)=z(\tau)$. T is continuous if

$$d(Tz_n,Tz)\longrightarrow 0 \; as\; n\longrightarrow\infty$$

Proof.

$$\begin{aligned}&|Tz_n(\tau)-Tz(\tau)|\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\,|[h(\tau)(z(\tau)-z_n(t))]|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\begin{pmatrix}|\mu_1|\int_0^x|\rho_1(\tau,t)||\mathrm{G}_1(z(t)-z_n(t))|dt+\\ |\mu_2|\int_0^1|\rho_2(\tau,t)||\mathrm{G}_2(z(t)-z_n(t))|dt\end{pmatrix}ds\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\max_{\tau\in[0,1]}|h(\tau)|\max_{\tau\in[0,1]}|(z(\tau)-z_n(t))|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau\\ &-s)^{\alpha-1}\begin{pmatrix}|\mu_1|\max_{\tau\in[0,1]}\int_0^\tau|\rho_1(\tau,t)|\max_{\tau\in[0,1]}|\mathrm{G}_1(z(t)-z_n(t))|dt+\\ |\mu_2|\max_{\tau\in[0,1]}\int_0^\tau|\rho_2(\tau,t)|\max_{\tau\in[0,1]}|\mathrm{G}_2(z(t)-z_n(t))|dt\end{pmatrix}ds\\ &d(Tz_n,Tz)\le\left(\frac{h^*+g_1^*+g_2^*}{\Gamma(\alpha+1)}\right)d(z_n,z)\end{aligned}$$

Since $d(z_n,z)\longrightarrow 0\; as\; n\longrightarrow\infty$ then $d(Tz_n,Tz)\longrightarrow 0\; as\; n\longrightarrow\infty$ therefore T is continuous.

3.2 Method of solution

Let the solution of equation (1) and equation (2) be approximated by

$$z(\tau)=\sum_{m=0}^{M}a_m\tau^m=\phi(\tau)A \tag{12}$$

where $\phi(\tau)=[1,\tau,\tau^2,\dots,\tau^M]$, $A=[a_0,a_1,\dots,a_M]^T$
Substituting equation (11) into equation (8) gives

$$\begin{aligned}&\phi(\tau)A\\ &=\Omega(\tau)-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\,(h(\tau)\phi(\tau)A)\\ &-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^\tau\rho_1(\tau,t)\mathrm{G}_1(\phi(t)A)dt+\\ \mu_2\int_0^1\rho_2(\tau,t)\mathrm{G}_2(\phi(t)A)dt\end{pmatrix}ds\end{aligned} \tag{13}$$

Collocating at τ_i in equation (12) gives

$$\begin{aligned}&\phi\phi(\tau_i)A\\ &=\Omega(\tau_i)-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\,(h(\tau_i)\phi(\tau_i)A)\\ &-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau_i}\rho_1(\tau_i,t)\mathrm{G}_1(\phi(t)A)dt+\\ \mu_2\int_0^1\rho_2(\tau_i,t)\mathrm{G}_2(\phi(t)A)dt\end{pmatrix}ds\end{aligned} \tag{14}$$

Extracting the value of A from equation (14) yields

$$\begin{aligned}&\left[\phi(\tau_i)+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\,(h(\tau_i)\phi(\tau_i))ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^{\tau_i}\rho_1(\tau_i,t)\mathrm{G}_1(\phi(t)A)dt+\\ \mu_2\int_0^1\rho_2(\tau_i,t)\mathrm{G}_2(\phi(t)A)dt\end{pmatrix}ds=\right]A\\ &=\Omega(\tau_i)\end{aligned} \tag{15}$$

Equation (14) can be put in the form

$$J(\tau_i)A=\Omega(\tau_i) \tag{16}$$

3.3 Convergence of the method

Theorem 4. (Berinde, 2007a) (Convergence of method) Let (X,ϱ) be a metric space and $T:X\longrightarrow X$ be a continuous mapping and $z_N(\tau),z_{N-1}(\tau)$ are approximate solutions of equation (7). Let $\Delta_N(\tau)=|z_N(\tau)-z_{N-1}(\tau)\,|$ if $lim_{N\to0}(\Delta_N(\tau))\to 0$, then the method converges to exact solution.

Proof. Let $z_1(\tau)\; and\; z_2(\tau)$ be approximated by

$$z_N(\tau)=\sum_{n=0}^{N}a_n\tau^n=\phi(\tau)A\; and\; z_{N-1}(\tau)=\sum_{n=0}^{N}b_n\tau^n=\phi(\tau)B$$

Substitute the approximate solution into equation (8) gives

$$\begin{aligned}Tz_N(\tau)=\Omega(\tau)&-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\,(h(\tau)\phi(\tau)A)ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau\\ &-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^\tau\rho_1(\tau,t)\mathrm{G}_1(\phi(t)A)dt+\\ \mu_2\int_0^1\rho_2(\tau,t)\mathrm{G}_2(\phi(t)A)dt\end{pmatrix}ds\end{aligned}$$

Similarly

$$\begin{aligned}Tz_{N-1}(\tau)=\Omega(\tau)&-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\,(h(\tau)\phi(\tau)B)ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau\\ &-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^\tau\rho_1(\tau,t)\mathrm{G}_1(\phi(t)B)dt+\\ \mu_2\int_0^1\rho_2(\tau,t)\mathrm{G}_2(\phi(t)B)dt\end{pmatrix}ds\end{aligned}$$

$$\begin{aligned}&|Tz_N(\tau)-Ta_{N-1}(\tau)|\\ &=\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\,h(\tau)\phi(\tau)|B-A|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}\begin{pmatrix}\mu_1\int_0^\tau\rho_1(\tau,t)\mathrm{G}_1(\phi(t))|B-A|dt+\\ \mu_2\int_0^1\rho_2(\tau,t)\mathrm{G}_2(\phi(t))|B-A|dt\end{pmatrix}ds\end{aligned}$$

Since$\tau\in[0,1]and\;|B-A|\neq 0$, hence $lim_{N\to0}(\Delta_N(\tau))\to 0$
Hence, the solution method is shown to converge.

4. Numerical Examples

This section presents numerical examples to evaluate the method's applicability and accuracy. Let the approximate and exact solutions be $z_n(\tau) and\ z(\tau)$ respectively.

$$Error_N = |z_n(\tau) - z(\tau)|$$

Example 1. (Hou, Niu, Xu, & Ngolo, 2021) considering first order Volterra-Fredholm integro-differential equation

$$z'(\tau) + z(\tau) + \frac{1}{2}\int_0^\tau \tau z^2(t)dt - \frac{1}{4}\int_0^1 tz^3(t)dt = 2\tau + \tau^2 + \frac{1}{10}\tau^6 - \frac{1}{32} \quad (17)$$

Subject to the initial condition

$$z(0) = 0$$

Exact solution: $z(\tau) = \tau^2$
The approximate solution of equation (17) with $N = 7$ gives

$$z_7 = (-2.13717932240343 \times 10^{-15}\tau^0 + 1.00000000002910\tau^2 + 0.0 \times \tau^3 + 9.31322574615479 \times 10^{-10}\tau^4 + 1.86264514923096 \times 10^{-9}\tau^5 + 1.86264514923096 \times 10^{-9}\tau^6 - 6.98491930961609 \times 10^{-10}\tau^7)$$

Example 2. Hou *et al*. (2021) considering first order Volterra-Fredholm integro-differential equation

$$z'(\tau) + z(\tau) - 2\int_0^\tau sin(\tau)z^2(t)dt = cos(\tau) + (1-\tau)\,sin(\tau) + cos(\tau)\,sin^2(\tau) \quad (18)$$

Subject to the initial conditions

$$y(0) = 0$$

Exact solution: $z(\tau) = \sin(\tau)$
The approximate solution of equation (18) with $N = 7$ yields
$$z_7 = (-5.34615685054973 \times 10^{-11}\tau^0 + 0.999999981849015\tau + 0.000366836415196303\tau^2 - 0.170942933182232\tau^3 + 0.0207648414652795\tau^4 - 0.0440900705289096\tau^5 + 0.0708730777259916\tau^6 - 0.0258030885597691\tau^7)$$

Example 3. Hou *et al*. (2021) Considering first order Volterra-Fredholm integro-differential equation

$$z'(\tau) + \int_0^\tau (\tau^2(t) - 2)dt = \frac{1}{5}\tau^2 \quad (19)$$

subject to the initial conditions
Exact solution: $z(\tau) = \tau^2$
The approximate solution of equation (19) with $N = 8$ yields

$$z_8 = (-4.99600361081320 \times 10^{-16}\tau^0 - 1.42108547152020 \times 10^{-14}\tau + 0.999999999934516\tau^2 + 5.82076609134674 \times 10^{-11}\tau^3 - 2.79396772384644 \times 10^{-9}\tau^4 + 6.05359673500061 \times 10^{-9}\tau^5 - 3.72529029846191 \times 10^{-9}\tau^6 + 2.32830643653870 \times 10^{-9}\tau^7 + 1.16415321826935 \times 10^{-10}\tau^8)$$

5. Results and Discussion

This section presents and discusses the numerical results obtained from the solved examples using the proposed numerical approach.

For Example 1, the results displayed in Table 1 indicate that the approximate solution with N = 7 yields

$$z_7 = (-2.13717932240343 \times 10^{-15}\tau^0 + 1.00000000002910\tau^2 + 0.0 \times \tau^3 + 9.31322574615479 \times 10^{-10}\tau^4 + 1.86264514923096 \times 10^{-9}\tau^5 + 1.86264514923096 \times 10^{-9}\tau^6 - 6.98491930961609 \times 10^{-10}\tau^7)$$

The numerical results converge to the exact solution, confirming that our method performs more favorably than the approach proposed by Hou *et al*. (2021).

In Numerical Example 2, as presented in Table 2, the approximate solution with N=7 yields

$$z_7 = (-5.34615685054973 \times 10^{-11}\tau^0 + 0.999999981849015\tau + 0.000366836415196303\tau^2 - 0.170942933182232\tau^3 + 0.0207648414652795\tau^4 - 0.0440900705289096\tau^5 + 0.0708730777259916\tau^6 - 0.0258030885597691\tau^7)$$

The numerical results demonstrate improved accuracy compared to those obtained by Hou *et al*. (2021) at N = 12.

Table 1. Exact values, approximate values and absolute error for example 1

τ	Exact values	Our method$_{N=7}$	Error$_7$	Hou *et al* (2021) Error$_{12}$
0.2	0.040000000000	0.040000000000	0.00	9.3974e-5
0.4	0.160000000000	0.160000000000	0.00	1.9813e-4
0.6	0.360000000000	0.360000000300	3.00e-10	2.8891e-4
0.8	0.640000000000	0.640000001400	1.40e-9	4.0786e-4
1.0	1.000000000000	1.000000004000	4.00e-9	5.1647e-4

The approximate solution obtained in Example 3 at N = 8 gives

$$z_8 = (-4.99600361081320 \times 10^{-16}\tau^0 - 1.42108547152020 \times 10^{-14}\tau + 0.999999999934516\tau^2 + 5.82076609134674 \times 10^{-11}\tau^3 - 2.79396772384644 \times 10^{-9}\tau^4 + 6.05359673500061 \times 10^{-9}\tau^5 - 3.72529029846191 \times 10^{-9}\tau^6 + 2.32830643653870 \times 10^{-9}\tau^7 + 1.16415321826935 \times 10^{-10}\tau^8)$$

The numerical result converged to the exact solution, confirming that our method performed favorably than the approach proposed in Hou *et al*. (2021), as shown in Table 3.

6. Conclusions

For the numerical solution of the Volterra-Fredholm integral-differential equations, the collocation approach was investigated in this paper. This approach is straightforward, reliable, and effective. For all computations in this work, Maple 18 was employed.

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Author Contributions

Aliyu Muhammad: Conceptualization, Software, Methodology. Aliyu Muhammad: Data curation, Writing - original draft. Solomon Ojobo: Visualization, Investigation. Adewale Jamse: Supervision. Adewale James: Software, Validation. Aliyu Muhammad: Writing - Reviewing and Editing.

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Table 2. Exact values, approximate values and absolute error for example 2

τ	Exact values	Our method$_{N=7}$	Error$_7$	Hou *et al* (2021) Error$_{12}$
0.2	0.1986693308	0.1986704468	1.1160e-6	8.53966e-5
0.4	0.3894183423	0.3894464566	2.81143e-5	1.79180e-4
0.6	0.5646424734	0.5650553891	4.129157e-4	2.5968e-4
0.8	0.7173560909	0.7199374758	2.5813849-3	3.71839e-4
1.0	0.8414709848	0.8511686449	9.6976601e-3	5.01073e-4

Table 3. Exact values, approximate values and absolute error for example 3

τ	Exact values	Our method$_{N=7}$	Error$_7$	Hou *et al* (2021) Error$_{12}$
0.2	0.040000000000	0.040000000000	0.00	4.6875e-17
0.4	0.160000000000	0.160000000000	0.00	2.77556e-17
0.6	0.360000000000	0.360000000000	0.00	1.11022e-16
0.8	0.640000000000	0.640000000300	3.0e-10	2.22045e-16
1.0	1.000000000000	1.000000002000	2.0e-9	1.11022e-15

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