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Magnetic field of direct current in heterogeneous ground

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Abstract

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We derive solutions of the steady state magnetic field due to a DC current source in four types of heterogeneous earth models. Four two-layered continuously earthed models are considered : an exponential earth with a homogeneous overburden, a homogeneous earth with an exponential overburden, a homogeneous earth with a linear varying overburden, and a linearly varying conductivity earth with a homogeneous overburden. These solutions are critical to interpret the magnetometric resistivity (MMR) data. Our solutions are achieved by solving a boundary value problem in the wave number domain and then transforming back to the spatial domain. The propagator matrix techniques are used. The curves of magnetic field are plotted to show the behavior of the field while some parameters are given approximately. To determine the conductivity parameter, the inverse problem is introduced via the use of optimization technique.

Key words : magnetometric resistivity, magnetic field, MMR

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บทคัดย่อ

สืบสกุล อยู่ยืนยง และ วรินทร์ ศรีปัญญา
สนามแม่เหล็กของไฟฟ้ากระแสตรงในโครงสร้างพื้นดินแบบวิวิธพันธุ์
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งานวิจัยนี้นำเสนอการศึกษาหาผลเฉลยของสนามแม่เหล็กในสภาวะคงตัว ที่เกิดจากขั้วไฟฟ้ากระแสตรงซึ่งถูกฝังลงในพื้นโลกที่มีลักษณะเป็นวิวิธพันธุ์และเป็นชั้น โดยที่สภาพนำไฟฟ้าของพื้นโลกในลักษณะ 2 ชั้นถูกนำมาพิจารณา 4 กรณีคือดินมีลักษณะการเปลี่ยนแปลงสภาพนำไฟฟ้าแบบฟังก์ชันเลขชี้กำลังโดยดินชั้นบนมีสภาพนำไฟฟ้าแบบฟังก์ชันคงที่ กรณีที่ดินมีสภาพนำไฟฟ้าแบบฟังก์ชันคงที่โดยที่ดินชั้นบนมีสภาพนำไฟฟ้าแบบฟังก์ชันเลขชี้กำลัง กรณีที่ดินมีสภาพนำไฟฟ้าแบบฟังก์ชันคงที่โดยดินชั้นบนมีสภาพนำไฟฟ้าแบบฟังก์ชันเชิงเส้น และกรณีสุดท้ายพื้นดินมีสภาพนำไฟฟ้าแบบฟังก์ชันเชิงเส้นโดยดินชั้นบนมีสภาพนำไฟฟ้าแบบฟังก์ชันคงที่ ผลเฉลยของสนามแม่เหล็กสามารถหาได้จากการแก้ปัญหาค่าขอบเขตและอาศัยการแปลงทางคณิตศาสตร์ วิธีการทางเมทริกซ์ถูกนำมาใช้ประกอบในการแก้ปัญหา โดยการทดลองเชิงตัวเลข เมื่อกำหนดค่าพารามิเตอร์ให้ จะสามารถเขียนกราฟความสัมพันธ์ระหว่างสนามแม่เหล็กและระยะห่างของเครื่องมือรับ-ส่งได้ นอกจากนี้โดยอาศัยการหาค่าเหมาะที่สุดนำมาแก้ปัญหาผกผัน ทำให้สามารถหาค่าพารามิเตอร์ของสภาพนำไฟฟ้าของโครงสร้างใต้พื้นโลกได้

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The magnetometric resistivity method has recently become an additional electrical prospecting technique used for finding mineral resources. This technique is based on the measurement of low-level, low-frequency magnetic fields associated with non-inductive current flow in the ground. Edwards *et al*. (1985) discussed a specific case where the upper half-space is conductive seawater, as encountered in the magnetometric offshore electrical sounding system. Edwards (1988) and Edwards and Nabighian (1991) concentrated the ratio of the magnetic fields below and above a known conductive layer to infer the basement resistivity. Sezginer and Habashy (1988) computed the static magnetic field due to an arbitrary current injected into a conducting uniform half-space. Inayat-Hussein (1989) gave a new proof that the magnetic field outside the one dimension medium. Veitch *et al*. (1990) pointed out that the general solution for the magnetic field within a layered earth due to a point source has not been fully explored. They, indirectly, derived the magnetic field by applying stoke's theorem and Ampare's law to the electric potential. Unfortunately, these works do not supply the amount of information about the magnetic field that is required for many current applications. Chen and Oldenburg (in press) derived the magnetic field directly from solving a boundary value problems which was similar to the approach used by Edwards (1988) and then discussed a homogeneous and a 2-layered earth model. The motivation of this study is to determine if the magnetometric resist-ivity method may have applications for salinity mapping in different parts of Thailand. The continuous varying conductivity ground profiles used in this paper are constant, linearly and exponentially with depth.

Magnetic field due to a semi-infinite source in a 1-dimension earth

A semi-infinite vertical wire carries an exciting current I and terminates at the electrode Q. The electrode Q is deliberately placed at the interface $z=z_s$ of layer s and layer $s+1$. Each layer has conductivity as a function of depth, $\sigma_j(z)$ with thickness h_j. The Maxwell's equations can be used to determine the magnetic field intensity $\overline{H}$ as

$$\nabla \times \overline{E} = \overline{0}, \qquad (1)$$

and $$\nabla \times \overline{H} = \sigma \overline{E}, \qquad (2)$$

where $\overline{E}$ is the electric field intensity, $\overline{H}$ is the magnetic field intensity and σ is the conductivity of the medium. Using (1) and (2), we have

$$\nabla \times \frac{1}{\sigma} \nabla \times \overline{H} = \bar{0}. \quad (3)$$

Since the problem is axi-symmetric, $\overline{H}$ has only an azimuthal component in cylindrical coordinate (r,ϕ,z). For simplicity, we use H to represent the azimuthal component in the following derivations. Expanding equation (3) yields

$$\frac{1}{\sigma}\frac{\partial^2 H}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z} + \frac{1}{\sigma}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}(rH) + \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial r}(rH)\right) + \frac{\partial}{\partial r}\left(\frac{1}{\sigma}\right)\frac{1}{r}\frac{\partial}{\partial r}(rH) = 0.$$

Since σ is a function of depth z only, the above equation becomes

$$\frac{\partial^2 H}{\partial z^2} + \sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z} + \left(\frac{\partial^2 H}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial H}{\partial r}\right) - \frac{1}{r^2}H = 0. \quad (4)$$

Taking the Hankel transform defined by

$$\tilde{H}(\lambda, z) = \int_0^\infty rH(r,z)J_1(\lambda r)dr \quad , \quad (5)$$

where J_1 is the Bessel function of the first kind of order one, to the equation (4), we have

$$\frac{\partial^2 \tilde{H}}{\partial z^2} + \sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial \tilde{H}}{\partial z} - \lambda^2 \tilde{H} = 0. \quad (6)$$

1. The magnetic field response from the exponential conductivity ground profile

Soil salinity profiles frequently display monotonically increasing or decreasing salt concentrations with depth, z. This salt concentration is strongly correlated with the conductivity of the ground and frequently can be represented by an equation

$$\sigma(z) = ae^{nbz}, \quad (7)$$

where a and b are greater than zero and $n \in \{-1,0,1\}$, and hence the equation (6) becomes

$$\frac{\partial^2 \tilde{H}}{\partial z} - nb\frac{\partial \tilde{H}}{\partial z} - \lambda^2 \tilde{H} = 0. \quad (8)$$

The solution of (8) is

$$\tilde{H}(\lambda, z) = Ae^{\alpha^- z} + Be^{\alpha^+ z}, \quad (9)$$

where A and B are arbitrary constants which can be determined from the boundary conditions and

$$\alpha^- = \frac{nb - \sqrt{(nb)^2 + 4\lambda^2}}{2},$$

and

$$\alpha^+ = \frac{nb + \sqrt{(nb)^2 + 4\lambda^2}}{2}.$$

2. The magnetic field response from conductive homogeneous ground profile

For the constant conductivity profile, the equation (6) becomes

$$\frac{\partial^2 \tilde{H}}{\partial z^2} - \lambda^2 \tilde{H} = 0, \quad (10)$$

and the solution is

$$\tilde{H}(\lambda,z) = Ce^{-\lambda z} + De^{\lambda z}, \qquad (11)$$

where C and D are arbitrary constants which can be determined from the boundary conditions.

3. The magnetic field response from the linearly conductivity ground profile

For the linearly varying conductivity ground profile, the equation represented the variation is denoted by

$$\sigma(z) = a(1+nbz)^m,$$

where $a,b>0, n \in \{-1,1\}$, and $m \in \{0,1\}$. Putting $\sigma(z)$ to the equation (6), we now have

$$\frac{\partial^2 \tilde{H}}{\partial z^2} - \frac{nmb}{1+nbz}\frac{\partial \tilde{H}}{\partial z} - \lambda^2 \tilde{H} = 0, \qquad (12)$$

and the solution is

$$\tilde{H}(\lambda,z) = \alpha^p(z) \left[EI_{-p}(\lambda\alpha(z)/b) + FK_{-p}(\lambda\alpha(z)/b)\right], \qquad (13)$$

where E and F are arbitrary constants which can be determined from the boundary conditions. I_p and K_p are the Modified Bessel functions of the first and second kind of order p. $\alpha(z)=1+nbz$ and $p=(1+m)/2$

Two layered earth models

Although stratified models are often relevant and can usually be applied to real geoelectric structures, few treatments of continuous geoelectric structures have been presented. Although stratified models with a large number of layers can represent identically a continuum's response to surface measurements, numerical computation usually takes quite a long time. A better way to handle certain continuous structures might be to solve the equations directly for the desired structure. In our work, we design the model as a two layered earth structure.

1. An exponential earth with a homogeneous overburden

In the case of an exponential earth with a homogeneous overburden, we denote the conductivity of the ground as

$$\sigma_1(z) = a, \qquad 0 \le z \le h,$$

$$\sigma_2(z) = ae^{nb(z-h)}, \qquad z > h,$$

where $a, b > 0$ and $n \in \{-1,1\}$. The magnetic fields in the first and second layers after using the boundary conditions at the interfaces are obtained from (10) and (8) as

$$H_1(r,z) = \int_0^\infty \lambda \left\{ \frac{I}{2\pi\lambda}\left[1 - \frac{sinh(\lambda z)}{sinh(\lambda h) - cosh(\lambda h)/\delta^-}\right]\right\} J_1(\lambda r)d\lambda, \quad 0 \le z \le h,$$

$$H_2(r,z) = \int_0^\infty \lambda \left\{ \frac{Ie^{\alpha^-(z-h)}}{2\pi\lambda(1 - tanh(\lambda h)\delta^-)}\right\} J_1(\lambda r)d\lambda, \quad z > h,$$

where $\delta^- = \alpha^-/\lambda$.

2. A homogeneous earth with an exponential overburden

In the case of a homogeneous earth with an exponential overburden, we denote the conductivity of the ground as

$$\sigma_1(z) = ae^{nbz}\ , 0 \le z \le h\ ,$$

$$\sigma_2(z) = \sigma_1(h), \quad z > h,$$

where a,b>0 and n $\in$ {-1,1}. The magnetic fields in the first and second layers after applying the boundary conditions at the interfaces can be obtained from (8) and (10), respectively, as

$$\mathrm{H}_1(r,z) = \int_0^\infty \lambda \left\{ \frac{I}{2\pi\lambda} \left[1 - \frac{e^{\alpha^- z} - e^{\alpha^+ z}}{(1+\delta^-)e^{\alpha^- h} - (1+\delta^+)e^{\alpha^+ h}} \right] \right\} J_1(\lambda r)d\lambda,\ 0 \le z \le h,$$

$$\mathrm{H}_2(r,z) = \int_0^\infty \lambda \left\{ \frac{I}{2\pi\lambda} \left[\frac{(\delta^- e^{\alpha^- h} - \delta^+ e^{\alpha^+ h})e^{-\lambda(z-h)}}{(1+\delta^-)e^{\alpha^- h} - (1+\delta^+)e^{\alpha^+ h}} \right] \right\} J_1(\lambda r)d\lambda,\ z > h,$$

where $\delta^- = \alpha^- / \lambda$ and $\delta^+ = \alpha^+ / \lambda$.

3. A homogeneous earth with a linear varying overburden

In the case of a homogeneous earth with a linear varying overburden, we denote the conductivity of the ground as

$$\sigma_1(\mathrm{z}) = a(1+nbz)\ ,\ 0 \le \mathrm{z} \le h\ ,$$

$$\sigma_2(\mathrm{z}) = \sigma_1(h),\ \mathrm{z} > h,$$

where a,b>0 and $n \in \{-1,1\}$. The magnetic fields in the first and second layers after applying the boundary conditions at the interfaces can be obtained from (12) and (10), respectively, as

$$H_1(\mathrm{r,z}) = \int_\circ^\infty \lambda \left\{ \frac{\mathrm{I}}{2\pi\lambda} \left[1 - \frac{\alpha(\mathrm{z})[\mathrm{I}_1(\lambda\alpha(z)/b) - \delta K_1(\lambda\alpha(z)/b)]}{\alpha(\mathrm{h})(\mathrm{I}^* + \delta \mathrm{K}^*)} \right] \right\} \mathrm{J}_1(\lambda \mathrm{r})\mathrm{d}\lambda,\ 0 \le \mathrm{z} \le \mathrm{h},$$

$$H_2(\mathrm{r,z}) = \int_\circ^\infty \lambda \left\{ \frac{I}{2\pi\lambda} \left[\frac{I_0(\lambda\alpha(h)/b) + \delta K_0(\lambda\alpha(h)/b)}{\mathrm{I}^* + \delta \mathrm{K}^*} \right] ne^{-\lambda(\mathrm{z-h})} \right\} J_1(\lambda r)\mathrm{d}\lambda, z > h,$$

where I_v and K_v are Modified Bessel function of the first and second kind of order v . $\delta = I_1(\lambda/b)/K_1(\lambda/b)$, $\alpha(z) = 1+nbz$, $I^* = nI_0(\lambda\alpha(h)/b) + I_1(\lambda\alpha(h)/b)$, and $K^* = nK_0(\lambda\alpha(h)/b) - K_1(\lambda\alpha(h)/b)$.

4. A linearly varying conductivity earth with a homogeneous overburden

In this case, we denote the conductivity of the ground as

$$\sigma_1(z) = a,\ 0 \le z \le h\ ,$$

$$\sigma_2(z) = a(1+nb(z-h)),\ z > h,$$

where $a > 0$, $n = 1$ and $b > 0$. The magnetic fields in the first and second layers after applying the boundary conditions at the interface can be obtained from (10) and (12), respectively, as

$$H_1(r,z) = \int_0^\infty \lambda \left[\frac{I}{2\pi\lambda} \left\{ 1 - \frac{sinh(\lambda z)}{sinh(\lambda h) + cosh(\lambda h) K_1(\lambda / b) / K_0(\lambda / b)} \right\} \right] J_1(\lambda r) d\lambda, 0 \le z \le h,$$

$$H_2(r,z) = \int_0^\infty \lambda \left[\frac{I}{2\pi\lambda} \frac{\alpha(z-h) K_1(\lambda\alpha(z-h)/b)}{(K_1(\lambda / b) + tanh(\lambda h) K_0(\lambda / b))} \right] J_1(\lambda r) d\lambda,\ z > h.$$

Numerical experiments

In our forward model examples, we compute the magnetic field due to direct current source on the ground surface of the 4 models in the previous section. The models are applied a current of 1-Ampere, injected by the probe length of 1 meter perpendicular to the ground surface. The depth of overburden for the entire example models is 1 meter under the ground surface. The conductivity used for the first example model is $\sigma_1(z) = e^{-0.1960475832}$ for overburden and $\sigma_2(z) = e^{-0.1960475832(z-h)}$ for the second layer. The conductivity used for the second example model is $\sigma_1(z) = e^{-0.1960475832z}$ for overburden and $\sigma_2(z) = \sigma_1(1)$ for the second layer. The conductivity used for the third example model is $\sigma_1(z) = 1-0.1960475832z$ for overburden and $\sigma_2(z)=\sigma_1(1)$ for the second layer. The conductivity used for the last example model is $\sigma_1(z) = e^{-0.1960475832z}$ for overburden and $\sigma_2(z)=(1-0.1960475832(z-h))\sigma_1$ for the second layer. The results are performed as the graphs in Figures 1, 2, 3 and 4, respectively. The graphs show the behavior of the magnetic field against source-receiver spacing (r) at different depth. The curves of each model at the same depth are not too much different, but they are quite different with varying depth. The magnetic field intensity drops very fast as we increase the source-receiver spacing to 10 meters.

In our inverse model example, we simulate reflection of radiation data from our forward model by injecting the 1-Ampere of current to the ground. The conductivity distribution below the ground surface is assumed to be continuous and depends only on depth. In our example, the model is given by

$$\sigma_1(z) = e^{-0.1960475832z},\ 0 \le z \le 1,$$

$$\sigma_2(z) = \sigma_1(1),\ z > 1.$$

The forward model to simulate the set of real data generates the magnetic field. Superimposing a Gaussian relative error to the 3 per cent level perturbs the theoretical values. The associated errors can be regarded as realizations of normal random variables with zero means and variances S_i^2; $i = 1, 2, \ldots, m$. Table 1 shows the result from our procedure. We start the model with initial guess b=1 and n = -1. The result from our procedure converges to b = 0.1960475832 which is the true value after using 7 iterations only.

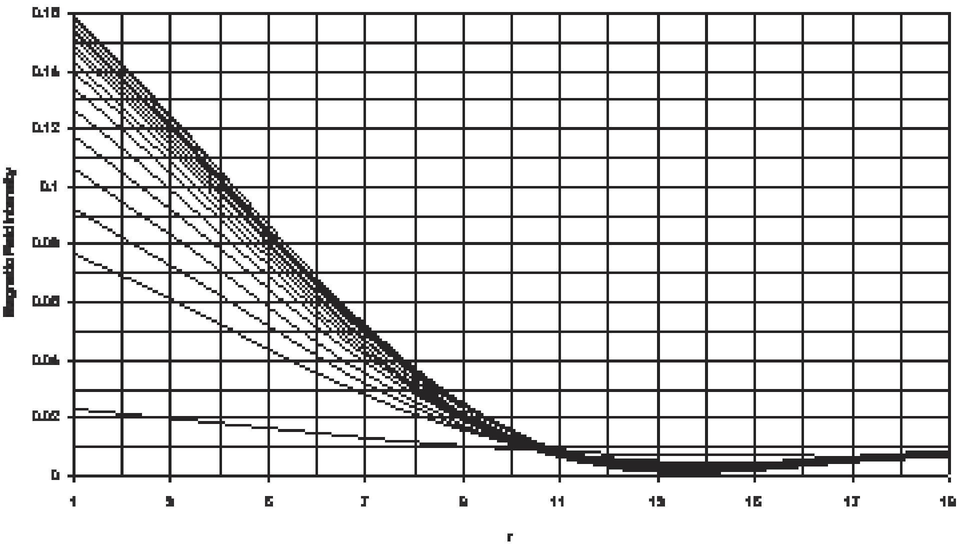

Figure 1. The behavior of magnetic fields (from model in section 3.1) against *r* at different depth z = 0, 0.2, 0.4,...,3.2,3.4

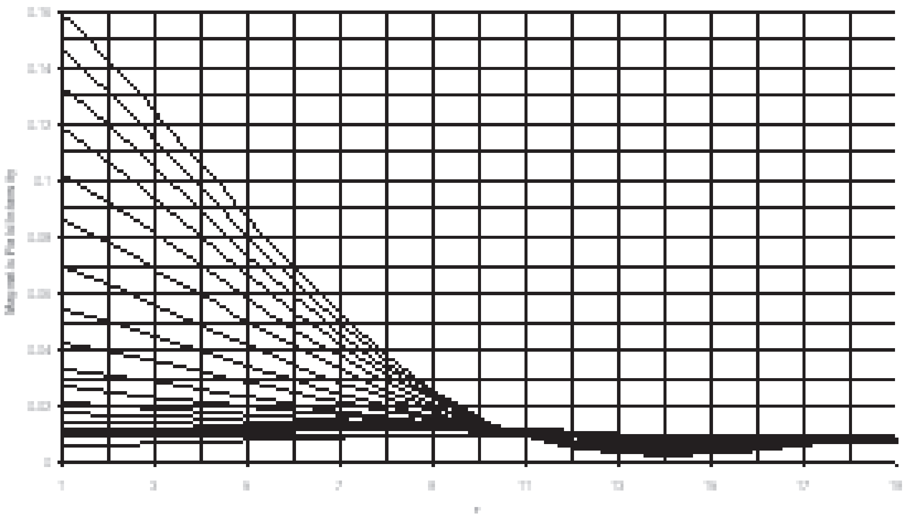

Figure 2. The behavior of magnetic fields (from model in section 3.2) against *r* at different depth z = 0, 0.2, 0.4,...,3.2,3.4

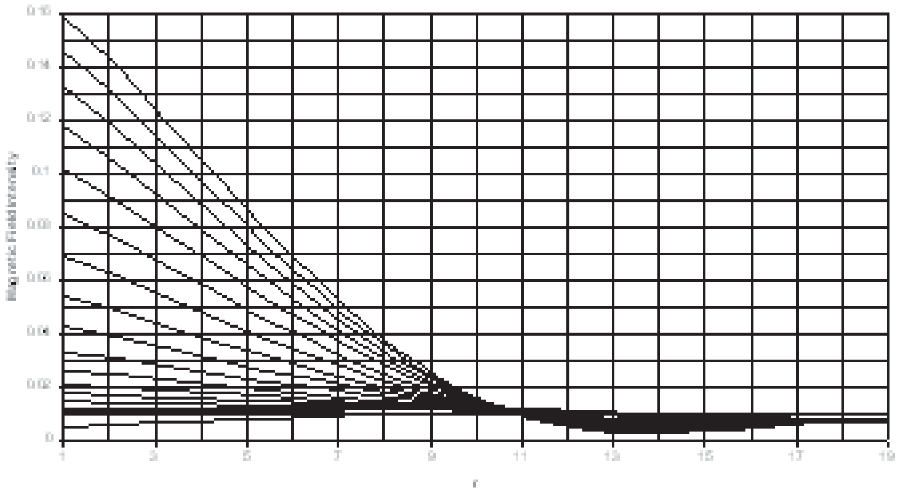

Figure 3. The behavior of magnetic fields (from model in section 3.3) against *r* at different depth z = 0, 0.2,

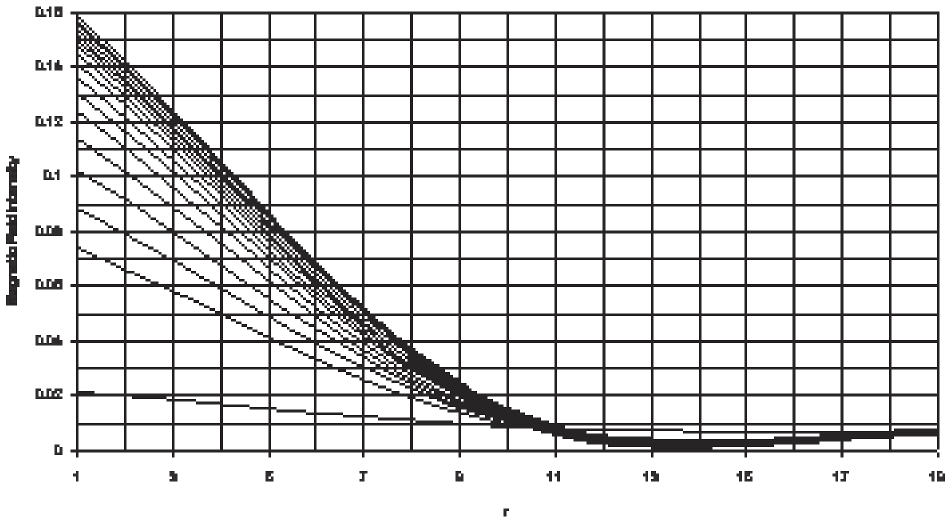

Figure 4. The behavior of magnetic fields (from model in section 3.4) against *r* at different depth z = 0, 0.2, 0.4, ...,3.2,3.4

Table 1. Successive iterates using initial estimates for *n* = -1, *b* = 1.0 in an exponentially decreasing ground profile with true values being *n* = -1 and *b* = 0.1960475832

i	0	1	2	3	4	5	6	7
b	1.000000000	0.496656754	0.196014757	0.196647589	0.196647583	0.196647584	0.196047583	0.196047583
ΔH_i	0.3713×10^{-3}	1.6661×10^{-8}	5.0911×10^{-11}	4.6283×10^{-12}	7.7139×10^{-13}	3.0855×10^{-12}	3.8569×10^{-13}	5.5523×10^{-17}

Conclusions

In this paper, we conducted a method to explore the parameter of the conductivity of ground. The method used the integral transform technique to produce the magnetic field which can be computed easily. The magnetic field is plotted against source-receiver spacing (r) at different depths and these can be used for comparing with the observed magnetic data to identify the earth structure. The inversion process is used to find out the parameter of the conductivity of ground. The Quasi-Newton method is used to construct the iterative procedure. The method produces a good result and shows the robustness of the procedure.

References

Chen, J. and Oldenburg, D.W., Magnetic and electrical fields of direct currents in layered earth. Geophysics (in press).

Edwards, R. N. 1988.A downhole MMR technique for electrical sounding beneath a conductive surface layer: Geophysics, 53: 528-536.

Edwards, R.N., Law, L. and DeLaurier, J. 1985. On measuring the electrical conductivity of the ocean crust by a modified magnetometric resistivity method: J. Geophys. Res., 86, B12, 11609-11615.

Edwards, R.N., and Nabighian, M.1991. The magnetometric resitivity method, in Nabighian, M.N., Ed., Electromagnetic Methods in Applied Geophysics: Investigations in Geophysics: Soc. Expl. Geophys., 47-104.

Inayat-Hussein, A.A. 1989. Magnetic fields of direct currents in horizontally stratified conductors: J. Appl. Phys., 65, 3731-3732.

Sezginer, A. and Habashy, T. 1988. An image method to compute the static magnetic field due to current injected into a homogeneous, conducting, and magnetically polarizable half-space: Radio Science, 23, 41-45.

Veitch, D., Asten, M.W., and Van Leeuwen, E.H.1990. Electrical and magnetometric fields in a layered earth containing buried electrodes: Geophysics, 55: 1605-1612.