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Original Article

Explicit analytical solutions for the average run length of modified EWMA control chart for ARX(p,r) processes

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Abstract

Statistical process control (SPC) plays a necessary role in manufacturing industry processes. An essential tool for SPC used for monitoring, measuring, controlling, and improving quality in various fields is the control chart. The modified exponentially weighted moving average (modified EWMA) control chart is widely used in various fields, and a measure commonly used to elucidate its efficiency is average run length (*ARL*). The main purpose of this study is to derive explicit formulas for the *ARL* to detect changes in the process mean of modified EWMA control chart for an autoregressive processes with explanatory variables (ARX(p,r)) with exponential white noise. In addition, the performances of the modified EWMA are compared with EWMA control charts based on the relative mean index (RMI). It was found that the explicit formulas for the *ARL* of the modified EWMA control chart performed better than on the EWMA control chart for monitoring process mean.

Keywords: average run length, modified EWMA control chart, ARX process, explanatory variables, Integral equation

1. Introduction

Statistical process control (SPC) plays a necessary role in manufacturing industry processes and is used for monitoring, measuring, controlling and improving quality in various fields (science, economics, engineering, finance, medicine, etc.). An important SPC tool is the control chart which is used for detecting changes in process means. The first control chart, introduced by Shewhart (1931), was used for detecting large shifts in process means $(>1.5\sigma)$ (Montgomery, 2012). The cumulative sum (CUSUM) control chart proposed by Page (1954) is a good alternative to the Shewhart control chart for detecting small shifts in process means $(<1.5\sigma)$ (Montgomery, 2012), as has been indicated by comparative studies on the two (Hawkins & Olwell, 1998; Lucas & Saccucci, 1990). In addition, another option for detecting small shifts is the exponentially weighted moving average (EWMA) control chart first presented by Roberts (1959), which has been used in various industries, However, these charts cannot be used directly for chemical and pharmaceutical processes due to the observations being frequently autocorrelated (Patel & Divecha, 2011). The EWMA technique is used in SPC to monitor the results of manufacturing processes by tracking the moving average of the efficiency throughout the lifetime of the process.

The modified EWMA control chart developed by Patel and Divecha (2011) is a simplified EWMA control chart for detecting shifts in the process mean regardless of size. It is used in various fields, especially in the chemical industry, in which the processes are frequently autocorrelated. Past observations are considered (similar to the EWMA scheme) along with past changes as well as the latest change in the process mean. Khan, Aslam and Jun (2016) developed a new EWMA control chart based upon a modified EWMA statistic that considers the past and current behavior of the process; they compared it with the existing one by Patel and Divecha (2011) and found that the proposed control chart had the ability to detect shifts more quickly.

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A commonly used measure for the efficiency of control charts is the average run length (*ARL*). Ryu, Wan and Kim (2010) applied in-control *ARL* (ARL_0), which refers to the average number of observations on the in-control process before a false out-of-control alarm is raised, as a measure of the false-alarm rate. The out-of-control *ARL* (ARL_1), is the average number of observations required to detect a specific process mean shift and represents the ability to detect shifts in the process mean.

Various methods that can be used to find the *ARL* of control charts have been proposed, such as Monte Carlo simulation, Markov chains, Martingales, numerical integration equations (NIEs), and explicit formulas. A NIE is a method for evaluating the *ARL* that has many rules, namely the midpoint rule, trapezoidal rule, Simson's rule, and the Gauss-Legendre rule. In this study, we used the Gauss-Legendre rule. An explicit formula is a method for evaluating the *ARL* that requires an integral equation for its derivation. In this study, we used the Fredholm integral equation of the second type (Mititelu, Areepong, Sukparungsee, & Novikov, 2010). In 1959, Robert (1959) proposed an EWMA control chart by using Monte Carlo simulations to estimate the *ARL*. Crowder (1987) used an NIE approach to find the *ARL* for a Gaussian distribution. Harris and Ross (1991) studied CUSUM with serially correlated observations via Monte Carlo simulations. Mititelu *et al.*, (2010) used a linear Fredholm-type integral equation approach to derive explicit formulas for the *ARL* in certain special cases. The *ARL* for a CUSUM control chart has been found when the random observations follow a hyperexponential distribution and the *ARL* for an EWMA control chart with observations following a Laplace distribution. Suriyakat, Areepong, Sukparungsee and Mititelu (2012) derived explicit formulas for the *ARL* of the EWMA statistic for first-order autoregressive (AR(1)) observations with errors following an exponential white noise process. Paichit (2016) used an NIE to find the exact expression for the *ARL* of an EWMA control chart for an AR process with exogenous input (ARX(p)). Paichit (2017) presented an exact expression for the *ARL* of the control chart for an ARX(p) procedure. Explicit formulas for the *ARL* of a modified EWMA control chart for an exponential AR(1) process were presented by Phanthuna, Areepong and Sukparungsee (2018).

The main purpose of this study is to derive explicit formulas for the *ARL* for detecting changes in the process mean of modified EWMA control chart based on Khan *et al.* (2016) for an autoregressive processes with explanatory variables (ARX(p,r)) with exponential white noise. In the present study, Fredholm-type integral equations are used to derive explicit formulas of ARL_0 and ARL_1. This paper is organized as follows. An introduction to the properties of control charts and the model for an ARX(p,r) process with exponential white noise is given in Section 2. The solutions for the *ARL*s of the EWMA and modified EWMA control charts for an ARX(p,r) process with exponential white noise are presented in Section 3. Next, the NIEs for the *ARL*s of the modified EWMA control charts are introduced in Section 4. Furthermore, numerical results for a comparison of the *ARL*s on the modified EWMA control charts for ARX(p,r) process with exponential white noise are offered in Section 5 and Section 6. The proposed explicit formulas are applied in Section 7. Finally, conclusions are given in Section 8.

2. The Properties of the Control Charts and ARX(p,r) Process with Exponential White Noise

2.1 The EWMA control chart

Roberts (1959) introduced the EWMA control chart for detecting small shifts in the process mean that is defined as

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t \qquad ;t = 1,2,3,... , \tag{2.1}$$

where Z_t is the EWMA statistic, Y_t is the sequence of the ARX(p,r) process with exponential white noise and λ is an exponential smoothing parameter $(0 < \lambda \le 1)$.

The stopping time will occur when an out-of-control observation is firstly detected, which is sufficient to decide that the process is out-of-control. The stopping time τ_b for the EWMA control chart can be written as

$$\tau_b = \inf\{t > 0;\ Z_t > b\}, \tag{2.2}$$

where b is a constant parameter known as the upper control limit $(b > 0)$. The upper side of the *ARL* for the ARX(p,r) process on the EWMA control chart with an initial value $(Z_0 = u)$ can be found. Now, the function $L(u)$ is defined as

$$L(u) = ARL = E_\infty(\tau_b) \ge T,\ Z_0 = u. \tag{2.3}$$

The mean and the variance of the EWMA control chart can be written as

$$E(Z_t) = \mu, \tag{2.4}$$

$$Var(Z_t) = \left(\frac{\lambda}{2-\lambda}\right)\sigma^2, \tag{2.5}$$

and the upper control limit (UCL) and the lower control limit (LCL) of the EWMA control chart is defined as follows:

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{2.6a}$$

$$CL = \mu_0, \tag{2.6b}$$

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{2.6c}$$

where μ_0 is the target mean, σ is the process standard deviation and L is an appropriate control width limit $(L > 0)$.

2.2 The modified EWMA control chart

Khan *et al.* (2016) developed a new EWMA control chart based upon the modified EWMA statistic that considers the past and current behavior of the process. They compared the proposed chart with the existing modified EWMA control chart developed by Patel and Divecha (2011) that is a simplified EWMA control chart for detecting shifts of all sizes in the process mean under the assumption that the observations follow a normal distribution. The modified EWMA control chart proposed by Khan *et al.* (2016) is defined as

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}) \qquad ;t = 1,2,3,..., \tag{2.7}$$

where M_t is the modified EWMA statistic, Y_t is the sequence of the ARX(p,r) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, and k is a constant $(k > 0)$.

The modified EWMA control chart is based on two constants, λ and k, and comprises an extension to the existing EWMA control chart. The modified EWMA control chart by Khan *et al.* (2016) is reduced to the original EWMA control chart by Roberts (1959) if $k = 0$ and is reduced to the control chart based on the modified EWMA control chart by Patel and Divecha (2011) if $k = 1$.

The stopping time τ_h for the modified EWMA control chart can be written as

$$\tau_h = \inf\{t > 0;\ M_t > h\}, \tag{2.8}$$

where h is a constant parameter known as the upper control limit $(h > 0)$. The upper side of the *ARL* for the ARX(p,r) process on the modified EWMA control chart with an initial value ($M_0 = u$) can be found. Now, we define the function $G(u)$ as

$$ARL = G(u) = E_\infty(\tau_h) \ge T,\ M_0 = u, \tag{2.9}$$

where *T* is a fixed number (should be large) and $E_\infty(.)$ is the expectation under the assumption that observations ε_t have the distribution $F(y_t,\alpha)$.

The value of the mean and the variance of the modified EWMA control chart is defined as

$$E(M_t) = \mu, \tag{2.10}$$

$$Var(M_t) = \frac{(\lambda + 2\lambda k + 2k^2)\sigma^2}{(2-\lambda)}, \tag{2.11}$$

and the *UCL* and the *LCL* of the modified EWMA control chart can be written as

$$LCL = \mu_0 - L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{2.12a}$$

$$CL = \mu_0, \tag{2.12b}$$

$$UCL = \mu_0 + L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{2.12c}$$

where μ_0 is the target mean, σ^2 is the process variance and L is an appropriate control width limit $(L > 0)$.

2.3 The ARX(p,r) process with exponential white noise

The ARX(p,r) process is defined as

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \sum_{j=1}^{r} \beta_j X_j + \varepsilon_t \qquad ;t = 1,2,3,..., \tag{2.13}$$

where δ is a constant $(\delta \ge 0)$, ϕ_i is an autoregressive coefficient for $i = 1,2,...,p$ $(-1 < \phi_i < 1)$; ε_t is an independent and identically distributed (iid) sequence; $\varepsilon_t \sim Exp(\alpha)$; X_j are explanatory variables of Y_t; and β_j are coefficients of

$X_j\ ; j=1,2,...,r$. The initial value for the ARX(p,r) process mean is $Y_{t-1},Y_{t-2},...,Y_{t-p}=1$ and the initial value for the explanatory variables $X_1,X_2,...,X_r=1$.

3. Explicit Formulas for the *ARL*s of the Modified EWMA Control Chart for an ARX(p,r) Process with Exponential White Noise

Explicit formulas for the *ARL* of the modified EWMA control chart for an ARX(p,r) process are derived as follows:

Substituting Y_t from Equation (2.13) into Equation (2.7), then

$$M_t=(1-\lambda)M_{t-1}+(\lambda+k)\delta+(\lambda+k)\phi_1 Y_{t-1}+...+(\lambda+k)\phi_p Y_{t-p} \\ +(\lambda+k)\sum_{j=1}^{r}\beta_j X_j+(\lambda+k)\varepsilon_t-kY_{t-1}\ .$$

If Y_1 gives the out-of-control state for M_1, $M_0=u$ and $Y_0=v$, then

$$M_1=(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1 v+...+(\lambda+k)\phi_p v+(\lambda+k)\sum_{j=1}^{r}\beta_j X_j+(\lambda+k)\varepsilon_1-kv$$

If ε_1 is the in-control limit for M_1, then $0\le M_1\le h$.

The function $G(u)$ can be derived by the Fredholm integral equation of the second type (Mititelu *et al.*, 2010), and thus $G(u)$ can be written as

$$G(u)=1+\int G(M_1)f(\varepsilon_1)d(\varepsilon_1)\ , \tag{3.1}$$

by substituting ε_1 with y. Therefore, the function $G(u)$ is obtained as

$$G(u)=1+\int_0^h L\left\{\begin{matrix}(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1 v+...+(\lambda+k)\phi_p v\\ +(\lambda+k)\sum_{j=1}^{r}\beta_j X_j-kv+(\lambda+k)y\end{matrix}\right\}f(y)dy\cdot$$

Let $w=(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1 v+...+(\lambda+k)\phi_p v+(\lambda+k)\sum_{j=1}^{r}\beta_j X_j-kv+(\lambda+k)y.$

By changing the integral variable, we obtain the integral equation as follows:

$$G(u)=1+\frac{1}{\lambda+k}\int_0^h G(w)f\left\{\frac{w-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}dw\cdot \tag{3.2}$$

If $Y_t \sim Exp(\alpha)$ the $f(y)=\frac{1}{\alpha}e^{\frac{-y}{\alpha}}\ ;\ y\ge 0$, so

$$G(u)=1+\frac{1}{\lambda+k}\int_0^h G(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}}dw\cdot$$

Let the function $D(u)=e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}$, then we have

$$G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\int_0^h L(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw \qquad ;0\le u\le h\cdot$$

Let $g=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw$, then $G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\cdot g$. Consequently, we obtain

$$G(u)=1+\frac{1}{\alpha(\lambda+k)}e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}\cdot g\cdot \tag{3.3}$$

Solving a constant g

$$g=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw \;=\int_0^h\left[1+\frac{g}{\alpha(\lambda+k)}D(w)\right]e^{\frac{-w}{\alpha(\lambda+k)}}dw$$

$$=\int_0^h e^{\frac{-w}{\alpha(\lambda+k)}}dw+\int_0^h\frac{g}{\alpha(\lambda+k)}D(w)\cdot e^{\frac{-w}{\alpha(\lambda+k)}}dw$$

$$=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)+\frac{ge^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\alpha(\lambda+k)}\int_0^h e^{\frac{-\lambda w}{\alpha(\lambda+k)}}dw$$

$$=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)-\frac{de^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)$$

$$g=\frac{-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)}{1+\frac{e^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)}. \tag{3.4}$$

Substituting g from Equation (3.4) into Equation (3.3), then

$$G(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}}\left[e^{\frac{-h}{\alpha(\lambda+k)}}-1\right]}{\lambda e^{\frac{kv}{\alpha(\lambda+k)}-\frac{\delta}{\alpha}-\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}-\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}+\left[e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right]}. \tag{3.5}$$

When the process is in the in-control state with exponential parameter $\alpha = \alpha_0$, we obtain the explicit solution for ARL_0 as follows:

$$ARL_0 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_0(\lambda+k)}}\left[e^{\frac{-h}{\alpha_0(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kv}{\alpha_0(\lambda+k)} - \frac{\delta}{\alpha_0} - \frac{v\sum_{i=1}^{p}\phi_i}{\alpha_0} - \frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_0}} + \left[e^{\frac{-\lambda h}{\alpha_0(\lambda+k)}} - 1\right]}. \tag{3.6}$$

Similarly, when the process is in the out-of-control state with exponential parameter $\alpha = \alpha_1$, the explicit solution for ARL_1 can be written as

$$ARL_1 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_1(\lambda+k)}}\left[e^{\frac{-h}{\alpha_1(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kv}{\alpha_1(\lambda+k)} - \frac{\delta}{\alpha_1} - \frac{v\sum_{i=1}^{p}\phi_i}{\alpha_1} - \frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_1}} + \left[e^{\frac{-\lambda h}{\alpha_1(\lambda+k)}} - 1\right]}. \tag{3.7}$$

4. The NIE for the *ARL* on the Modified EWMA Control Chart

An integral equation of the second type for the *ARL* on the modified EWMA control chart for the ARX(p,r) process in Equation (3.5) can be approximated by using the quadrature formula. In this study, the Gauss-Legendre quadrature rule is applied as follows:

Given $$f\left(a_j\right) = f\left\{\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}. \tag{4.1}$$

The approximation for the integral is in the form

$\int_0^h G(w)\, f(w)\,dw \approx \sum_{j=1}^{m} w_j f\left(a_j\right)$, where $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{h}{m}; j = 1,2,...,m$.

Using the quadrature formula, the numerical approximation $\tilde{G}(u)$ for the integral equation can be found as a solution of the linear equations as follows:

$$\tilde{G}\left(a_i\right) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}\left(a_j\right) f\left\{\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}; i = 1,2,...,m.$$

Thus,

$$\tilde{G}\left(a_1\right) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}\left(a_j\right) f\left\{\frac{a_j - (1-\lambda)a_1}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\},$$

$$\tilde{G}\left(a_2\right) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}\left(a_j\right) f\left\{\frac{a_j - (1-\lambda)a_2}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\},$$

$$\tilde{G}(a_3)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_3}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\},$$

$$\vdots$$

$$\tilde{G}(a_m)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_m}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}.$$

The set of m equations with m unknowns can be rewritten in matrix form. The column vector of $\tilde{G}(a_i)$ is $\mathbf{G}_{m\times 1}=(\tilde{G}(a_1),\tilde{G}(a_2),...,\tilde{G}(a_m))'$. Since $\mathbf{1}_{m\times 1}=(1,1,...,1)'$ is a column vector of ones and $\mathbf{R}_{m\times m}$ is a matrix, we can define m to m^{th} as elements of matrix $\mathbf{R}$ as follows:

$$\left[R_{ij}\right]\approx\frac{1}{\lambda+k}w_j\,f\left\{\frac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\},$$

and $\mathbf{I}_m=diag(1,1,...,1)$ is a unit matrix of order m. If $(\mathbf{I}-\mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation in terms of the matrix can be written as

$$\mathbf{G}_{m\times 1}=\left(\mathbf{I}_m-\mathbf{R}_{m\times m}\right)^{-1}\mathbf{1}_{m\times 1}.$$

Finally, by substituting a_i by u in $\tilde{G}(a_i)$, the numerical integration equation for function $\tilde{G}(u)$ can be derived as

$$\tilde{G}(u)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}. \tag{4.2}$$

5. Comparison of the NIE Method and the Explicit Formulas

Here, a comparison of the efficacies of the NIE method ($\tilde{G}(u)$) and the explicit formulas ($G(u)$) for the *ARL* of an ARX(p,r) process on the modified EWMA control chart is presented. The parameter values were set as $ARL_0=370$ and 500; $\lambda=$ 0.01, 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0=1$; and the shift size was varied as 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.3, and 0.5. In general, the popular setting of the initial value is equal to the expected value of the distribution. For setting the in-control parameter, $\alpha_0=1$, which is the initial value as 1. The coefficient has a value from -1 to 1, which can be specified as any value. The configuration does not affect the accuracy of the explicit formulas and the NIE methods. The absolute percentage difference to measure the accuracy of the *ARL* is defined as

$$Diff\,(\%)=\frac{\left|G(u)-\tilde{G}(u)\right|}{G(u)}\times 100. \tag{5.1}$$

Equations (3.5) and (4.2) are used to evaluate the *ARL* on the modified EWMA control chart for an ARX(p,r) process with exponential white noise. The number of nodes equal to 500 iterations was used to obtain the *ARL* results from the NIE method. The computations for the NIE method were carried out on a Windows 7 Professional 32-bit PC System with RAM of 2 GB and an AMD E1-1200 CPU.

The results in Tables 1–3 report the numerical values of the *ARL* derived from the explicit formulas and NIE method, and the absolute percentage difference between them. From the results, we can see that the *ARL* values derived from the explicit formulas give the same results as the NIE method. The numerical approximations had an absolute percentage difference of less than 0.003%. However, the computational time of the NIE method was 13.42–13.58 s whereas that of the explicit formulas was < 1 s.

Table 1. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(1,1) with $u=1,$ $v=1,$ $k=1,$ $\beta_1=0.2,$ and $ARL_0=370$.

λ	δ	ϕ_1	h	shift	Explicit	NIE	Time[a]	Diff%
0.05	0	0.1	2.11284	0.00	370.514622	370.514167	13.468	0.000123
				0.01	185.632808	185.632635	13.555	0.000093
				0.02	123.215119	123.215018	13.494	0.000082
				0.03	91.885107	91.885037	13.428	0.000076
				0.04	73.065410	73.065357	13.551	0.000072
				0.05	60.520993	60.520951	13.552	0.000068
				0.10	32.116753	32.116734	13.504	0.000058
				0.30	10.731776	10.731772	13.546	0.000037
				0.50	6.457709	6.457707	13.440	0.000026
		-0.1	2.61195	0.00	370.424900	370.424156	13.511	0.000201
				0.01	274.686377	274.685905	13.563	0.000172
				0.02	214.349128	214.348800	13.478	0.000153
				0.03	173.294237	173.293995	13.511	0.000140
				0.04	143.823485	143.823298	13.446	0.000130
				0.05	121.811602	121.811453	13.445	0.000122
				0.10	64.531777	64.531713	13.421	0.000098
				0.30	17.824627	17.824616	13.541	0.000060
				0.50	9.519466	9.519462	13.452	0.000042
0.10	1	0.2	0.69141	0.00	370.555941	370.555865	13.502	0.000021
				0.01	90.098635	90.098626	13.498	0.000010
				0.02	51.385975	51.385971	13.460	0.000008
				0.03	35.997312	35.997309	13.434	0.000007
				0.04	27.736604	27.736602	13.467	0.000007
				0.05	22.584474	22.584472	13.485	0.000006
				0.10	11.823565	11.823565	13.518	0.000005
				0.30	4.369675	4.369675	13.506	0.000003
				0.50	2.902154	2.902154	13.467	0.000002
		-0.2	1.04870	0.00	370.273926	370.273736	13.537	0.000052
				0.01	105.208634	105.208608	13.519	0.000025
				0.02	61.437849	61.437836	13.532	0.000020
				0.03	43.452288	43.452281	13.542	0.000018
				0.04	33.653645	33.653639	13.449	0.000017
				0.05	27.489870	27.489866	13.471	0.000016
				0.10	14.478770	14.478768	13.508	0.000013
				0.30	5.327500	5.327500	13.488	0.000008
				0.50	3.494115	3.494115	13.434	0.000005

[a]The computational times for the NIE methods in seconds (PC System: Windows 7 Professional 32-bit, RAM: 2 GB and CPU: AMD E1-1200)

Table 2. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(2,1) with $u=1,$ $v=1,$ $\delta=0,$ $k=1,$ $\beta_1=0.2,$ and $ARL_0=370$.

λ	ϕ_1	ϕ_2	h	shift	Explicit	NIE	Time	Diff%
0.05	0.1	0.1	1.90196	0.00	370.104536	370.104178	13.492	0.000097
				0.01	164.156587	164.156470	13.512	0.000072
				0.02	105.192627	105.192560	13.467	0.000063
				0.03	77.258066	77.258020	13.551	0.000059
				0.04	60.970239	60.970204	13.466	0.000056
				0.05	50.307351	50.307324	13.532	0.000054
				0.10	26.685128	26.685116	13.545	0.000046
				0.30	9.216663	9.216660	13.436	0.000029
				0.50	5.688211	5.688210	13.529	0.000020
		-0.1	2.34842	0.00	370.111274	370.110694	13.520	0.000157
				0.01	218.233870	218.233600	13.487	0.000124
				0.02	153.301535	153.301368	13.495	0.000109
				0.03	117.350069	117.349952	13.416	0.000100
				0.04	94.565641	94.565553	13.479	0.000094
				0.05	78.865508	78.865438	13.492	0.000089
				0.10	41.847694	41.847663	13.492	0.000074
				0.30	13.172037	13.172030	13.492	0.000047
				0.50	7.603031	7.603028	13.517	0.000033

Table 2. Continued.

λ	ϕ_1	ϕ_2	h	shift	Explicit	NIE	Time	Diff%
0.10	0.1	0.2	1.78838	0.00	370.299172	370.298515	13.519	0.000178
				0.01	144.464904	144.464768	13.518	0.000094
				0.02	89.685034	89.684969	13.459	0.000073
				0.03	65.007924	65.007883	13.455	0.000063
				0.04	50.973254	50.973225	13.535	0.000057
				0.05	41.920612	41.920590	13.490	0.000052
				0.10	22.222112	22.222103	13.425	0.000041
				0.30	7.893825	7.893823	13.482	0.000024
				0.50	4.987173	4.987172	13.499	0.000016
		-0.2	2.78993	0.00	370.115966	370.114002	13.501	0.000531
				0.01	286.090889	286.089671	13.542	0.000426
				0.02	228.394081	228.393273	13.492	0.000354
				0.03	187.010941	187.010376	13.531	0.000302
				0.04	156.285272	156.284860	13.569	0.000264
				0.05	132.822686	132.822375	13.438	0.000234
				0.10	70.157635	70.157527	13.440	0.000153
				0.30	18.713286	18.713273	13.557	0.000072
				0.50	9.832924	9.832919	13.520	0.000047
0.20	0.2	0.1	1.95666	0.00	370.295141	370.292831	13.437	0.000624
				0.01	137.992552	137.992202	13.490	0.000253
				0.02	84.820251	84.820109	13.462	0.000168
				0.03	61.246548	61.246469	13.427	0.000130
				0.04	47.941844	47.941792	13.486	0.000108
				0.05	39.398626	39.398589	13.534	0.000093
				0.10	20.908407	20.908394	13.466	0.000060
				0.30	7.523560	7.523557	13.512	0.000028
				0.50	4.802271	4.802270	13.539	0.000018
		-0.1	2.49307	0.00	370.002909	369.998734	13.478	0.001128
				0.01	183.558057	183.556989	13.456	0.000582
				0.02	121.435201	121.434716	13.457	0.000399
				0.03	90.409051	90.408773	13.539	0.000307
				0.04	71.822583	71.822402	13.520	0.000252
				0.05	59.455236	59.455108	13.515	0.000215
				0.10	31.519801	31.519760	13.498	0.000129
				0.30	10.562568	10.562562	13.519	0.000054
				0.50	6.382784	6.382781	13.433	0.000033

Table 3. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(2,1) with $u=1,\ v=1,\ \delta=0,\ k=1,\ \phi_1=0.1,\ \phi_2=-0.2,\ \beta_1=0.2,$ and $ARL_0=500$.

shift	$\lambda=0.01,\ h=2.48512$				$\lambda=0.05,\ h=2.61385$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.00	500.488509	500.487979	13.458	0.000106	500.640854	500.639672	13.476	0.000236
0.01	331.637724	331.637386	13.484	0.000102	340.395524	340.394878	13.463	0.000190
0.02	243.153820	243.153579	13.448	0.000099	252.391615	252.391203	13.531	0.000163
0.03	189.153991	189.153809	13.490	0.000096	197.352918	197.352629	13.536	0.000146
0.04	153.035120	153.034976	13.469	0.000094	160.016146	160.015932	13.469	0.000134
0.05	127.342375	127.342258	13.468	0.000092	133.230089	133.229922	13.533	0.000125
0.10	64.915512	64.915458	13.506	0.000083	67.592092	67.592025	13.497	0.000099
0.30	17.675818	17.675808	13.491	0.000056	18.040792	18.040781	13.502	0.000060
0.50	9.459167	9.459163	13.546	0.000040	9.576707	9.576703	13.510	0.000042
shift	$\lambda=0.1,\ h=2.79202$				$\lambda=0.2,\ h=3.22379$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.00	500.386457	500.383045	13.496	0.000682	500.609662	500.595309	13.455	0.002867
0.01	358.298531	358.296709	13.537	0.000508	434.146693	434.136038	13.552	0.002454
0.02	272.247864	272.246767	13.458	0.000403	368.724286	368.716686	13.482	0.002061

Table 3. Continued.

shift	$\lambda=0.1,\ h=2.79202$				$\lambda=0.2,\ h=3.22379$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.03	215.455768	215.455048	13.496	0.000334	310.585722	310.580379	13.495	0.001720
0.04	175.682830	175.682330	13.486	0.000285	261.603168	261.599405	13.447	0.001438
0.05	146.584892	146.584527	13.469	0.000249	221.388663	221.385984	13.530	0.001210
0.10	73.817022	73.816906	13.579	0.000157	108.236492	108.235857	13.506	0.000586
0.30	18.955138	18.955124	13.516	0.000072	23.381342	23.381309	13.507	0.000139
0.50	9.895165	9.895160	13.461	0.000047	11.381464	11.381456	13.513	0.000072

6. Comparison of the *ARL*s on the EWMA with modified EWMA control charts

After verifying the accuracy of the explicit formulas in the previous section, we used simulated data and the relative mean index (RMI) to compare the performances of the *ARL* of an ARX(p,r) process on EWMA and modified EWMA control charts. The RMI is defined as

$$RMI=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i}-Min\left[ARL_{shift,i}\right]}{Min\left[ARL_{shift,i}\right]}\right). \tag{6.1}$$

where $ARL_{shift,i}$ is the *ARL* of the control chart when the position process shift, $shift,i$ is the shift size for $i=1,2,...,n$, $Min\left[ARL_{shift,i}\right]$ denotes the smallest *ARL* of two control charts in comparison when the position process shift. The control chart with the smallest RMI performs the best in detecting mean changes on the whole.

For the comparison of the *ARL*s on the EWMA and modified EWMA control charts for an ARX(1,1) process, the parameter values were set as $ARL_0=370$; $\lambda=$ 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0=1$; the shift size was varied as 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07 and 0.09. The results are reported in Table 4.

Table 4. Comparison of the *ARL* of EWMA and modified EWMA control charts using explicit formulas for an ARX(1,1) with $u=1,\ v=1,\ \delta=0,\ k=40\lambda,\ \beta_1=0.2,$ and $ARL_0=370$.

shift	$\lambda=0.05,\ \phi_1=0.2$		$\lambda=0.1,\ \phi_1=0.2$		$\lambda=0.2,\ \phi_1=0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=2.5496\times10^{-8}$	$h=1.3590441$	$b=0.00107964$	$h=2.7665764$	$b=0.04441$	$h=5.734902$
0.000	370.071291	370.0768919	370.004307	370.003373	370.722568	370.715572
0.001	362.264617	257.030787*	365.787507	239.166659*	362.075941	233.363671*
0.003	347.186399	159.726241*	357.521610	140.347811*	345.767724	134.327916*
0.005	332.792484	115.984299*	349.473732	99.484050*	330.655613	94.494822*
0.007	319.049437	91.124031*	341.637371	77.151829*	316.615806	72.994088*
0.009	305.925567	75.090081*	334.006238	63.075235*	303.540954	59.537517*
0.010	299.586374	69.033841*	330.265724	57.823254*	297.335678	54.535706*
0.030	198.799944	26.743266*	264.852791	22.105174*	206.492192	20.787108*
0.050	134.056104	16.821242*	214.163689	13.957965*	153.166554	13.153305*
0.070	91.808369	12.393346*	174.541323	10.348427*	118.601180	9.778282*
0.090	63.828457	9.886695*	143.313881	8.311410*	94.690406	7.875239*
RMI	3.763440	0	7.445925	0	5.824404	0
shift	$\lambda=0.05,\ \phi_1=-0.2$		$\lambda=0.1,\ \phi_1=-0.2$		$\lambda=0.2,\ \phi_1=-0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=3.8036\times10^{-8}$	$h=2.0452044$	$b=0.0016149$	$h=4.200333$	$b=0.067334$	$h=8.88252$
0.000	370.075490	370.074437	370.035318	370.040056	370.017072	370.021854
0.001	362.413120	272.589846*	365.971526	257.425111*	362.407567	254.808465*
0.003	347.604772	178.698291*	358.000435	160.267524*	347.947355	157.288491*
0.005	333.457209	133.038467*	350.232942	116.523643*	334.419802	113.936839*

Table 4. Continued.

shift	$\lambda=0.05,\ \phi_1=-0.2$		$\lambda=0.1,\ \phi_1=-0.2$		$\lambda=0.2,\ \phi_1=-0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=3.8036\times10^{-8}$	$h=2.0452044$	$b=0.0016149$	$h=4.200333$	$b=0.067334$	$h=8.88252$
0.007	319.938646	106.040681*	342.663148	91.643263*	321.740040	89.431083*
0.009	307.018935	88.205073*	335.285344	75.588906*	309.833149	73.676321*
0.010	300.774653	81.379575*	331.666704	69.523408*	304.148484	67.735772*
0.030	201.120167	32.306069*	268.119397	27.141638*	218.327779	26.405507*
0.050	136.616346	20.420367*	218.476210	17.187944*	165.516539	16.743211*
0.070	94.217093	15.067503*	179.369496	12.742094*	130.215179	12.432517*
0.090	65.938916	12.022993*	148.318390	10.222777*	105.253345	9.990907*
RMI	3.062808	0	6.078151	0	4.802528	0

*The smallest *ARL* on each shift size according to the case.

For the *ARL* comparison for an ARX(2,1) process on the EWMA and modified EWMA control charts, the parameter values were set as $ARL_0=500$; $\lambda=$ 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0=1$; shift sizes of 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07 and 0.09. The results are reported in Table 5.

Table 5. Comparison of the *ARL* of EWMA and modified EWMA control charts using explicit formulas for an ARX(2,1) with $u=1,\ v=1,$ $\delta=2,\ k=50\lambda,\ \beta_1=0.3$ and $ARL_0=500$.

shift	$\lambda=0.05,\ \phi_1=0.1,\ \phi_1=-0.2$		$\lambda=0.1,\ \phi_1=0.1,\ \phi_1=-0.1$		$\lambda=0.2,\ \phi_1=0.1,\ \phi_1=0.1$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=1.5491\times10^{-8}$	$h=0.7568155$	$b=0.00058356$	$h=1.378502$	$b=0.014952$	$h=2.277413$
0.000	500.060978	500.076069	500.324113	500.360304	500.029224	500.022472
0.001	489.114475	267.090908*	494.127012	239.961773*	480.592452	221.762202
0.003	467.997069	138.496237*	481.998019	117.893325*	445.500726	105.295630
0.005	447.870153	93.629467*	470.213191	78.325628*	414.691287	69.234304
0.007	428.683824	70.801160*	458.761545	58.745762*	387.433378	51.680381
0.009	410.390862	56.975292*	447.632483	47.062407*	363.153404	41.295851
0.010	401.565333	51.923525*	442.185713	42.825869*	351.983744	37.546730
0.030	262.306360	19.020934*	347.882344	15.647366*	210.659066	13.700820
0.050	174.194272	11.857221*	276.201620	9.822354*	143.484292	8.635854
0.070	117.530522	8.724851*	221.192183	7.284761*	104.938276	6.434339
0.090	80.526574	6.969808*	178.593387	5.865033*	80.339327	5.203939
RMI	7.449113	0	14.096343	0	9.179088	0
shift	$\lambda=0.05,\ \phi_1=0.2,\ \phi_1=-0.1$		$\lambda=0.1,\ \phi_1=0.2,\ \phi_1=0.1$		$\lambda=0.2,\ \phi_1=0.2,\ \phi_1=0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=1.2685\times10^{-8}$	$h=0.6187533$	$b=0.00043188$	$h=1.0175587$	$b=0.0122187$	$h=1.85691$
0.000	500.141339	500.150484	500.174945	500.189478	500.055307	500.024614
0.001	489.095541	259.298533*	493.822612	228.318970*	479.729360	214.054265*
0.003	467.792945	132.316088*	481.395919	109.704783*	443.211639	100.188030*
0.005	447.497611	88.958619*	469.329752	72.374872*	411.339382	65.592229*
0.007	428.158412	67.081769*	457.612366	54.098310*	383.287724	48.862371*
0.009	409.726985	53.891761*	446.232438	43.253693*	358.416006	38.999212*
0.010	400.837290	49.083498*	440.665594	39.332505*	347.010479	35.444488*
0.030	260.830440	17.912907*	344.589520	14.318623*	204.669047	12.911676*
0.050	172.580252	11.160506*	272.008910	8.990081*	138.296605	8.143100*
0.070	116.034271	8.212396*	216.634577	6.672959*	100.601078	6.072667*
0.090	79.236668	6.562091*	173.992869	5.378034*	76.705241	4.916320*
RMI	7.883248	0	15.215424	0	9.476407	0

*The smallest *ARL* on each shift size according to the case.

From the results in Tables 4 and 5, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, in Table 4, when $\phi_1 = 0.2$, $\lambda = 0.05$ and shift = 0.05, the *ARL* is 332.792484 from the EWMA control chart while the *ARL* is 115.984299 from the modified EWMA control chart, which corresponds to the RMI values for the modified EWMA control chart being less than those for the EWMA control chart.

7. Application

In Section 6, we compared the performance of the *ARL* of an ARX(p,r) process on EWMA and modified EWMA control charts by using simulation data. The results show that the *ARL* values derived from the explicit formulas for the modified EWMA control chart were shorter than those for the EWMA control chart in every case. Hence, we applied the explicit formulas for the *ARL*s on the EWMA and modified EWMA control charts for an ARX(1,1) process using 55 real-world data observations on the value of exports and imports of agricultural products to and from Thailand (Unit: Ten billion baht) from January 2016 to July 2019, where the value of the imports is the explanatory variable (data from the Office of Agricultural Economics of Thailand (2019)) to confirm the above results. The parameters were set as $\lambda = 0.05$, 0.1, and 0.2; $\alpha_0 = u = 0.589259$; shift size values of 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07, and 0.09; and autoregressive coefficients $\phi_1 = 0.326152$, $\delta = 6.652233$, $v = 10.4918$, $\beta_1 = 0.933313$, and $X_1 = 4.1439$. The results are given in Table 6.

Another *ARL* comparison for an ARX(2,1) process on the modified EWMA control charts was conducted using real-world data on the price of cassava (unit: Baht per kilogram, data from the Office of Agricultural Economics of Thailand (2019)) and diesel oil (unit: Baht per liter, data from Petroleum Authority of Thailand (2019)), with the latter being the explanatory variable. The parameters used were $\lambda = 0.1$, 0.15 and 0.2; $\alpha_0 = u = 0.136281$; shift size values of 0.0001, 0.0003, 0.0005, 0.0007, 0.0009, 0.001, 0.003, 0.005, 0.007, and 0.009; and autoregressive coefficients $\phi_1 = 0.623567$ and $\phi_2 = 0.292098$, $\delta = 0$, $v = 1.88$, $\beta_1 = 0.064905$, and $X_1 = 25.62$. The results are summarized in Table 7.

Table 6. Comparison of the *ARL* of ARX(1,1) of EWMA and modified EWMA control charts for the value of exports and imports of agricultural products for $ARL_0 = 370$.

shift	$\lambda = 0.05$ $(k = 100\lambda)^a$		$\lambda = 0.1$ $(k = 40\lambda)$		$\lambda = 0.2$ $(k = 60\lambda)$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 3.27104 \times 10^{-18}$	$h = 0.007128814$	$b = 1.3617 \times 10^{-13}$	$h = 0.0044711692$	$b = 5.446519 \times 10^{-12}$	$h = 0.0153495067$
0.000	374.505189	374.510271	370.015353	370.013368	370.026479	370.026204
0.001	348.463281	80.027591*	348.920418	79.529679*	276.359424	71.606027*
0.003	301.920274	31.405097*	310.520857	31.227647*	178.925791	27.747820*
0.005	261.864143	19.690008*	276.644980	19.575453*	128.916300	17.383968*
0.007	227.356776	14.422145*	246.734101	14.334171*	98.643720	12.748838*
0.009	197.600498	11.428518*	220.280627	11.355250*	78.453653	10.121645*
0.010	184.287726	10.369330*	208.221352	10.301213*	70.713067	9.193284*
0.030	48.289170	3.899445*	70.946841	3.863411*	17.089659	3.535219*
0.050	14.280238	2.596183*	26.449591	2.568076*	6.766353	2.397466*
0.070	5.028239	2.046506*	10.869219	2.022750*	3.406272	1.917426*
0.090	2.310792	1.748461*	5.049280	1.727746*	2.092499	1.656939*
RMI	8.975697	0	11.224773	0	4.159967	0

k for the modified EWMA control chart. [a]a constant value. *The smallest *ARL* on each shift size according to the case.

Table 7. Comparison of the *ARL* of ARX(2,1) of EWMA and modified EWMA control charts for the price of cassava and diesel oil for $k = 40\lambda$ and $ARL_0 = 500$.

shift	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.2971 \times 10^{-14}$	$b = 6.399136 \times 10^{-6}$	$b = 2.1239 \times 10^{-13}$	$b = 9.600579 \times 10^{-6}$	$b = 4.0308 \times 10^{-13}$	$b = 1.2801949 \times 10^{-5}$
0.0000	500.039396	500.039009	500.179242	500.174921	500.226696	500.227872
0.0001	486.931023	108.038894*	473.487004	101.389201*	417.839400	98.180795*
0.0003	461.702687	42.328712*	426.334701	39.364297*	311.758448	37.955938*
0.0005	437.968562	26.452894*	385.987063	24.571033*	246.391305	23.680218*

Table 7. Continued.

shift	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.2971\times10^{-14}$	$b = 6.399136\times10^{-6}$	$b = 2.1239\times10^{-13}$	$b = 9.600579\times10^{-6}$	$b = 4.0308\times10^{-13}$	$b = 1.2801949\times10^{-5}$
0.0007	415.503914	19.307837*	351.191004	17.937223*	202.151348	17.289440*
0.0009	394.328865	15.245271*	320.856371	14.171951*	170.215918	13.665109*
0.0010	384.125979	13.807393*	307.136133	12.840435*	157.387407	12.383952*
0.0030	230.568928	5.011479*	146.767998	4.707794*	54.533946	4.564598*
0.0050	141.203211	3.229155*	81.398115	3.062145*	27.781114	2.983359*
0.0070	88.045417	2.471605*	48.796600	2.362680*	16.357609	2.311262*
0.0090	55.861517	2.057219*	30.752391	1.979945*	10.453770	1.943442*
RMI	24.968946	0	18.129071	0	8.344739	0

*The smallest *ARL* on each shift size according to the case.

From the results in Tables 6 and 7, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, in Table 6 when $\lambda = 0.05$ and shift = 0.009, the *ARL* is 197.600498 from the EWMA control chart while the *ARL* is 11.428518 from the modified EWMA control chart. This corresponds to an RMI value of 0 for the modified EWMA control chart, which is less than that for the EWMA control chart. The results from Tables 6 and 7 are plotted on the charts in Figures. 1 and 2, respectively.

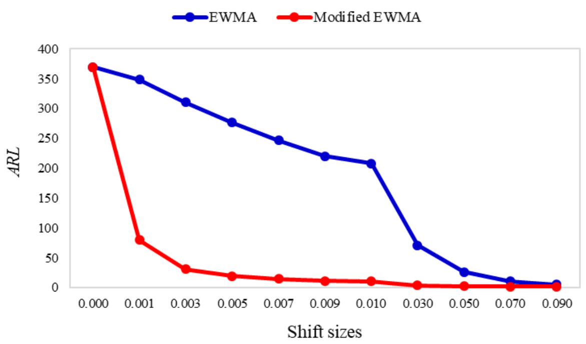

Figure 1. Comparison of the *ARL* for an ARX(1,1) on EWMA and modified EWMA control charts for real data in table 6, where $\lambda = 0.10$.

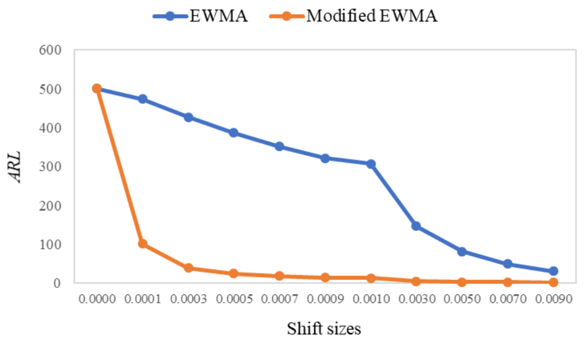

Figure 2. Comparison of the *ARL* for an ARX(2,1) on EWMA and modified EWMA control charts for real data in table 7, where $\lambda = 0.15$.

From Figures 1 and 2, it can be seen that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every case. For example, in Figure 1, when shift = 0.009, the *ARL* from the modified EWMA control chart (*ARL* = 11.355250) is less than that of the EWMA control chart (*ARL* = 220.280627).

From Tables 6 and 7 and Figures 1 and 2, it is evident that the *ARL* for the modified EWMA control chart is smaller than that of the EWMA control chart for every case. Such that the *ARL* values derived from the explicit formulas for the modified EWMA control chart outperformed that for the EWMA control chart.

8. Conclusions

In this study, we derived explicit formulas for the *ARL*s on the EWMA and modified EWMA control charts for an ARX(p,r) process with exponential white noise using real-world data observations and compared the performance of the *ARL* of an ARX(p,r) process on both control charts using the RMI. The suggested formulas are easy to calculate and program. The explicit formulas clearly take much less computational time than the numerical Integral Equation method (NIE). Our results show that they performed better for an ARX(p,r) process on the modified EWMA control chart compared to the EWMA control chart for the case of a one-sided shift with constant *k*. However, the conclusions drawn in this study are only applicable to an ARX(p,r) process and may not be relevant for other processes. In future work, it would be of interest to derive explicit formulas for the *ARL* of other control charts and processes using the Fredholm integral equation of the second type technique. Based on the findings, the *ARL* explicit formula for an ARX(p,r) process on the modified EWMA control chart outperformed the EWMA control chart. Thus, the modified EWMA control chart could be applied to other processes.

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