SHORT COMMUNICATION

Boundedness and continuity of superposition operator on $E_r(p)$ and $F_r(p)$

Assa-aree Sama-ae

Abstract

Sama-ae, A.

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Let $X \in \{E_r(p)V , F_r(p)\}$, in this research, necessary and sufficient conditions are given for superposition operator to act from X into the space ℓ_1. Moreover, necessary and sufficient conditions are obtained for superposition operator acting from X into ℓ_1 to be locally bounded, bounded, and continuous.

Suppose that P_f is a superposition operator which acts from X into ℓ_1, it is found that

1. P_f is locally bounded if and only if f satisfies the condition $A(2^{/})$,

2. if P_f is bounded then f satisfies the condition $A(2^{/})$,

3. P_f is continuous if and only if f satisfies the condition A(2) .

Key words : sequence space, superposition operator, locally bounded function, bounded function, continuous function

M.S. (Mathematics), Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Muang, Pattani 94000 Thailand.

Corresponding e-mail : sassaare@bunga.pn.psu.ac.th

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บทคัดย่อ

อัสอารีย์ สมาแอ

การมีขอบเขตและความต่อเนื่องของตัวดำเนินการซูเปอร์โพซิชันบนปริภูมิ $E_r(p)$ และ $F_r(p)$

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กำหนดให้ $X \in \{E_r(p)V, F_r(p)\}$ ในงานวิจัยนี้ได้ศึกษาเงื่อนไขที่จำเป็นและเพียงพอให้กับตัวดำเนินการซูเปอร์โพซิชันที่ส่งจากปริภูมิ X ไปยังปริภูมิ ℓ_1 นอกจากนี้ยังได้ศึกษาเงื่อนไขที่จำเป็นและเพียงพอให้กับตัวดำเนินการซูเปอร์โพซิชันที่ส่งจากปริภูมิ X ไปยังปริภูมิ ℓ_1 เพื่อให้เป็นตัวดำเนินการที่มีขอบเขตเฉพาะที่ มีขอบเขตและต่อเนื่องตามลำดับ

ถ้า P_f เป็นตัวดำเนินการซูเปอร์โพซิชันที่ส่งจาก X ไปยัง ℓ_1 พบว่า

1. P_f มีขอบเขตเฉพาะที่ก็ต่อเมื่อ f สอดคล้องสมบัติ A(2′)

2. ถ้า P_f มีขอบเขตแล้ว f สอดคล้องสมบัติ A(2′)

3. P_f ต่อเนื่องก็ต่อเมื่อ f สอดคล้องสมบัติ A(2)

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยสงขลานครินทร์ อำเภอเมือง จังหวัดปัตตานี 94000

The superposition operator plays an important role in studying the theory of representation of orthogonality in additive functionals on sequence spaces. There are many mathematicains trying to characterize the superposition operator acting from a sequence space into ℓ_1. For example, Chew (1990) gave necessary and sufficient conditions for superposition operator acting from w_0 into ℓ_1, and Pluciennik (1990, 1991) characterized local boundedness, boundedness of superposition operator acting from w_0 into ℓ_1, and continuity of superposition operator acting from w_0 and W_0 into ℓ_1.

In this research, we shall give necessary and sufficient conditions for superposition operator acting from $E_r(p)$ and $F_r(p)$ into ℓ_1. Moreover, we shall characterize local boundedness, boundedness, and continuity of superposition operator acting from $E_r(p)$ and $F_r(p)$ into ℓ_1.

Let N and R stand for the set of natural numbers and the set of real numbers respectively. Let x be a sequence. For $k \in N$, the k^{th} term of the sequence x is denoted by x_k and we write $x = (x_k) = (x_1, x_2, \ldots, x_n, \ldots)$. For a non-empty subset A of the set of natural numbers N, $x_A = \{x_{\lambda_1(A)}, x_{\lambda_2(A)}, \ldots, x_{\lambda_n(A)}, \ldots\}$ is a sequence with

$$x_{\lambda_k A} = \begin{cases} x_k & : k \in A \\ 0 & : k \notin A \end{cases}. \text{ A sequence } e^{(n)} \text{ is one with } e_k^{(n)} = \begin{cases} 0 : k \neq n \\ 1 : k = n \end{cases}.$$

Let Φ denote the space of finite sequences, S be the space of real sequences and

$\ell_1 = \left\{(x_k) \middle| \sum_{k=1}^{\infty} |x_k| < \infty \right\}$ with the norm || || defined by $||x|| = \sum_{k=1}^{\infty} |x_k|$.

Throughout this research, we let $p = (p_k)$ be a bounded sequence of positive numbers and $M = \max\left\{1, \sup_k p_k\right\}$.

For any $r > 0$ the space $E_r(p)$ and space $F_r(p)$ are defined as follows:

$E_r(p) = \{(x_k) : \exists A > 0\ \forall k \in N, |x_k|^{p_k} < Ak^r\}$,

$F_r(p) = \left\{(x_k) \middle| \sum_{k=1}^{\infty} k^r |x_k|^{p_k} < \infty\right\}$.

Let $\|\ \|_1 : E_r(p) \to R$ be defined by $\|x\|_1 = \sup \frac{|x_k|^{\frac{p_k}{M}}}{k^r}$. $\|\ \|_1$ can not be a norm; however, by its properties we can define a metric d on $E_r(p)$ by letting $d(x, y) = \|x-y\|_1$ for each $x, y \in E_r(p)$.

Let $\|\ \|_2 : F_r(p) \to R$ be defined by $\|x\|_2 = \left(\sum_{k=1}^{\infty} k^r |x_k|^{p_k}\right)^{\frac{1}{M}}$; we find that $\|\ \|_2$ is not a norm. Suppose that $d : F_r(p) \times F_r(p) \to R$ is defined by $d(x, y) = \|x-y\|_2$ for each $x, y \in F_r(p)$. By using properties of $\|\ \|_2$ and the inequality,

$$\left(\sum_{k=1}^{\infty} |a_k + b_k|^{p_k}\right)^{\frac{1}{M}} \leq \left(\sum_{k=1}^{\infty} |a_k|^{p_k}\right)^{\frac{1}{M}} + \left(\sum_{k=1}^{\infty} |b_k|^{p_k}\right)^{\frac{1}{M}}$$

shown by Maddox(1970), it is not difficult to verify that d is a metric on $F_r(p)$.

Let $f : N \times R \to R$ and X, Y be two sequence spaces. A superposition operator P_f on X is a mapping from X into S defined by $P_f(x) = (f(k, x_k))_{k=1}^{\infty}$. P_f acts from X into Y, denoted by $P_f : X \to Y$, if $P_f(x) \in Y$ for all $x \in X$. We say that the function f satisfies the following conditions:

A(2) if $f(k, .)$ is continuous for every positive $k \in N$ and

A(2′) if $f(k, .)$ is bounded on every bounded subset of real numbers for all $k \in N$.

It is obvious that f satisfies the condition A(2′) if it satisfies the condition A(2).

Let $X = (X, d)$ and $Y = (Y, d^*)$ be two metric sequence spaces. An operator $F : X \to Y$ is bounded if F(A) is bounded for every bounded subset A of X. An operator F is said to be locally bounded at $x_0 \in X$ if there exist $\alpha, \beta > 0$ such that $F(x) \in \overline{B}_{d^*}(F(x_0), \beta)$ whenever $x \in \overline{B}_d(x_0, \alpha)$ and an operator F is locally bounded if F is locally bounded at each $x \in X$.

Throughout this research, we defined $\|\ \|_1 : E_r(p) \to R$ and $\|\ \|_2 : F_r(p) \to R$ by $\|x\|_1 = \sup \frac{|x_k|^{\frac{p_k}{M}}}{k^r}$ and $\|x\|_2 = \left(\sum_{k=1}^{\infty} k^r |x_k|^{p_k}\right)^{\frac{1}{M}}$ respectively. Suppose that $P_f : X \to \ell_1$, where $X \in \{E_r(p), F_r(p)\}$, by utilizing the metric induced by previous associated function $\|\ \|_i$ $i = 1, 2$, we obtain that for each i, P_f is bounded if and only if for each $\alpha > 0$ there exists $\beta > 0$ such that $\|P_f(x)\| \leq \beta$ whenever $\|x\|_i \leq \alpha$ and P_f is locally bounded at $x_0 \in X$ if there exist $\alpha, \beta > 0$ such that $\|P_f(x)-P_f(x_0)\| \leq \beta$ whenever $\|x-x_0\|_i \leq \alpha$.

It is easy to verify that if P_f is bounded then it is locally bounded. Finally, we note that if $f : R \to R$ is locally bounded then f satisfies the condition $A(2')$. This was justified by Tainchai (1996).

Superposition operator on $E_r(p)$

Firstly, we now give necessary and sufficient conditions for superposition operator acting from $E_r(p)$ into ℓ_1.

Theorem 1

Let $f : N \times R \to R$ satisfy the condition $A(2')$. The superposition operator P_f acts from $E_r(p)$ to ℓ_1 if and only if for all $\alpha > 0$ there exists a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, t)| \le c_k$ whenever $|t|^{\frac{p_k}{M}} \le k^r\alpha$.

Proof.

($\Leftarrow$) Let $x = (x_k) \in E_r(p)$. We now prove that P_f acts from $E_r(p)$ to ℓ_1. Since $(x_k) \in E_r(p)$, there exists $A > 0$ such that $|x_k|^{p_k} \le Ak^r$ for all $k \in N$, and thus we get $|x_k|^{\frac{p_k}{M}} = \left(|x_k|^{p_k}\right)^{\frac{1}{M}} \le (Ak^r)^{\frac{1}{M}} = A^{\frac{1}{M}}k^{\frac{r}{M}} \le A^{\frac{1}{M}}k^r$. It follows from the assumption that there exists a sequence $(c_k) \in \ell_1$ such that $|f(k, x_k)| \le c_k$ for all $k \in N$ and consequently we have $\sum_{k=1}^{\infty}|f(k, x_k)| \le \sum_{k=1}^{\infty} c_k < \infty$. This implies $P_f(x) = \in \ell_1$ and thereby shows that P_f acts from $E_r(p)$ to ℓ_1.

($\Rightarrow$) Suppose that P_f acts from $E_r(p)$ to ℓ_1. For each $\alpha > 0$ and for each $k \in N$, we define

$$A(k, \alpha) = \{t \in R : |t|^{p_k} \le (k^r\alpha)^M\}$$

and $B(k, \alpha) = \sup\{|f(k, t)| : t \in A(k, \alpha)\}$.

We note that $|f(k, t)| \le B(k, \alpha)$ whenever $|t|^{\frac{p_k}{M}} \le k^r\alpha$. Next, we are going to show that $B(k, \alpha))_{k=1}^{\infty} \in \ell_1$ for each $\alpha > 0$. Suppose that there exists $\alpha_1 > 0$ such that $\sum_{k=1}^{\infty} B(k, \alpha_1) = \infty$. Then there is a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ and the least positive integer n_i such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, \alpha_1) > 1$. For each $i \in N$, there is an $\varepsilon_i > 0$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k, \alpha_1) - \varepsilon_i(n_i - n_{i-1}) > 1 . \qquad (1)$$

Let $i \in N$ be fixed. As f satisfies the condition $A(2')$, $0 \le B(k, \alpha_1) < \infty$ for all $k \in N$ with $n_{i-1} + 1 \le k \le n_i$. It follows from the definition of $B(k, \alpha_1)$ that for each $k \in N$ with $n_{i-1} + 1 \le k \le n_i$ there is $x_k \in A(k, \alpha_1)$ such that

$$|f(k, x_k)| > B(k, \alpha_1) - \varepsilon_i . \qquad (2)$$

From (1) and (2), for each $i \in N$,

$$\sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| > \sum_{k=n_{i-1}+1}^{n_i} B(k, \alpha_1) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i$$
$$= \sum_{k=n_{i-1}+1}^{n_i} B(k, \alpha_1) - \varepsilon_i(n_i - n_{i-1}) > 1 .$$

Thus $\sum_{k=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| = \sum_{k=1}^{\infty} |f(k, x_k)| = \infty$, and hence $(f(k, x_k))_{k=1}^{\infty} \notin \ell_1$. Since $x_k \in A(k, \alpha_1)$ for all $k \in N$ with $n_{i-1} + 1 \le k \le n_i$ and for every $i \in N$, we obtain that the sequence (x_k) is in $E_r(p)$ and therefore $P_f(x) = (f(k, x_k))_{k=1}^{\infty} \in \ell_1$, which is a contradiction. Accordingly, $B(k, \alpha))_{k=1}^{\infty} \in \ell_1$ for every $\alpha > 0$. Put $c_k = B(k, \alpha))$ for all $k \in N$. Then for each $\alpha > 0$ there exists a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, t)| \le c_k$ whenever $|t|^{\frac{p_k}{M}} \le k^r\alpha$.

Theorem 2

Let $f : N \times R \to R$ Suppose that P_f is a superposition operator which acts from $E_r(p)$ to ℓ_1. Then P_f is locally bounded if and only if f satisfies the condition $A(2')$.

Proof.

($\Leftarrow$) Suppose that f satisfies the condition $(A2')$ and P_f is superposition operator from $E_r(p)$ into ℓ_1. Let $x = (x_k) \in E_r(p)$ be given. We now show that P_f is locally bounded at x. Let $\alpha > 0$ and $y \in E_r(p)$ with $\|y - x\| \le \alpha$. Then $\|y\| \le \alpha + \|x\|$ and this implies $|y_k|^{\frac{p_k}{M}} \le k^r (\alpha + \|x\|)$ for all $k \in N$ and by applying Theorem 1, there is a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, y_k)| \le c_k$ and hence $\|P_f(y)\| = \sum_{k=1}^{\infty} |f(k, y_k)| \le \sum_{k=1}^{\infty} c_k = \|(c_k)\|$. Therefore the operator P_f is locally bounded at x, as $\|P_f(y) - P_f(x)\| \le \|P_f(y)\| + \|P_f(x)\| < \|P_f(x)\| + \|(c_k)\|$.

($\Rightarrow$) Suppose that $P_f : E_r(p) \to \ell_1$ is locally bounded. We shall prove that f satisfies the condition $A(2')$; it suffices to show that f(k,.) is locally bounded for all $k \in N$. Let $k \in N$ and $b \in R$. We define a sequence $y = (y_k)$ with $y_n = \begin{cases} b , & n = k \\ 0 , & n \ne k \end{cases}$, and observe that $y \in E_r(p)$. By the assumption, there exist $\alpha , \beta > 0$ such that

$$\|P_f(x) - P_f(y)\| \le \beta \text{ whenever } \|x - y\| \le \alpha . \quad (3)$$

Let $a \in R$ with $|a - b| \le (k^r\alpha)^{\frac{M}{p_k}}$ and let $x = (x_n)$ be a sequence with $x_n = \begin{cases} a , & n = k \\ 0 , & n \ne k \end{cases}$. We see that $x \in E_r(p)$ and $\|x - y\| = \sup \frac{|x_n - y_n|^{\frac{p_n}{M}}}{n^r} = \frac{|a - b|^{\frac{p_k}{M}}}{k^r} \le \alpha$. It follows from (3) that $\|P_f(x) - P_f(y)\| \le \beta$ and thus $|f(k, a) - f(k, b)| \le \sum_{k=1}^{\infty} |f(n, x_n) - f(n, y_n)| = \|P_f(x) - P_f(y)\| \le \beta$. Therefore f(k , .) is locally bounded at $b \in R$.

Corollary 3

Let $f : N \times R \to R$. Suppose that P_f is a superposition operator which acts from $E_r(p)$ to ℓ_1. Then P_f is bounded if and only if for each $\alpha > 0$ there exist a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, t)| \le c_k$ whenever $|t|^{\frac{p_k}{M}} \le k^r\alpha$.

Proof .

($\Rightarrow$) The result follows directly from Theorem 1 and 2.

($\Leftarrow$) Suppose the sufficient condition holds. Let $\alpha > 0$ and $x \in E_r(p)$ with $\|x\| \le \alpha$. Thus $\sup \frac{|x_k|^{\frac{p_k}{M}}}{k^r} \le \alpha$, that is $|x_k|^{\frac{p_k}{M}} \le k^r\alpha$ for every $k \in N$. By the assumption there exists a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, x_k)| \le c_k$. This implies $\|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| \le \sum_{k=1}^{\infty} c_k = \|(c_k)\| < \infty$. Accordingly P_f is bounded.

The following two corollaries are easily verified by utilizing Theorems 1 and 2 and Corollary 3.

Corollary 4

Let $f : N \times R \to R$. Suppose that P_f is a superposition operator which acts from $E_r(p)$ to ℓ_1. Then P_f is bounded if and only if f satisfies the condition $A(2')$.

Corollary 5

Let $f : N \times R \to R$. Suppose that P_f is a superposition operator which acts from $E_r(p)$ to ℓ_1. Then P_f is locally bounded if and only if for each $\alpha > 0$ there exists a sequence $(c_k) \in \ell_1$ such that for each $k \in N$, $|f(k, t)| \le c_k$ whenever $|t|^{\frac{p_k}{M}} \le k^r\alpha$.

Theorem 6

Let $f : N \times R \to R$. The superposition operator $P_f : E_r(p) \to \ell_1$ is continuous if and only if f satisfies the condition A(2) .

Proof .

($\Rightarrow$) Suppose that P_f is continuous on $E_r(p)$. Let $k \in N$, $t_0 \in R$ and $\varepsilon > 0$ be given. As P_f is continuous at $t_0e^{(n)} \in E_r(p)$, there is a $\delta > 0$ such that for each $z = (z_k) \in E_r(p)$,

$$\|P_f(z) - P_f(t_0e^{(n)})\| < \varepsilon \text{ whenever } \|z - t_0e^{(n)}\| < \delta . \quad (4)$$

Let $t \in R$ be such that $|t - t_0| < (\delta k^r)^{\frac{M}{p_k}}$ and $y_n = \begin{cases} 0 , & n \ne k \\ t , & n = k \end{cases}$. Observe that $y = (y_n) \in E_r(p)$ and $\|y - t_0e^{(n)}\| = \frac{|t - t_0|^{\frac{p_k}{M}}}{k^r} < \delta$. Employing (4), we have $|P_f(y) - P_f(t_0e^{(n)})\| < \varepsilon$ and hence $|f(k, t) - f(k, t_0)| \le \|P_f(y) - P_f(t_0e^{(n)})\| < \varepsilon$. Therefore f satisfies the condition A(2).

($\Leftarrow$) Suppose that f satisfies the condition A(2). We are going to justify that P_f is continuous on $E_r(p)$. Let $x = (x_k) \in E_r(p)$ and $\varepsilon > 0$. It follows from Theorem 1 that for each $\alpha > 0$ there exists a sequence $(c_k) \in \ell_1$ such that for every $k \in N$,

$$|f(k, t)| \le c_k \text{ whenever } |t|^{\frac{p_k}{M}} \le k^r\alpha . \quad (5)$$

As (x_k) is in $E_r(p)$, there is $\beta > 0$ such that $|x_k|^{p_k} < \left(k^r \frac{\beta}{2}\right)^M$ for all $k \in N$. By (5), there is a sequence $(c_k) \in \ell_1$ such that for all $k \in N$,

$$|f(k, x_k)| \le c_k \quad (6)$$

and a sequence $(c_k^*) \in \ell_1$ such that for each $k \in N$,

$$|f(k, t)| \le c_k^* \text{ whenever } |t|^{\frac{p_k}{M}} \le k^r\beta . \quad (7)$$

Because (c_k) and (c_k^*) are in ℓ_1, there is $N_1 \in N - \{1\}$ such that

$$\sum_{k=N_1}^{\infty} c_k < \frac{\varepsilon}{3} \text{ and } \sum_{k=N_1}^{\infty} c_k^* < \frac{\varepsilon}{3} . \quad (8)$$

As a result f(k,.) is continuous at x_k for all $k \in \{1, 2,..., N_1-1\}$. This implies that there exists $\delta_k > 0$ with $\delta_k \le \min\left\{1, \frac{\beta}{2}\right\}$ such that for each $k \in \{1, 2,..., N_1-1\}$ and for every $t \in R$,

$$|f(k, t) - f(k, x_k)| < \frac{\varepsilon}{3(N_1 - 1)} \text{ whenever } |t - x_k| < \delta_k . \quad (9)$$

Let $y = (y_k) \in E_r(p)$ be such that $\|y - x\| < \delta$ with $\delta \le \min\left\{\frac{\delta_k}{k^r} | k = 1, 2, \ldots, N_1 - 1\right\}$. Since $\|y - x\| < \delta$, $\sup \frac{|y_k - x_k|^{\frac{p_k}{M}}}{k^r} < \delta$ and hence $|y_k - x_k|^{\frac{p_k}{M}} < k^r\delta$ for all $k \in N$. Thus for all k is in $\{1,2,...,N_1-1\}$, $|y_k - x_k| < (k^r\delta)^{\frac{M}{p_k}} \le (\delta_k)^{\frac{M}{p_k}} < \delta_k$. By (9), we have

$|f(k, y_k) - f(k, x_k)| < \dfrac{\varepsilon}{3(N_1 - 1)}$ for every $k \in \{1, 2, ..., N_1 - 1\}$, and consequently

$$\sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3} \cdot \quad (10)$$

As $|y_k| \le |y_k - x_k| + |x_k| < \left(\frac{\beta}{2}\right)^{\frac{M}{p_k}} + \left(k^r \frac{\beta}{2}\right)^{\frac{M}{p_k}} \le \frac{1}{2}\left(k^r \beta\right)^{\frac{M}{p_k}} + \frac{1}{2}\left(k^r \beta\right)^{\frac{M}{p_k}} = \left(k^r \beta\right)^{\frac{M}{p_k}}$, it follows from (7) that for all $k \in N$,

$$|f(k, y_k)| \le c_k^*. \quad (11)$$

Utilizing (6), (8), and (11) we get

$$\sum_{k=N_1}^{\infty} |f(k, x_k)| \le \sum_{k=N_1}^{\infty} c_k < \frac{\varepsilon}{3} \text{ and } \sum_{k=N_1}^{\infty} |f(k, y_k)| \le \sum_{k=N_1}^{\infty} c_k^* < \frac{\varepsilon}{3} . \quad (12)$$

Finally, by using (10) and (12) we obtain that

$$\begin{aligned} \|P_f(y) - P_f(x)\| &= \sum_{k=1}^{\infty} |f(k, y_k) - f(k, x_k)| \\ &= \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=N_1}^{\infty} |f(k, y_k) - f(k, x_k)| \\ &\le \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=N_1}^{\infty} |f(k, y_k)| + \sum_{k=N_1}^{\infty} |f(k, x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon . \end{aligned}$$

The proof of Theorem 6 is then complete.

Theorem 7

Let $f : N \times R \to R$. The superposition operator $P_f : E_r(p) \to \ell_1$ is uniformly continuous on every bounded subset of $E_r(p)$ if and only if f satisfies the condition A(2).

Proof.

($\Rightarrow$) The result is an immediate consequence of Theorem 6.

($\Leftarrow$) Assume that f satisfies A(2). To show that P_f is uniformly continuous on every bounded subset of $E_r(p)$, it is enough to prove that P_f is uniformly continuous on $\overline{B}_d(0, r)$ for all $r > 0$. Let $\sigma > 0$ and $\varepsilon > 0$ be given. Since f satisfies the condition A(2), f satisfies the condition A(2$'$) and hence it follows from Theorem 1 that there exists a sequence $(c_k) \in \ell_1$ such that for each $k \in N$,

$$|f(k, t)| \le c_k \text{ whenever } |t|^{\frac{p_k}{M}} \le k^r \sigma . \quad (13)$$

Because $(c_k) \in \ell_1$, there is $N_1 \in N - \{1\}$ such that $\sum_{k=N_1}^{\infty} c_k < \frac{\varepsilon}{3}$. As f(k,.) is uniformly continuous on $\left[-(k^r\sigma)^{\frac{M}{p_k}}, (k^r\sigma)^{\frac{M}{p_k}}\right]$ for every $k \in \{1, 2, ..., N_1 - 1\}$, there is $0 < \delta_k < 1$ such that for each $k \in \{1, 2,..., N_1-1\}$ and for all $a, b \in \left[-(k^r\sigma)^{\frac{M}{p_k}}, (k^r\sigma)^{\frac{M}{p_k}}\right]$,

$$|f(k, a) - f(k, b)| < \frac{\varepsilon}{3(N_1-1)} \text{ whenever } |a - b| < \delta_k . \quad (14)$$

Let x and $y \in \overline{B}_d(0, \sigma)$ be such that $\|y - x\| < \delta$, where $\delta \le \min\left\{\frac{\delta_k}{k^r} | k = 1, 2, \ldots, N_1 - 1\right\}$. Hence $|x_k| \le (k^r\sigma)^{\frac{M}{p_k}}$ and $|y_k| \le (k^r\sigma)^{\frac{M}{p_k}}$ for all $k \in N$ and $|y_k - x_k| < (k^r\delta)^{\frac{M}{p_k}} \le (\delta_k)^{\frac{M}{p_k}} \le \delta_k$ for all $k \in \{1, 2,..., N_1-1\}$. Employing (14), we obtain that for all $k \in \{1, 2, ..., N_1-1\}$, $|f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3(N_1-1)}$ and therefore

$$\sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3} . \quad (15)$$

Because $|x_k| \le (k^r\sigma)^{\frac{M}{p_k}}$ and $|y_k| \le (k^r\sigma)^{\frac{M}{p_k}}$ for all $k \in N$, we apply (13) to get $|f(k, x_k)| \le c_k$ and $|f(k, y_k)| \le c_k$ for all $k \in N$. So we have

$$\sum_{k=N_1}^{\infty} |f(k, x_k)| \le \sum_{k=N_1}^{\infty} c_k < \frac{\varepsilon}{3} \text{ and } \sum_{k=N_1}^{\infty} |f(k, y_k)| \le \sum_{k=N_1}^{\infty} c_k < \frac{\varepsilon}{3} . \quad (16)$$

Therefore, utilizing (15) and (16), we obtain

$$\begin{aligned} \|P_f(y) - P_f(x)\| &= \sum_{k=1}^{\infty} |f(k, y_k) - f(k, x_k)| \\ &= \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=N_1}^{\infty} |f(k, y_k) - f(k, x_k)| \\ &\le \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=N_1}^{\infty} |f(k, y_k)| + \sum_{k=N_1}^{\infty} |f(k, x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon . \end{aligned}$$

Hence the operator P_f is uniformly continuous on every bounded subset of $E_r(p)$.

Superposition operator on $F_r(p)$

Theorem 8

Let $f : N \times R \to R$ satisfy the condition $A(2')$. The superposition operator P_f acts from $F_r(p)$ to ℓ_1 if and only if the following condition is satisfied (*) there exist $\alpha, \beta > 0$ and a sequence $(c_k) \in \ell_1$ such that for each $k \in N$,

$$|f(k, t)| \le c_k + \alpha k^r |t|^{p_k} \text{ whenever } k^r |t|^{p_k} \le \beta^M .$$

Proof.

($\Leftarrow$) Suppose that the condition (*) holds. Let $x = (x_k) \in F_r(p)$. By the assumption, there exist $\alpha, \beta > 0$ and $(c_k) \in \ell_1$ such that for each $k \in N$,

$$|f(k, t)| \le c_k + \alpha k^r |t|^{p_k} \text{ whenever } k^r|t|^{p_k} \le \beta^M .$$

Since $(x_k) \in F_r(p)$, $\sum_{k=1}^{\infty} k^r |x_k|^{p_k} < \infty$, so we get that $\lim_{k\to\infty} k^r |x_k|^{p_k} = 0$ and then there is an integer $i \in N$ such that $k^r |x_k|^{p_k} < \beta^M$ for all $k \ge i$. It follows from the assumption that for each $k \ge i$, $|f(k, x_k)| \le c_k + \alpha k^r |x_k|^{p_k}$. Therefore $\sum_{k=i}^{\infty} |f(k, x_k)| \le \sum_{k=i}^{\infty} c_k + \alpha \sum_{k=i}^{\infty} k^r |x_k|^{p_k} \le \|(c_k)\| + \alpha \sum_{k=i}^{\infty} k^r |x_k|^{p_k} < \infty$ and then we have $P_f(x) = (f(k, x_k))_{k=1}^{\infty} \in \ell_1$. Hence P_f acts from $F_r(p)$ to ℓ_1.

($\Rightarrow$) Suppose that P_f acts from $F_r(p)$ to ℓ_1. For each $\alpha, \beta > 0$ and for each positive integer k , we define

$$A(k, \alpha, \beta) = \left\{ t \in R \,\middle|\, |t|^{p_k} \le \min\left\{ \frac{\beta^M}{k^r}, \frac{|f(k,t)|}{\alpha k^r} \right\} \right\}$$

and $B(k, \alpha, \beta) = \sup\{|f(k, t)| \,;\, t \in A(k, \alpha, \beta)\}$.

If $k^r|t|^{p_k} \le \beta^M$ and $t \in A(k, \alpha, \beta)$ then $|f(k, t)| \le B(k, \alpha, \beta)$. And if $k^r|t|^{p_k} \le \beta^M$ and $t \notin A(k, \alpha, \beta)$ then $|f(k, t)| \le \alpha k^r |t|^{p_k}$. Thus we have that

$$|f(k, t)| \le B(k, \alpha, \beta) + \alpha k^r |t|^{p_k} \text{whenever } k^r|t|^{p_k} \le \beta^M .$$

Next we shall show that $(B(k, \alpha_1, \beta_1))_{k=1}^{\infty} \in \ell_1$ for some α_1 , $\beta_1 > 0$. Suppose that for each α , $\beta > 0$, $\sum_{k=1}^{\infty} B(k, \alpha,\beta) = \infty$. We get that for each integer $i \in N$, $\sum_{k=1}^{\infty} B(k, 2^i, \frac{1}{2^i}) = \infty$. Then there is a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ and the least positive integer n_i such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) > 1$. Therefore for each $i \in N$, there is $\varepsilon_i > 0$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) - \varepsilon_i(n_i - n_{i-1}) > 1 . \qquad (17)$$

Let $i \in N$ be fixed. Since f satisfies the condition $A(2')$, $0 \le B(k, 2^i, \frac{1}{2^i}) < \infty$ for all $k \in N$ with $n_{i-1} + 1 \le k \le n_i$. It follows from the definition of $B(k, 2^i, \frac{1}{2^i})$ that for each $k \in N$ with $n_{i-1} + 1 \le k \le n_i$, there exists $x_k \in B(k, 2^i, \frac{1}{2^i})$ such that

$$|f(k, x_k)| > B(k, 2^i, \frac{1}{2^i}) - \varepsilon_i . \qquad (18)$$

From (17) and (18), for each $i \in N$,

$$\sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| > \sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i = \sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) - e_i(n_i - n_{i-1}) > 1 .$$

Thus $\sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| = \sum_{k=1}^{\infty} |f(k, x_k)| = \infty$, and hence we have $(f(k, x_k))_{k=1}^{\infty} \notin \ell_1$. As $x_k \in B(k, 2^i, \frac{1}{2^i})$ for all $k \in N$ with $n_{i-1} + 1 \le k \le n_i$, we obtain that for each $k \in N$ with $n_{i-1} + 1 \le k \le n_i$,

$$|x_k|^{p_k} \le \frac{1}{k^r}\left(\frac{1}{2^i}\right)^M \le \frac{1}{k^r 2^i} \text{ and } |x_k|^{p_k} \le \frac{|f(k, x_k)|}{k^r 2^i} . \qquad (19)$$

Since n_i is the least positive integer such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) > 1$, we obtain $\sum_{k=n_{i-1}+1}^{n_i - 1} B(k, 2^i, \frac{1}{2^i}) \le 1$. It follows from (19),

$$\sum_{k=n_{i-1}+1}^{n_i} k^r |x_k|^{p_k} = \sum_{k=n_{i-1}+1}^{n_i-1} k^r |x_k|^{p_k} + n_i^r |x_{n_i}|^{p_{n_i}}$$

$$\leq \sum_{k=n_{i-1}+1}^{n_i-1} k^r \left(\frac{|f(k, x_k)|}{2^i k^r}\right) + n_i^r \left(\frac{1}{2^i n_i^r}\right) \leq \sum_{k=n_{i-1}+1}^{n_i-1} \frac{|f(k, x_k)|}{2^i} + \frac{1}{2^i}$$

$$\leq \sum_{k=n_{i-1}+1}^{n_i-1} \frac{B(k, 2^i, \frac{1}{2^i})}{2^i} + \frac{1}{2^i} \leq \frac{1}{2^i} + \frac{1}{2^i} = \frac{2}{2^i} .$$

Thus $\sum_{k=1}^{\infty} k^r |x_k|^{p_k} = \sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} k^r |x_k|^{p_k} \leq \sum_{i=1}^{\infty} \frac{2}{2^i} = 2$ and consequently (x_k) is in $F_r(p)$. By the assumption we have that $P_f(x) = (f(k, x_k))_{k=1}^{\infty} \in \ell_1$, which is a contradiction. Accordingly $(B(k, \alpha, \beta))_{k=1}^{\infty} \in \ell_1$. For all $k \in N$, we put $c_k = B(k, \alpha, \beta)$ then we have (*).

Before we prove Theorem 9, we need the result that, if $f : R \to R$ is locally bounded then f satisfies the condition $A(2')$, which was proved by Tainchai (1990).

Theorem 9

Let $f : N \times R \to R$. Suppose that P_f is a superposition operator which acts from $F_r(p)$ to ℓ_1. Then P_f is locally bounded if and only if f satisfies the condition $A(2')$.

Proof.

($\Leftarrow$) Suppose that f satisfies the condition $A(2')$. Let $z = (z_k) \in F_r(p)$. We now prove that P_f is locally bounded at z. Since f satisfies the condition $A(2')$ and P_f acts from to $F_r(p)$ to ℓ_1, it follows from Theorem 8, there are $\alpha, \beta > 0$ and a sequence $(c_k) \in \ell_1$ such that for each $k \in N$,

$$|f(k, t)| \leq c_k + \alpha k^r |t|^{p_k} \text{ whenever } k^r |t|^{p_k} \leq \beta^M . \quad (20)$$

Let $\eta = \frac{\beta}{2}$ and $(x_k) \in F_r(p)$ with $\|x - z\| \leq \eta$. Since $\|z\| < \infty$, there is an $i \in N$ such that

$$\left\| z_{\{i, i+1, i+2, \dots\}} \right\| = \left(\sum_{k=i}^{\infty} k^r |z_k|^{p_k} \right)^{\frac{1}{M}} \leq \eta . \quad (21)$$

Because $\|x - z\| \leq \eta$, we have

$$\left(\sum_{k=i}^{\infty} k^r |x_k - z_k|^{p_k} \right)^{\frac{1}{M}} \leq \eta . \quad (22)$$

It follows from (21) and (22),

$$\left(\sum_{k=i}^{\infty} k^r |x_k|^{p_k} \right)^{\frac{1}{M}} = \left(\sum_{k=i}^{\infty} k^r |x_k - z_k + z_k|^{p_k} \right)^{\frac{1}{M}} \leq$$

$$\left(\sum_{k=i}^{\infty} k^r |z_k|^{p_k} \right)^{\frac{1}{M}} + \left(\sum_{k=i}^{\infty} k^r |x_k - z_k|^{p_k} \right)^{\frac{1}{M}} \leq \eta + \eta = \beta$$

and then $k^r |x_k|^{p_k} \leq \beta^M$ for all $k \geq i$. By (20), we get that for each $k \geq i$, $|f(k, x_k)| \leq c_k + \alpha k^r |x_k|^{p_k}$. Consequently,

$$\sum_{k=i}^{\infty} |f(k, x_k)| \leq \sum_{k=i}^{\infty} c_k + \alpha \sum_{k=i}^{\infty} k^r |x_k|^{p_k} \leq \|(c_k)\| + \alpha \beta^M . \quad (23)$$

For each $k \in N$, let $m_k = \sup_{|t - z_k| \leq \left(\frac{\eta^M}{k^r}\right)^{\frac{1}{p_k}}} |f(k, t)|$. As f satisfies the condition A(2'), we have that $m_k < \infty$ for every $k \in N$. And since $\|x - z\| \le \eta$, $|x_k - z_k| \le \left(\frac{\eta^M}{k^r}\right)^{\frac{1}{p_k}}$ for every $k \in N$ and then

$$|f(k, x_k)| \le m_k \text{ for every } k \in N . \qquad (24)$$

It follows from (23) and (24) ,

$\|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| = \sum_{k=1}^{i-1} |f(k, x_k)| + \sum_{k=i}^{\infty} |f(k, x_k)| \le \sum_{k=1}^{i-1} m_k + \|(c_k)\| + \alpha\beta^M$, and hence $\|P_f(x) - P_f(z)\| \le \|P_f(x)\| + \|P_f(z)\| \le \|P_f(x)\| + \sum_{k=1}^{i-1} m_k + \|(c_k)\| + \alpha\beta^M$. We choose $\gamma = \|P_f(x)\| + \sum_{k=1}^{i-1} m_k + \|(c_k)\| + \alpha\beta^M$, so we obtain $\|P_f(x) - P_f(z)\| \le \gamma$. The operator P_f is then locally bounded at z.

($\Rightarrow$) Suppose that P_f acting from $F_r(p)$ to ℓ_1 is locally bounded. We are going to show that f satisfies the condition A(2') by proving that f(k , .) is locally bounded for all $k \in N$. Now let $k \in N$ and $b \in R$. We define a sequence $y = (y_k)$ with $y_n = \begin{cases} b, & n = k \\ 0, & n \ne k \end{cases}$. Observe that $y \in P_f(x)$ and by assumption, there exist $\alpha, \beta > 0$ such that

$$\|P_f(x) - P_f(y)\| \le \beta \text{ whenever } \|x - y\| \le \alpha . \qquad (25)$$

Let $a \in R$ with $|a - b| \le \left(\frac{\alpha^M}{k^r}\right)^{\frac{1}{p_k}}$ and let $x = (x_k)$ be a sequence with $x_n = \begin{cases} a, & n = k \\ 0, & n \ne k \end{cases}$. We suddenly have $x \in F_r(x)$ and $\|x - y\| = \left(\sum_{n=1}^{\infty} n^r |x_n - y_n|^{p_n}\right)^{\frac{1}{M}}$. It follows from (25) that $\|P_f(x) - P_f(y)\| \le \beta$ and thus

$$|f(k, a) - f(k, b)| \le \sum_{n=1}^{\infty} |f(n, x_n) - f(n, y_n)| = \|P_f(x) - P_f(y)\| \le \beta .$$

Hence f(k ,.) is locally bounded at $b \in R$.

The following corollary is readily verified by employing Theorem 9.

Corollary 10

Let $f : N \times R \to R$ satisfy the condition A(2). If the superposition operator P_f acts from $F_r(p)$ to ℓ_1 then the operator P_f is locally bounded.

The next corollary is a sequence of above corollary.

Corollary 11

Let $f : N \times R \to R$. If the superposition operator P_f acting from $F_r(p)$ to ℓ_1 is bounded then f satisfies the condition A(2').

Lemma 12

Let $f : N \times R \to R$ satisfy the condition A(2'). If for each $\beta > 0$ there is an $\alpha(\beta) > 0$ such that for any finite sequence (x_k) ,

$$\sum_{k=1}^{\infty} |f(k, x_k)| \le \alpha(\beta) \text{ provided } \sum_{k=1}^{\infty} k^r |x_k|^{p_k} \le \beta^M .$$

Then there exists a sequence $c(\beta) = (c_k(\beta)) \in \ell_1$ with $c_k(\beta) \ge 0$ for all $k \in N$ and $\|c(\beta)\| \le \alpha(\beta)$ such that for each $k \in N$,

$|f(k,t)| \le c_k(\beta) + 2\dfrac{\alpha(\beta)}{\beta^M}k^r|t|^{p_k}$ whenever $k^r|t|^{p_k} \le \beta^M$.

Proof.

Let $\beta > 0$. By the assumption, there is $\alpha(\beta) > 0$ such that for any finite sequence (x_k) ,

$$\sum_{k=1}^{\infty} |f(k, x_k)| \le \alpha(\beta) \text{ provided } \sum_{k=1}^{\infty} k^r|x_k|^{p_k} \le \beta^M. \quad (26)$$

For each $k \in N$, we define

$$h_\beta(k,t) = \max\left\{0,\ |f(k,t)| - 2\frac{\alpha(\beta)}{\beta^M}k^r|t|^{p_k}\right\} \text{ and}$$

$$c_k(\beta) = \sup\{h_\beta(k,t)\ ;\ k^r|t|^{p_k} \le \beta^M\}\ .$$

Let $k \in N$ and $t \in R$ be such that $k^r|t_k|^{p_k} \le \beta^M$. If $h_\beta(k, t) = 0$, $|f(k, t)| \le c_k(\beta) + 2\dfrac{\alpha(\beta)}{\beta^M}k^r|t|^{p_k}$; otherwise, $h_\beta(k,t) = |f(k,t)| - 2\dfrac{\alpha(\beta)}{\beta^M}k^r|t|^{p_k}$. Therefore for each $\beta > 0$ and $k \in N$, $|f(k, t)| \le c_k(\beta) + 2\dfrac{\alpha(\beta)}{\beta^M}k^r|t|^{p_k}$ whenever $k^r|t_k|^{p_k} \le \beta^M$. Next we shall show that $c(\beta) = (c_k(\beta)) \in \ell_1$ and $\|c(\beta)\| \le \alpha(\beta)$. Since f satisfies the condition A(2′), $h_\beta(k, t)$ is bounded on every bounded subset of real numbers for all $k \in N$, so we have that $0 \le c_k(\beta) < \infty$ for all $k \in N$. By the definition of $c_k(\beta)$, for each $\varepsilon > 0$ there exists the sequence $y = (y_k)$ with $k^r|y_k|^{p_k} \le \beta^M$ for all $k \in N$ and

$$c_k(\beta) < h_\beta(k, y_k) + \frac{\varepsilon}{2^k}\ . \quad (27)$$

Let the sequence y' be defined as follow :

$$y'_k = \begin{cases} y_k\ ,\ h_\beta(k,t) > 0 \\ 0\ ,\ h_\beta(k,t) = 0 \end{cases}.$$

For any $m \in N$, a finite sequence (m_i) with $m_1 = 1 < m_2 < \ldots < m_s = m$ can be found such that $\sum_{k=1}^{m} k^r|y'_k|^{p_k} = \sum_{k=1}^{m_2-1} k^r|y'_k|^{p_k} + \sum_{k=m_2}^{m_3-1} k^r|y'_k|^{p_k} + \ldots + \sum_{k=m_{s-1}}^{m} k^r|y'_k|^{p_k}$ with $\dfrac{\beta^M}{2} \le \sum_{k=m_i}^{m_{i+1}-1} k^r|y'_k|^{p_k} \le \beta^M$ for all $i \in \{1, 2, \ldots , s-2\}$ and $0 \le \sum_{k=m_{s-1}}^{m} k^r|y'_k|^{p_k} \le \beta^M$. We let $z^{(i)} = y'_{\{m_i, m_{i+1}, \ldots, m_{i+1}-1\}}$ for all $i \in \{1, 2, \ldots , s-2\}$ and $z^{(s-1)} = y'_{\{m_{s-1}, m_{s-1}+1, \ldots, m_s\}}$. Consequently for each $i \in \{1, 2, \ldots , s-1\}$, $z^{(i)} \in \Phi$ and $\sum_{k=1}^{\infty} k^r|z_k^{(i)}|^{p_k} \le \beta^M$. From (26), we obtain that $\sum_{k=1}^{\infty} |f(k, z_k^{(i)})| \le \alpha(\beta)$ for all $i \in \{1, 2, \ldots , s-2\}$. Accordingly for each $i \in \{1, 2, \ldots , s-2\}$,

$$\sum_{k=m_i}^{m_{i+1}-1} |f(k, z_k^{(i)})| \le \alpha(\beta) \text{ and}$$

$$\sum_{k=m_{s-1}}^{m} |f(k, z_k^{(s-1)})| \le \alpha(\beta)\ . \quad (28)$$

We define $f_1(k,t) = \begin{cases} f(k,t), & h_\beta(k,t) > 0 \\ 0, & h_\beta(k,t) = 0 \end{cases}$. It follows from (28),

$$\sum_{k=m_i}^{m_{i+1}-1} |f_1(k, z_k^{(i)})| \le \alpha(\beta) \text{ for every } i \in \{1, 2, \ldots , s-2\}. \quad (29)$$

and

$$\sum_{k=m_{s-1}}^{m} |f_1(k, z_k^{(s-1)})| \le \alpha(\beta)\ . \quad (30)$$

By definitions of y' , $f_1(k, t)$ and $h_\beta(k, t)$ we obtain

$$h_\beta(k, y_k) = |f_1(k, y'_k)| - 2\frac{\alpha(\beta)}{\beta^M}k^r|y'_k|^{p_k} \tag{31}$$

and hence it follows from (27), (29), (30) and (31),

$$\begin{aligned}
\sum_{k=1}^{m} c_k(\beta) &< \sum_{k=1}^{m} h_\beta(k, y_k) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&= \sum_{k=1}^{m_2-1} h_\beta(k, y_k) + \sum_{k=m_2}^{m_3-1} h_\beta(k, y_k) + \dots + \sum_{k=m_s-1}^{m} h_\beta(k, y_k) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&= \sum_{k=1}^{m_2-1}\left(|f_1(k, y'_k)| - 2\frac{\alpha(\beta)}{\beta^M}k^r|y'_k|^{p_k}\right) + \sum_{k=m_2}^{m_3-1}\left(|f_1(k, y'_k)| - 2\frac{\alpha(\beta)}{\beta^M}k^r|y'_k|^{p_k}\right) + \dots + \\
&\quad \sum_{k=m_s-1}^{m}\left(|f_1(k, y'_k)| - 2\frac{\alpha(\beta)}{\beta^M}k^r|y'_k|^{p_k}\right) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&= \sum_{k=1}^{m_2-1}\left(|f_1(k, z_k^{(1)})| - 2\frac{\alpha(\beta)}{\beta^M}k^r|z_k^{(1)}|^{p_k}\right) + \sum_{k=m_2}^{m_3-1}\left(|f_1(k, z_k^{(2)})| - 2\frac{\alpha(\beta)}{\beta^M}k^r|z_k^{(2)}|^{p_k}\right) + \dots + \\
&\quad \sum_{k=m_{s-1}}^{m}\left(|f_1(k, z_k^{(s-1)})| - 2\frac{\alpha(\beta)}{\beta^M}k^r|z_k^{(s-1)}|^{p_k}\right) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&\le (s-1)\alpha(\beta) - 2\frac{\alpha(\beta)}{\beta^M}\left(\sum_{k=1}^{m_2-1}k^r|z_k^{(1)}|^{p_k} + \sum_{k=m_2}^{m_3-1}k^r|z_k^{(2)}|^{p_k} + \dots + \sum_{k=m_{s-1}}^{m}k^r|z_k^{(s-1)}|^{p_k}\right) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&\le (s-1)\alpha(\beta) - 2\frac{\alpha(\beta)}{\beta^M}(s-2)\frac{\beta^M}{2} + \sum_{k=1}^{m}\frac{\varepsilon}{2^k} \\
&= \alpha(\beta) + \sum_{k=1}^{m}\frac{\varepsilon}{2^k}
\end{aligned}$$

and thus $\sum_{k=1}^{\infty} c_k(\beta) = \lim_{m\to\infty}\sum_{k=1}^{m} c_k(\beta) \le \alpha(\beta) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\sum_{k=1}^{\infty} c_k(\beta) \le \alpha(\beta)$ and thus we get that $\|c(\beta)\| \le \alpha(\beta)$. Hence the lemma is proved.

Theorem 13

Let $f : N \times R \to R$. The superposition operator P_f acting from $F_r(p)$ to ℓ_1 is bounded if and only if for every $\rho > 0$ there exist $\alpha(\rho) > 0$ and a sequence $c(\rho) = (c_k(\rho)) \in \ell_1$ such that for each $k \in N$, $|f(k, t)| \le c_k(\rho) + 2\frac{\alpha(\rho)}{\rho^M}k^r|t|^{p_k}$ whenever $k^r|t|^{p_k} \le \rho^M$.

Proof .

($\Leftarrow$) Suppose that the condition holds. Let $\rho > 0$ and $x \in F_r(p)$ with $\|x\| \le \rho$. Then $\left(\sum_{k=1}^{\infty} k^r|x_k|^{p_k}\right)^{\frac{1}{M}} \le \rho$ and consequently $k^r|x_k|^{p_k} \le \rho^M$ for every $k \in N$. By the assumption, there is an $\alpha(\rho) > 0$ and a sequence $c(\rho) = (c_k(\rho)) \in \ell_1$ such that for each

$k \in N$, $|f(k, x_k)| \le c_k(\rho) + 2\frac{\alpha(\rho)}{\rho^M} k^r |x_k|^{p_k}$. As $P_f(x) = (f(k, x_k))_{k=1}^{\infty}$, $\|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| \le \sum_{k=1}^{\infty} c_k(\rho) + 2\frac{\alpha(\rho)}{\rho^M}\rho^M \le \sum_{k=1}^{\infty} |c_k(\rho)| + 2\alpha(\rho) = \|c(\rho)\| + 2\alpha(\rho)$. Hence P_f is bounded.

($\Rightarrow$) Suppose that the superposition operator P_f acting from $F_r(p)$ to ℓ_1 is bounded. Now let $\rho > 0$. For each $x \in \Phi$ with $\|x\| \le \rho$, we have $\left(\sum_{k=1}^{\infty} k^r |x_k|^{p_k}\right)^{\frac{1}{M}} \le \rho$. Since P_f is bounded, there exists an $\alpha(\rho) > 0$ such that $\|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| \le \alpha(\rho) < \infty$. By employing Corollary 11, we obtain that f satisfies the condition $A(2')$. Finally, apply Lemma 12 then there is $c(\rho) = (c_k(\rho)) \in \ell_1$ with $\|c_k(\rho)\| \le \alpha(\rho)$ such that for each $k \in N$,

$|f(k, t)| \le c_k(\rho) + 2\frac{\alpha(\rho)}{\rho^M} k^r |t|^{p_k}$ whenever $k^r |t|^{p_k} \le \rho^M$.

This completes the proof of the theorem.

Theorem 14

Let $f : N \times R \to R$. The superposition operator P_f acting from $F_r(p)$ to ℓ_1 is continuous if and only if f satisfies the condition A(2).

Proof .

($\Rightarrow$) Suppose that P_f is continuous on $F_r(p)$. Let $k \in N$, $t_0 \in R$ and $\varepsilon > 0$ be given. Since P_f is continuous at $t_0 e^{(n)} \in F_r(p)$, there is a $\delta > 0$ such that for each $z = (z_k) \in F_r(p)$,

$$\|P_f(z) - P_f(t_0 e^{(n)})\| < \varepsilon \text{ whenever } \|z - t_0 e^{(n)}\| < \delta . \quad (32)$$

Let $t \in R$ be such that $|t - t_0| < \left(\frac{\delta^M}{k^r}\right)^{\frac{1}{p_k}}$ and $y_n = \begin{cases} 0, & n \ne k \\ t, & n = k \end{cases}$. Then $y = (y_n) \in F_r(p)$ and $\|y - t_0 e^{(n)}\| = \left(k^r |t - t_0|^{p_k}\right)^{\frac{1}{M}} < \delta$. By (32), we have $\|P_f(y) - P_f(t_0 e^{(n)})\| < \varepsilon$ and thus $|f(k, t) - f(k, t_0)| \le \|P_f(z) - P_f(t_0 e^{(n)})\| < \varepsilon$. Hence the function f(k,.) is continuous on R for all $k \in N$. That is f satisfies the condition A(2).

($\Leftarrow$) Suppose that f(k,.) is continuous on R for all $k \in N$. We shall show that P_f is continuous on $F_r(p)$. Let $x = (x_k) \in F_r(p)$ and $\varepsilon > 0$. Since f satisfies the condition A(2), so we get that f satisfies the condition $A(2')$. As P_f acts from $F_r(p)$ to ℓ_1, so we apply Theorem 8, then there exist $\alpha, \beta > 0$ and a sequence $(c_k) \in \ell_1$ such that for each $k \in N$,

$$f(k, t)| \le c_k + \alpha k^r |t|^{p_k} \text{ whenever } k^r |t|^{p_k} \le \beta^M. \quad (33)$$

As $(x_k) \in F_r(p)$, we get $\sum_{k=1}^{\infty} k^r |x_k|^{p_k} < \infty$ and therefore $\lim_{k\to\infty} k^r |x_k|^{p_k} = 0$. And since $(c_k) \in \ell_1$, we obtain $\sum_{n=1}^{\infty} |c_k| < \infty$. Accordingly there is $N_i \in N - \{1\}$ such that

$$\|(x_k)_{k=N_i}^{\infty}\| = \left(\sum_{k=N_i}^{\infty} k^r |x_k|^{p_k}\right)^{\frac{1}{M}} < \frac{1}{2}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}}, \quad (34)$$

$$|x_k|^{p_k} < \frac{1}{k^r}\left(\frac{\beta}{2}\right)^M \text{ for all } k \geq N_i \text{ and } \sum_{k=N_i}^{\infty} c_k < \frac{\varepsilon}{6}. \tag{35}$$

Using (33), we have that $|f(k, x_k)| \leq c_k + \alpha k^r |x_k|^{p_k}$ for all $k \geq N_i$ and hence

$$\sum_{k=N_i}^{\infty} |f(k, x_k)| \leq \sum_{k=N_i}^{\infty} c_k + \alpha \sum_{k=N_i}^{\infty} k^r |x_k|^{p_k} < \frac{\varepsilon}{6} + \alpha \frac{1}{12^M}\left(\frac{\varepsilon}{\alpha}\right) < \frac{\varepsilon}{3}. \tag{36}$$

Since $f(k,.)$ is continuous at x_k for all $k \in \{1, 2, \ldots, N_i - 1\}$, there exist $\delta > 0$ with $\delta \leq \min \left\{1, \left(\frac{\beta}{2}\right)^M, \frac{1}{12}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}}\right\}$ such that for each $k \in \{1, 2, \ldots, N_i - 1\}$ and $t \in R$

$$|f(k, t) - f(k, x_k)| < \frac{\varepsilon}{3(N_i - 1)} \text{ whenever } |t - x_k| < \delta. \tag{37}$$

Let $z = (z_k) \in F_r(p)$ be such that $\|x - z\| < \delta$ hence

$$|z_k - x_k| < \left(\frac{\delta^M}{k^r}\right)^{\frac{1}{p_k}} \leq \delta \text{ for all } k \in N. \tag{38}$$

By (37), we get that $|f(k, z_k) - f(k, x_k)| < \frac{\varepsilon}{3(N_i - 1)}$ for every $k \in \{1, 2, \ldots, N_i - 1\}$ and thus

$$\sum_{k=1}^{N_i - 1} |f(k, z_k) - f(k, x_k)| < \frac{\varepsilon}{3}. \tag{39}$$

By choosing $\delta \leq \min\left\{1, \left(\frac{\beta}{2}\right)^M, \frac{1}{12}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}}\right\}$ and $\|x - z\| < \delta$, we obtain $\|(z_k)_{k=N_i}^{\infty} - (x_k)_{k=N_i}^{\infty}\| \leq \|z - x\| < \delta < \frac{1}{12}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}}$. Employing (34),

$$\left(\sum_{k=N_i}^{\infty} k^r |z_k|^{p_k}\right)^{\frac{1}{M}} = \|(z_k)_{k=N_i}^{\infty}\| \leq \|(x_k)_{k=N_i}^{\infty}\| + \|(z_k)_{k=N_i}^{\infty} - (x_k)_{k=N_i}^{\infty}\| < \frac{1}{12}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}} + \frac{1}{12}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}} = \frac{1}{6}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{M}},$$

this implies $\sum_{k=N_i}^{\infty} k^r |z_k|^{p_k} \leq \left(\frac{1}{6^M}\right)\frac{\varepsilon}{\alpha} \leq \frac{\varepsilon}{6\alpha}$. For each $k \geq N_i$, by utilizing (35) and (38),

$$|z_k| \leq |x_k| + |z_k - x_k| < \left(\frac{\beta^M}{2^M k^r}\right)^{\frac{1}{p_k}} + \left(\frac{\delta^M}{k^r}\right)^{\frac{1}{p_k}} \leq \frac{1}{2}\left(\frac{\beta^M}{k^r}\right)^{\frac{1}{p_k}} + \frac{1}{2}\left(\frac{\beta^M}{k^r}\right)^{\frac{1}{p_k}} = \left(\frac{\beta^M}{k^r}\right)^{\frac{1}{p_k}},$$

and hence $k^r |z_k|^{p_k} \leq \beta^M$. It follows from (33), $|f(k, z_k)| \leq c_k + \alpha k^r |z_k|^{p_k}$ for all $k \geq N_i$ and then

$$\sum_{k=N_i}^{\infty} |f(k,\ z_k)| \le \sum_{k=N_i}^{\infty} c_k + \alpha \sum_{k=N_i}^{\infty} k^r |z_k|^{\frac{p_k}{M}} < \frac{\varepsilon}{6} + \alpha \frac{\varepsilon}{6\alpha} = \frac{\varepsilon}{3} \ . \tag{40}$$

By using (36), (39) and (40), we have

$$\|P_f(z) - P_f(x)\| = \sum_{k=1}^{\infty} |f(k,z_k) - f(k,x_k)|$$

$$= \sum_{k=1}^{N_i-1} |f(k,z_k) - f(k,x_k)| + \sum_{k=N_i}^{\infty} |f(k,z_k) - f(k,x_k)| \le \sum_{k=1}^{N_i-1} |f(k,z_k) - f(k,x_k)| +$$

$$\sum_{k=N_i}^{\infty} |f(k,z_k)| + \sum_{k=N_i}^{\infty} |f(k,x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ .$$

The proof of this theorem is then completed.

The last of our results is Corollary 15 which follows from Theorem 9 and 14.

Corollary 15

Let $f : N \times R \to R$. If the superposition operator P_f acting from $F_r(p)$ to ℓ_1 is continuous then P_f is locally bounded.

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