ORIGINAL ARTICLE

Linear system design using Routh Column polynomials

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Abstract

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In this paper, a novel approach for estimating the range of parameters, which exist as coefficients in the characteristic polynomial of a given Linear Time Invariant system, is presented. The approach involves the application of lower degree column polynomials obtained from the Routh table, which are termed as Pseudo Routh Column Polynomials. The approximate ranges of parameters obtained from the proposed approach are further sharpened to achieve marginal stability with the help of Routh's table formed for the original higher degree characteristic polynomial. The proposed approach is simple and straightforward in applications and is illustrated with suitable examples.

Key words : Routh table, Pseudo Routh Column Polynomials, Bisection Principle, marginal stability, design of parameters

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In general, the design of single and multi parameters existing as coefficients in the characteristic polynomial of a Linear Time Invariant system can be performed using the methods proposed by Nyquist (Nyquist, 1932), Bode (Bode, 1940), Neimark (Porter, 1967), Mikailov (Stojic & Siljak, 1965), Evans (Evans, 1948, 1950, 1954), Mitrovic (Thaler and Brown, 1960), Siljak (Stojic and Siljak, 1965), Sturm (Gantmacher, 1959), Maxwell (Porter, 1967), Clifford (Porter, 1967), Routh (Routh, 1905), Hurwitz (Hurwitz, 1964), Lienard-Chipart (Gantmacher, 1959), Wall (Wall, 1948), Chebyshev (Gantmacher, 1964), Lyapunov (Gantmacher, 1964), Hermite (Gantmacher, 1964), Schwarz (Gantmacher, 1964), Ralston (Ralston, 1962), Anderson-Jury (Jury and Anderson, 1972) and Barnett (Barnett, 1970). From the above methods, it can be easily observed that the Routh test (Chang and Chen, 1974, Ferrante, Lepschy, Viaro, 1999) is simple but the exact value of the interested parameter in the system cannot be predicted. As a result, various methods were brought out for the design of Linear Time Invariant Continuous system.

On viewing the design aspect, one can select Hurwitz method (Michael Margaliot, Crisloeon Laugholz, 2000) or Markov parameters (Wagner, 1959) or Ralston's method (Ralston, 1962). Among these three methods, Hurwitz method is comparatively simpler than Markov and Ralston's methods. Lienard-Chipart (Gantmacher, 1964) formulated a simplified version of Hurwitz method by reducing the evaluation of number of determinants. The disadvantage in this case is that it is not well suited for design purpose. Also, it should be noted that in the case of the higher order characteristic polynomial with unknown design parameters, the application of Routh table becomes tedious. To circumvent this situation, in this paper, the lower order polynomials termed as Pseudo Routh Column Polynomials are formed from the Routh table and applied further for extracting the approximate range of values of the design parameters. The forthcoming section discusses the Routh table in terms of array formulation, which provides a convenient method of determining control systems stability. This method may also be used to establish limiting values for a variable factor beyond which the system would become unstable.

Routh's Table (Kuo, Goharaghi, 2003)

The characteristic equation of a Linear Time Invariant Continuous system is written as,

$$F(s) = \sum_{i=0}^{n} a_i s^i = 0 \tag{1}$$

where 'n' is the degree of F(s) and all a_i are present with same sign for the system to be stable. The Routh's array is constructed using Routh's algorithm where the first two rows contain the coefficients of F(s) as shown in Table 1. The remaining elements in Table 1 are calculated using Routh's Multiplication rule given in equation (2) below.

$$\left.\begin{aligned} b_n &= \frac{a_{n-1} \times a_{n-2} - a_n \times a_{n-3}}{a_{n-1}} \\ b_{n-2} &= \frac{a_{n-1} \times a_{n-4} - a_n \times a_{n-5}}{a_{n-1}} \\ b_{n-1} &= \frac{b_n \times a_{n-3} - a_{n-1} \times b_{n-2}}{b_n} \end{aligned}\right\} \tag{2}$$

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The element from third rows onwards are calculated based on Routh's multiplication rule given in equation (2). For the Table 1, the following properties are true (Nagrath & Gopal, 2000).

Property 1: The number of coefficients in alternate rows is reduced by one.

Property 2: Because of the triangular shape of the Routh table, the constant term appears as the last element in alternate rows.

Based on Routh table, for a system to be stable, the elements in the first column must have the same sign. If there exists sign changes in the first column of the Routh table, then the system is unstable. The marginal stability condition can be identified by the presence of a zero element in the

Table 1. Routh Table

Column/ Row	1	2	3	4		(m-1)	M
s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	• • •	a_3	a_1
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	• • •	a_2	a_0
s^{n-2}	b_n	b_{n-2}	b_{n-4}	• • •	• • •	b_1	
s^{n-3}	b_{n-1}	b_{n-3}	b_{n-5}	• • •	• • •	b_0	
s^{n-4}	c_n	c_{n-2}	c_{n-4}	• • •	• • •		
s^{n-5}	c_{n-1}	c_{n-3}	c_{n-5}	• • •	• • •		
•	•	•	•	•	•		
•	•	•	•	•	•		
•	•	•	•	•	•		
s^1	R_1						
s^0	a_0						

Table 2. Routh Table (n = odd, m=)

Column/ Row	1	2	3	4		(m-2)	(m-1)	M
s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	• • •	a_5	a_3	a_1
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	• • •	a_4	a_2	a_0
s^{n-2}	b_n	b_{n-2}	b_{n-4}	• • •	• • •	b_3	b_1	
s^{n-3}	b_{n-1}	b_{n-3}	b_{n-5}	• • •	• • •	b_2	b_0	
s^{n-4}	c_n	c_{n-2}	c_{n-4}	• • •	• • •	c_1		
s^{n-5}	c_{n-1}	c_{n-3}	c_{n-5}	• • •	• • •	c_0		
•	•	•	•	•	•			
•	•	•	•	•	•			
•	•	•	•	•	•			
s^3	y_3	y_1						
s^2	y_2	y_0						
s^1	z_1							
s^0	z_0							

s^1-th row of the first column.

Formation of Pseudo Routh Column Polynomials (S.N. Sivanandam and S.N. Deepa, 2005)

Consider the characteristic polynomial,

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \ldots$$
$$+ a_3 s^3 + a_2 s^2 a_1 s + a_0 \qquad (3)$$

Assume that the given polynomial to be of odd degree (i.e. n is odd) and formulate Routh's table as shown in Table 2.

Depending upon the value of 'n', $\left(\frac{n+1}{2}\right)$ columns are formed for an odd polynomial. Thus, the maximum number of columns is given by,

$$m = \left(\frac{n+1}{2}\right) \qquad (4)$$

In Table 2 b_0, c_0 and z_0 are all equal to a_0 by property 2. It can be noted that in an odd polynomial, the last term of original polynomial F(s) appears as the last term in all columns.

From Table 2, it should be noted that the computation starts from the mth column, (m-1)th column, (m-2)th column and so on i.e., in the (m-1)th column, b1 is calculated first and is given by,

$$b_1 = a_1\left[1-\left(\frac{a_n}{a_{n-1}}\right)\times\left(\frac{a_0}{a_1}\right)\right] \quad (5)$$

By property 2, the last element in the (m-1)th column is just a shift from the last element of m-th column. In a similar manner, all the elements of the (m-2)th column are generated and so on. Thus, the Routh table is computed starting from last column onwards.

Once the column elements are generated, then the Pseudo Routh Column Polynomials are formed. For a polynomial of odd degree n, $\left(\frac{n-1}{2}\right)$ number of Pseudo Routh Column Polynomials can be computed. The formulation of Pseudo Routh Column Polynomials from Table 2 are as follows:

i) mth column : $P_m(s)=a_1s+a_0$ (6)

ii) (m-1)th column : $P_{(m-1)}(s) = a_3s^3+a_2s^2+b_1s+b_0$ (7)

iii) (m-2)th column : $P_{(m-2)}(s) = a_5s^5+a_4s^4+b_3s^3+b_2s^2+c_1s+c_0$ (8)

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It is found that the above Pseudo Routh Column Polynomials in equations (6-8) are very helpful in determining the instability of higher order systems leading to minimum computations (S.N. Sivanandam and S.N. Deepa, 2005). Also, the design of the unknown parameters existing in the original characteristic polynomial are easily dealt with the help of Pseudo Routh Column Polynomials $P_m(s)$ or $P_{m-1}(s)$ or $P_{m-2}(s)$... . In case of design, these polynomials help in extracting an approximated range for the design parameters (say gain 'K') and further sharpening process is carried out with Routh table for the original characteristic polynomial F(s) to achieve marginal stability (s1-th row element tends to zero). The Pseudo Routh Column Polynomial approach reduces the computational burden in designing the parameters. The same concept is applicable for 'n' even also. The proposed Pseudo Routh Column Polynomial approach presented in the algorithmic form is shown in section 4.

Proposed Algorithm

The various steps involved in the algorithm are as follows:

Step 1: Read the given characteristic polynomial F(s).
(Containing a parameter, say, 'K' to be designed for stability)

Step 2: Compute the Routh table for F(s) starting from last column onwards as shown in Table 2.

Step 3: Build the first Pseudo Routh Column Polynomial $P_m(s)$ starting from last column.

Step 4: If $P_m(s)$ is of 3rd degree
Goto Step5
Else
Goto Step12

Step 5: Construct Routh's array for $P_m(s)$ in terms of 'K'

Step 6: Extract the approximated range for 'K', from s^1-th row and first column of Routh table of $P_m(s)$.
If extracted range of 'K' has lowest and highest real values
Goto Step 7
Else
Goto Step 12

Step 7: Take, lowest real value of $K=K_1$ and highest real value of $K=K_2$
Thus, $K_1 < K < K_2$

Step 8: Choose a mid point between (K_1, K_2) to be K_3 to get two sub ranges (K_1,K_3) and (K_3, K_2).

Step 9: Form Routh's table for F(s) with $K=K_1$ and $K=K_3$. Apply bisection principle (mid point between any two given values), sharpen this range by checking if the coefficient of s^1-th term of Routh's Table tends to zero, to get lower limit K_l.

Step 10: Similarly, form Routh table for F(s) with $K=K_2$ and $K=K_3$. Applying principle of bisection sharpen the range (K_3, K_2) to get upper limit Kh.

Step 11: Thus final sharpened values of K are,

$$Kl < K < Kh \qquad (9)$$

For practical implementation, any value of K in the above range may be chosen.

Goto Step 13.

Step 12: Construct the next Pseudo Routh Column Polynomial say $P_{(m-1)}(s)$ or $P_{(m-2)}(s)$ or $P_{(m-3)}(s)$ or .... and perform steps 4 to 11.

Step 13: Stop

The above proposed algorithm is used for the design of single and two parameters in automatic voltage regulator system (Battison and Multineux, 1965) in the following section.

Illustrations

1. Example 1 (Battison and Multineux, 1965) - Single parameter design

The characteristic polynomial of a feedback control system incorporated with an automatic voltage regulator is given by,

$$F(s) = \sum_{i=0}^{9} a_i s^i = 0 \qquad (10)$$

where,

$a_9 = 2.06$ $a_4 = 11850+4620K$
$a_8 = 62.4$ $a_3 = 4160+6700K$
$a_7 = 713$ $a_2 = 1523K-38$
$a_6 = 3896$ $a_1 = 285-43.2K$
$a_5 = 10260+824.5K$ $a_0 = 16.6$

and the voltage regulator parameter is K > 0.

On applying the proposed algorithm to the automatic voltage regulator problem, we get,

Step 1: The given characteristic polynomial F(s) is read and is found to contain the design parameter 'K'.

Step 2: The Routh table for F(s) is constructed starting from last column onwards. (First two rows are scaled for convenience) and is shown in Table 3.

Step 3: Formulating first Pseudo Routh Column Polynomial $P_5(s)$ from last column, we get,

$$P_5(s) = (138-21K)s+0.266 \qquad (11)$$

Step 4: Obtained $P_5(s)$ is not of 3rd degree hence goto step 12.

(But from $P_5(s)$, we get K < 138/61 i.e., K<6.6, thus the design

Table 3. Routh Table for F(s) (starting from last column onwards)

Column/ Row	1	2	3	4	5	
s^9	1	346	(4980+400K)	(2020+3250K)	(138-21K)	
s^8	1	62.5	(190+73.2K)	(24.4K-0.61)	0.266	→ $P_5(s)$
s^7	•	•	•	(137.734-21K)		
s^6	•	•	•	0.266		→ $P_4(s)$
s^5	•	•	•			
s^4	•	•	•			
s^3	•	•				
s^2	•	•				
s^1	•					
s^0	0.266					

approach should satisfy this condition)

Step 12: Build the next Pseudo Routh Column Polynomial from last but one column, say, P4(s)

$P_4(s) = (2020+3250K)s^3 + (24.4K -0.61)s^2 + (137.734-21K)s + 0.266$ (12)

Perform Steps 4 to 11.

Step 4: $P_4(s)$ is of 3rd degree, so goto Step 5

Step 5: Constructing Routh's table for $P_4(s)$ in terms of 'K' as shown in Table 4

Step 6: Now, extracting the approximated range of K, from s^1-th row and first column of Routh table of $P_4(s)$, we get,

$(-512.4K^2 + 2509.0196K - 621.3377) > 0$ (13)

As a result, the limits of K obtained are,

K = 0.2616 and 4.6350.

The extracted range for 'K' are real, Goto Step 7

Step 7: Take, lowest real value of K = 0.2616 and highest real value of K = 4.6350

Thus, $0.2616<K<4.6350$ (14)

Step 8: Choosing a mid point between (0.2616, 4.6350) to be 2.4483, to get two sub ranges,

(0.2616, 2.4483) and

(2.4483, 4.6350)

Step 9: The values for K_l are sharpened. The design process is indicated in the Table 5.

Thus, sharpened value of lower limit is, $K_l = 0.6800$

Step 10: Now, sharpening the K_h values as shown in Table 6 between (2.4483, 4.6350), we get,

$K_h = 4.6265$ (occurrence of marginal stability)

Step 11: Hence, the final sharpened values of K are,

$0.6800 < K < 4.6265$ (15)

Goto Step 13.

Step 13: Stop.

Thus $K_l = 0.6800$ and $K_h = 4.6265$ obtained using the proposed algorithm are in agreement with that given in (Battison and Multineux, 1965).

2. Example 2 (Kabriel, 1967) - Parameter Design with Damping ratio

Considering practical systems, the absolute stability is not the only criterion; on the other hand, a stable system should meet the specification of relative damping margin, which is a quantitative measure of how quickly the transient leaves the system. Even a small disturbance in the system may develop oscillations of decreasing magnitude and then return to its equilibrium position. Here, the damping ratio ξ can be defined for such oscillations.

For carry out the relative stability study, the variable substitution (Siljak, 1969),

Table 4. Routh Table for P_4(s)

Column/ Row	1	2
s^3	(2020+3250K)	(137.734-21K)
s^2	(24.4K-0.61)	0.266
s^1	$\frac{(-512.4K^2+2509.0196K-621.3377)}{(24.4K-0.61)}$	
s^0	0.266	

Table 5. Design for Lower Limit K_l

Approximate range of K_l	Condition in Routh Table of F(s) (s1-th term)
0.2616	Unstable
2.4483	Stable
1.3550	Stable
0.8083	Stable
0.5350	Unstable
0.6717	Unstable
0.7400	Stable
0.7059	Stable
0.6900	Stable
0.6800	Marginally Stable

Table 6. Design for Upper Limit K_h

Approximate range of K_h	Condition in Routh Table of F(s) (s1-th term)
4.6350	Unstable
2.4483	Stable
3.5417	Stable
4.0884	Stable
4.3617	Stable
4.4984	Stable
4.5667	Stable
4.6009	Stable
4.6180	Stable
4.6265	Marginally Stable

$$s = S \mid \theta \quad (16)$$

where, $\xi = \sin\theta$ (damping ratio) is performed.

In the given characteristic polynomial F(s)=0, substituting the above equation (16), an equation with complex coefficient is formed, which is multiplied by its conjugate to get the real coefficients. This transformed equation obtained is of twice the degree of F(s) =0, and the proposed scheme can be applied over it. The following illustration depicts the application of the proposed procedure.

The characteristic polynomial of a voltage regulator system with two variable parameter K and K' (Kabriel, 1967) is given by,

$$F(s) = \sum_{i=0}^{4} a_i s^i = 0 \quad (17)$$

where, a_4 = 0.0288, a_3 = 0.3140, a_2 = 3.8300, a_1 = (K' + 8.7) and a_0 = (K-2.5).

With damping margin ξ = 0.1, the transformed characteristic polynomial corresponding to equation (17) is,

$$F(s) = \sum_{i=0}^{8} a_i s^i = 0 \quad (18)$$

where,
a_8 = 1
a_7 = 21.7000
a_6 = 379.7300
a_5 = (66.34K' + 3464.05)
a_4 = (742.34K' + 63.98K + 23993)
a_3 = (9193.73K' + 723.76K + 78176.12)

Table 7. Design Table for F(s) in equation (18)

Trial Value of K	Approximated Range of K' from step 7 of newly proposed algorithm	Sharpened range of K' from step 11 of newly proposed algorithm
80	-38.9173 < K' < 68.3902	2.3188 < K' < 15.0000
75	-38.4746 < K' < 68.6334	1.5209 < K' < 15.5240
65	-37.6115 < K' < 68.9894	0.0803 < K' < 16.4308
45	-35.7707 < K' < 70.0602	-2.5000 < K' < 17.8813
25	-33.8185 < K' < 70.9823	-5.0000 < K' < 19.0553

$$a_2 = (1206.27K'^2 + 20989.14K' + 9055.25K + 68664.66)$$
$$a_1 = (2400.24KK' - 6000.6K' + 20880.58K - 52205.07)$$
$$a_0 = (1206.27K^2 - 6031.36K + 7539.2)$$

From the coefficients of F(s) = 0 in equation (18), it can be found that K > 2.5 and K' > -8.7, form the necessary condition.

Hence, by applying the newly proposed scheme and for the trial value of K' = 0, the approximated range for K is found to be -392.9734 < K < 384.9665.

Utilizing the necessary condition (K>2.5) as well as the approximate range (-392.9734 < K < 384.9665), the region of K for operating situation may be written as (2.5 < K < 384.9665). Within this range, the trial values of K are chosen and the characteristic polynomial given in equation (18) is formed in terms of K'. Then, the newly proposed algorithm is applied and K' ranges are obtained. The approximated and sharpened ranges of K' obtained by the newly proposed approach are tabulated in Table 7.

Discussions

In this paper, a new algebraic method comprising of Pseudo Routh Column Polynomials along with Routh's table is presented. The method is straight forward in applications and simple in approach for designing a single as well as multiple parameters residing as coefficients in the characteristic polynomial F(s). These lower degree Pseudo Routh Column Polynomials are good enough in evaluating the approximate range of the design parameters, which are further sharpened employing Routh's table. This newly proposed scheme reduces the computational complexity compared to the direct application of Routh's table to the original higher degree characteristic polynomial.

In case of Linear Time Invariant Discrete System, the characteristic polynomial in z-plane can be transformed to r-plane using Bilinear Transformation (Kuo, 2003); the transformed characteristic polynomial can be dealt with the newly proposed scheme for design purpose.

Conclusions

In this paper, a novel approach employing Pseudo Routh Column Polynomials and Routh table for obtaining the approximated ranges of single and multi parameters residing as coefficients in the characteristic polynomial is presented. Further sharpening of these parameters is carried out with the help of the bisection principle and Routh's table. An algorithm is also depicted for carrying out the design steps. The illustrations show the applicability of the new procedure.

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