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Original Article

Development of strut and tie models for simply supported deep beams using topology optimization

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Abstract

Generally, structural members can be broadly divided into two regions, namely B, or Bernoulli regions, where the strain distributions are linear and D, or Disturbed regions, where the strain distributions are nonlinear While well defined theories are available for designing B regions, rules-of-thumb or empirical equations are still being used to design D regions although B and D regions are equally important. It has been recently understood that the strut and tie model is an effective tool for the design of both B and D regions. Since this method is a realistic approach, this has found place in many codes like Euro code, American code, Canadian code, Australian code, New Zealand code etc. In a deep beam, the distribution of strain across depth of the cross section will be nonlinear and hence these structural elements belong to D regions. The existing code provisions for the design of simply supported deep beams are inadequate and are empirical in nature. In this paper, the development of strut and tie models for simply supported deep beams using topology optimization is discussed. The design of deep beams using topology optimization is illustrated using an example and is compared with available code recommendations.

Keywords: deep beam, reinforced concrete, strut and tie model, topology optimization

1. Introduction

Concrete structural members having depth comparable to span are generally termed as deep beams. In these members, the distribution of strains across depth of the cross section will be nonlinear and the significant amount of load is carried to the supports by a compression strut joining the load and the reaction. These structural elements belong to D (disturbed) regions, which have traditionally been designed using empirical formulae or using past experience. As per the Indian code IS 456: 2000, when the ratio of the effective span to overall depth is less than 2.0 for a simply supported beam and less than 2.5 for a continuous beam, then they can be considered as deep beams. However, as per the American code ACI 318-08 (2008), when the clear span to over all depth is less than or equal to 4.0, then they can be treated as deep beams. Further, those regions of the beams loaded with concentrated loads within twice the member depth from the face of support should be designed as deep beams. Clearly there exists a disparity in the definition of deep beams as per various design codes.

ACI 318-08 (2008) does not contain any recommendations for designing deep beams for flexure and it recommends to either use a non linear analysis or strut and tie model (STM) for designing deep beams. Many codes have adopted the design recommendations for deep beams given in CEB (1970) which is based on the experimental investigations conducted by Leonhardt and Walter at University of Stuttgart (SP: 24, 1983). For example, IS 456: 2000 recommends this procedure for the design of deep beams. These provisions are valid for deep beams subjected to uniformly distributed loads (UDL) (Park and Paulay 1975). However, in actual practice, we may also come across concentrated

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loads, trapezoidal loads, triangular loads etc. Hence a general method which takes into account different types of loading is always preferred.

STM is a recent development in the analysis and design of reinforced concrete structural elements. STM provides design engineers with a more flexible and intuitive option for designing structural elements. In this method, complex stress flows in a cracked concrete structure are approximated with simple truss elements that can be analysed and designed using basic structural mechanics.

The concept of using uni-axially stressed truss member to model the complex stress flows in cracked reinforced concrete was used by the pioneers of reinforced concrete design, such as Ritter and Mörsch (Park and Paulay, 1975). Schliach, Schäfer and Jennewein (Schliach *et al.*, 1987; Schliach and Schäfer, 1991) have made significant contributions to this approach for the design of D regions. Following this work, STM began appearing in many international codes of practice.

Though STM is effective for the design of D regions, the method has not yet been widely implemented. One of the possible reasons could be the difficulty in fixing an optimum truss configuration for a given structural member with given loading. For this, it is essential that the designer should have a minimum level of experience to predict the structural response. Traditionally STM has been developed using load path method or with the aid of stress trajectories. However, the STM thus obtained is not unique and varies with the designer's intuition and past experience. The limitations of these conventional methods can be overcome if the development of STM for structural concrete is treated as a topology optimization problem (Liang *et al.*, 2000; 2001; 2002). In this paper, the development of STM for simply supported deep beams subject to different types of loadings like UDL, two point loading, central point loading using topology optimization are discussed. Finally the design of deep beams is illustrated using examples and is compared with the current IS 456 code recommendations.

2. Review of strut and tie model

In STM, a reinforced concrete member is replaced by an equivalent truss, which can resist the applied loads. In STM, the compression and tension zones are converted into equivalent struts and ties which are in turn connected at the nodes to form a truss. The STM is based on the lower bound theorem of plasticity (Nielsen, 1984; Muttoni *et al.*, 1997). Therefore the actual capacity of the structure is considered to be equal to or greater than that of the idealized truss, i.e. STM underestimates the strength of the reinforced concrete member. Hence designs based on this method will be always on the safer side. These models are generally used for the analysis, design and detailing of D regions, such as, vicinities of point loads, corners of frames, corbels and also where sudden changes in the cross section occur. The various components in a STM for reinforced concrete elements are given below.

2.1 Struts

Struts are compression members in a STM. They represent concrete stress fields whose principal compressive stresses are predominantly along the centreline of the strut. The idealized shape of the concrete stress field surrounding a strut in a plane (2-D) member can be prismatic, bottle-shaped or fan-shaped. The struts are classified, as mentioned above, into three types depending upon the geometry of the struts and are shown in Figure 1.

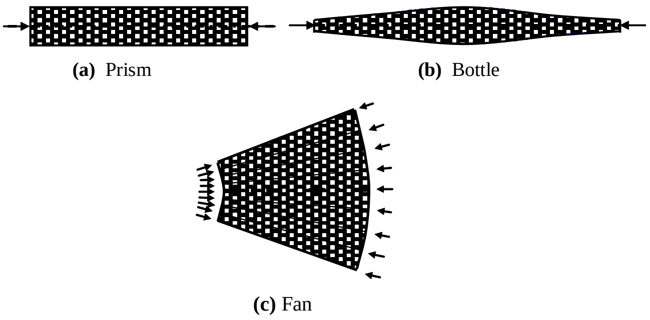


Figure 1. Different types of struts

The simplest type is called the prism, which has a constant width. The second form is the bottle in which the strut expands or contracts along its length. The final type is the fan where an array of struts with varying inclination meet at, or radiate from a single node.

2.2 Ties

Ties are tension members and they represent reinforcing steel.

2.3 Nodes

Nodes in STM are the intersection points of three or more straight struts and ties. They are analogous to joints in a conventional truss. Depending on the nature of forces, nodes can be classified as CCC, CCT, CTT and TTT nodes (Figure 2). C is used to denote the compression force and T is used to denote tension force.

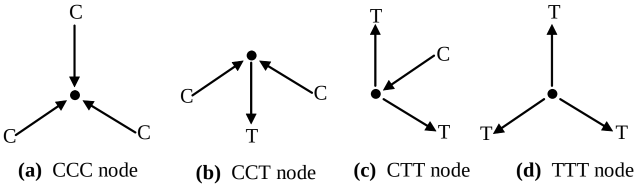


Figure 2. Different types of nodes

2.4 Dimensioning of struts and nodes

In the case of a real truss, the identification of member areas and joint details and their design is fairly straight forward. However, in the case of an implicit truss embedded in concrete, the determination of appropriate member cross sectional areas and node dimensions is not so simple, especially for the determination of the concrete strut and node dimensions. Although IS 456: 2000 recommends the use of the STM, no guidelines are given for the determination of the dimensions of the struts and nodes, and for the permissible stresses in these elements. Hence, the design recommendations given in ACI 318-08 (2008) are used in this paper and the salient details are given below. The recommendations are slightly modified by incorporating the safety factors and notations followed in IS 456: 2000.

The cross-sectional area of the strut can be computed based on the guidelines given in ACI 318-08 (2008). The area of the strut is calculated as the product of width of the strut W_S and the strut thickness. The strut thickness is equal to the width of the beam. The width of the strut W_S for a CCT node is shown in Figure 3.

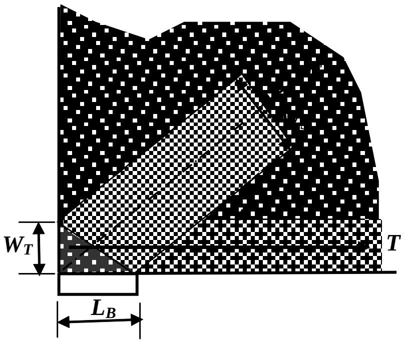


Figure 3. Effective width of strut and tie for a CCT node

From Figure 3, width of the strut can be evaluated as:

$$W_S = W_T \cos\theta + L_B \sin\theta \tag{1}$$

where W_T is the width of the tie, L_B is the length of the bearing plate and θ is the strut inclination with the horizontal. W_T can be taken as twice the effective cover to the steel reinforcement.

2.5 Permissible stresses in struts and nodes

The permissible stresses in different types of struts (f_{cs}) is given as

$$f_{cs} = 0.45 f_{ck} \beta_s \tag{2}$$

where f_{ck} is the characteristic compressive strength of concrete cube of size 150 mm and β_s is a stress reduction factor to account for the different types of struts. The values of β_s are given in Table 1.

Table 1. β_s for different types of struts

Type of strut	β_S
Prismatic	1
Bottle shaped (with crack control reinforcement)	0.75
Bottle shaped (with no crack control reinforcement)	0.6

The permissible stresses in different types of nodes (f_{cn}) is given as

$$f_{cn} = 0.45 f_{ck} \beta_n \tag{3}$$

where β_n is a stress reduction factor to account for the different types of nodes and its values are given in Table 2.

Table 2. β_n for different types of nodes

Type of node	β_n
CCC	1
CCT	0.8
CTT, TTT	0.6

2.6 Crack control reinforcement

ACI 318-08 (2008) recommends an orthogonal grid of bars (Figure 4) on each face to control the crack width of bottle shaped struts, and it should satisfy the criteria given below.

$$\sum \frac{A_{si}}{W_b S_i} \sin\phi_i \geq 0.003 \tag{4}$$

where A_{si} refers to the crack control reinforcement provided at a spacing of S_i in a layer of reinforcement with bars at an angle of ϕ_i to the axis of the strut and W_b is the width of the beam.

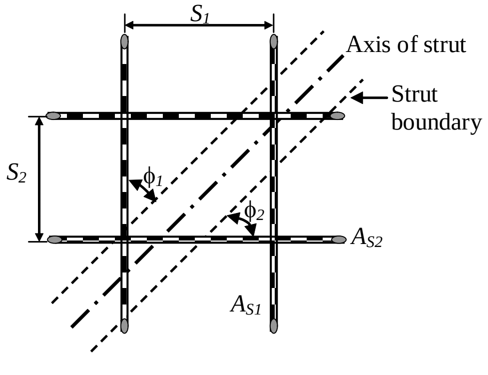


Figure 4. Crack control reinforcement crossing a bottle shaped strut

3. Identification of disturbed regions

The disturbed regions in a reinforced concrete member can be identified using Saint Venant’s principle which suggests that the localized effect of a disturbance vanishes at a distance of about one member depth from the point of disturbance. On this basis, disturbed regions are assumed to extend one member depth from the point of disturbance. Figure 5 shows Bernoulli and disturbed regions for simply supported beam subject to a central point load.

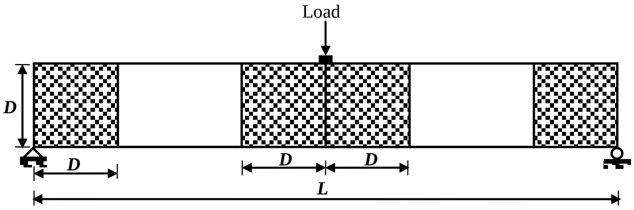


Figure 5. Simply supported beam subject to a central point load

In Figure 5, *D* is the overall depth of the beam and *L* is the span. The shaded region is the disturbed region and the unshaded region is Bernoulli region. The disturbed region can be designed using STM and the Bernoulli region can be designed using conventional design methods for shallow beams recommended by the various design codes. From the figure it can be seen that if $L/D \le 4$, the flexural member becomes a deep beam and hence, the entire member can be treated as a disturbed region. Thus if $L/D \le 4$, for a simply supported beam subject to a central point load, then, it can be considered a deep beam. This is the limit given in ACI 318-08 (2008). Figure 6 shows the disturbed region for a simply supported beam subject to UDL.

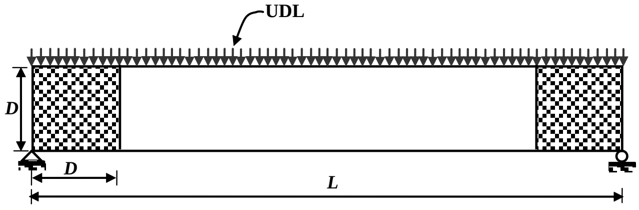


Figure 6. Simply supported beam subject to UDL

It can be seen that if $L/D \le 2$, for a simply supported beam subject to UDL then, it can be treated as a deep beam. This is the limit given in IS 456: 2000. Similarly, it can be seen that if $L/D \le 6$, for a simply supported beam subject to two point loading, then, it can be treated as a deep beam (Figure 7).

Hence, the limiting value of the *L/D* ratio so that a simply supported beam can be treated as deep beam depends on the type of loading and, hence, there is no uniform definition to classify a flexural member as deep beam.

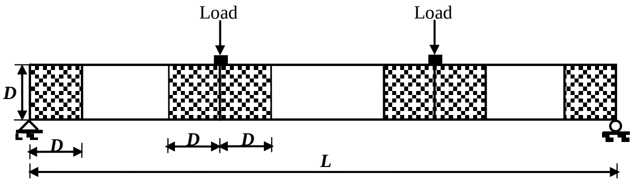


Figure 7. Simply supported beam subject to two point loading

4. IS 456: 2000 code provisions for the design of simply supported deep beams

As per the code, when the ratio of the effective span (*L*) to the overall depth (*D*) of a simply supported beam is less than or equal to 2.0, then the beam can be treated as a deep beam. The lever arm (*Z*) is given as

$$\begin{aligned} Z &= 0.2\,(L + 2\,D);\; 1 < \frac{L}{D} < 2 \\ &= 0.6\,L;\; < 1 \end{aligned} \tag{5}$$

where, *L* is the effective span taken as centre to centre distance between the supports or 1.15 times the clear span, which ever is smaller, and *D* is the overall depth. The tensile reinforcement, A_{st}, required to resist the positive bending moment can be calculated using the expression

$$M_U = \frac{W_U L^2}{8} = TZ = \frac{f_y}{\gamma_s} A_{st} Z = 0.87 f_y A_{st} Z$$

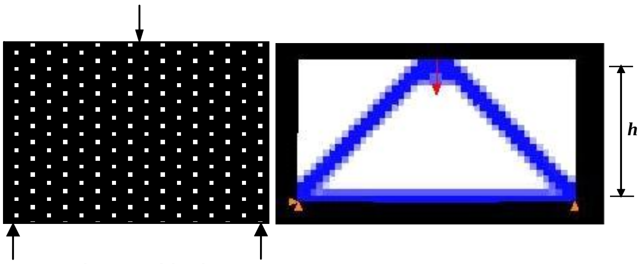
$$A_{st} = \frac{W_U L^2}{0.87 f_y \times 8Z} \tag{6}$$

where M_U is the factored bending moment, W_U is the factored UDL applied on the beam, *T* is the tension force and f_y is the yield stress of the steel used. γ_s is the partial (material) safety factor for steel and which is equal to 1.15.

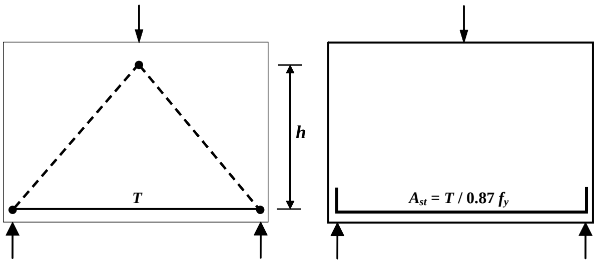
5. Development of STM using topology optimization

The use of topology optimization in the development of STM is based on the premise that some parts of a structural member are not effective in resisting loads when compared with the other parts. By eliminating these under-utilized portions from the member, the actual load path in the member can be determined and this load path can be taken as the STM. Topology optimization method solves the problem of distributing a given amount of material in a design domain for a given load and support condition such that the stiffness of the structure is maximized. From the final topology we get the path of effective material utilization and the STM is assumed to follow the shape of the final topology. In this study, the topology optimization is carried out using the software Topopt (Tcherniak and Sigmund, 2001).

The development of STM using topology optimiza-



(a) Simply supported deep beam **(b)** Final Topology



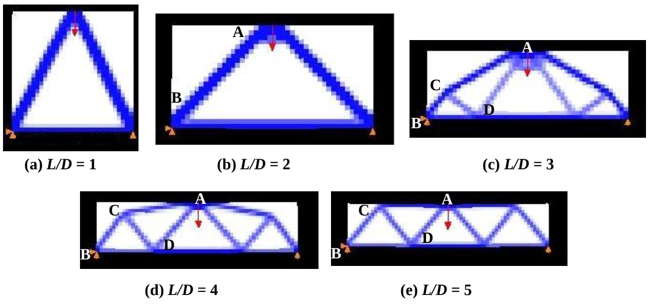
(c) Strut and tie model **(d)** Layout of reinforcement

Figure 8. Strut and tie model for simply supported beam subject to central point loading

tion is illustrated using an example. Figure 8 shows a simply supported deep beam subjected to central point load. The final topology and STM are shown in the Figures 8(b) and 8(c) respectively. The struts are shown by dotted lines and ties are shown as solid lines.

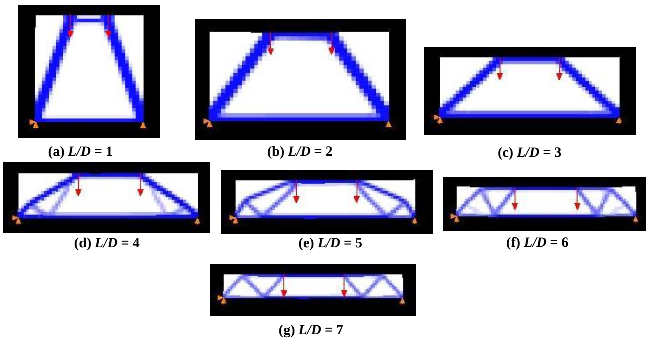
The final topology for simply supported deep beams for various span (*L*) to depth (*D*) ratios subject to different types of loading like central point load, two point load and UDL are shown in Figures 9-11.

From the final topology of the deep beams subject to a central point load shown in Figure 9, it can be seen that for small values of the *L/D* ratio (Figure 9(a) and 9(b)) a single strut AB is sufficient to transfer the load from the support to the reaction. As the value of the *L/D* ratio increases, though additional truss members are required for the transfer of load (member CD in Figures 9(c) and 9(d)), the shape of the final topology still resembles the shape of an arch, i.e. it reflects deep beam behaviour. For large values of *L/D* ratio (Figure 9(e)), the final topology is a Warren truss and is similar to the truss used by Mörsch for a shallow beam (Park and Paulay 1975) and this can be used to find the limiting value of the *L/D* ratio for which a beam will behave as a deep beam. From Figures 9-11, it can be seen that, the beam behaves as deep beam up to $L/D \approx 4$ in the case of central point loading, $L/D \approx 6$ in the case of two point loading and $L/D \approx 5$ in the case of UDL. This is the same limit for the *L/D* ratio (except for beams subject to UDL) obtained using Saint Venant's principle.



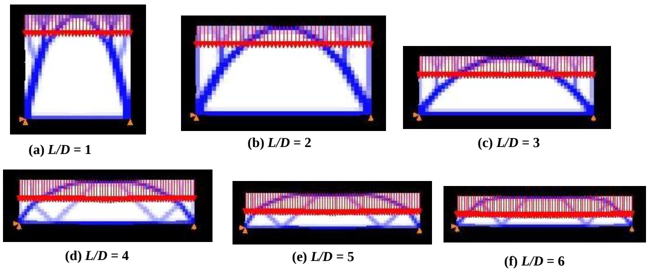
(a) ***L/D*** **= 1** **(b)** ***L/D*** **= 2** **(c)** ***L/D*** **= 3** **(d)** ***L/D*** **= 4** **(e)** ***L/D*** **= 5**

Figure 9. Final topology for simply supported beam subject to central point loading



(a) ***L/D*** **= 1** **(b)** ***L/D*** **= 2** **(c)** ***L/D*** **= 3** **(d)** ***L/D*** **= 4** **(e)** ***L/D*** **= 5** **(f)** ***L/D*** **= 6** **(g)** ***L/D*** **= 7**

Figure 10. Final topology for simply supported beam subject to two point loading



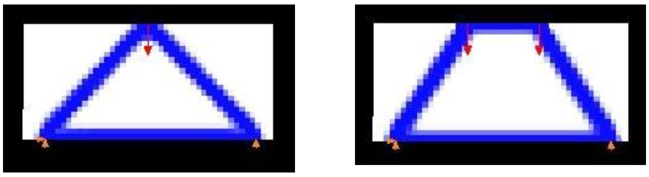
(a) ***L/D*** **= 1** **(b)** ***L/D*** **= 2** **(c)** ***L/D*** **= 3** **(d)** ***L/D*** **= 4** **(e)** ***L/D*** **= 5** **(f)** ***L/D*** **= 6**

Figure 11. Final topology for simply supported beam subject to UDL

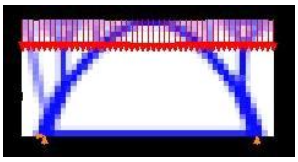
6. Design example

The design of simply supported deep beam is illustrated using an example. For this purpose, a simply supported deep beam of span (*L*) 6 m is considered. The depth of the beam (*D*) is taken as 3 m and the width (W_b) as 0.5 m. Concrete of grade M20 (f_{ck} = 20 MPa) and Fe 415 grade steel (f_y = 415 MPa) is used. The beam is designed to resist a total factored load W_u (including dead load) of 1500 kN. The final topology for the deep beam subject to different types of loads are shown in Figure 12. The STM for these cases are shown in Figure 13.

In Figure 13, the height of truss (h_i) and the strut inclination (θ_i) are taken from the final topology. The STM for the deep beam subject to UDL was developed by replacing the UDL by six point loads as shown in Figure 13(c). The truss can be analyzed to get the force in the tie (*T*) and the area of steel can be determined. The summary of the design for different loading conditions is shown in Table 3. The force

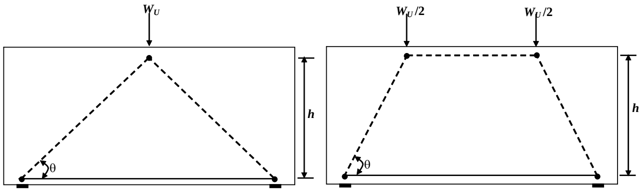


(a) Central point load **(b)** Two point load

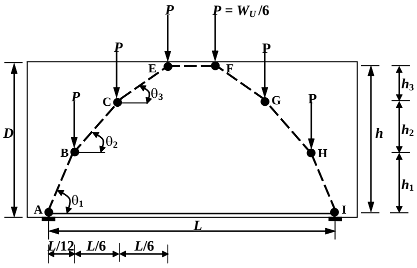


(c) Uniformly distributed load

Figure 12. Final topology for simply supported deep beam



(a) Central point load **(b)** Two point load



(c) Uniformly distributed load

Figure 13. Strut and tie model for simply supported deep beam

in the struts (C) can be determined, and the dimensions of struts and nodes can be determined using Figure 3 and eqn. 1. The stress in the struts and nodes can be evaluated, and it should be less than the permissible stress limits (eqns. 2-3). If required, the crack control reinforcement (Figure 4) should be provided.

From Table 3, it can be seen that the area of steel required for a deep beam subject to UDL obtained from the STM developed using topology optimization is less than that required according to IS 456: 2000 code recommendations.

7. Conclusions

From the above study, the following conclusions can be drawn

- STM can be easily developed by using topology optimization and it can be used to design deep beams subjected to any type of loading
- Even though STM is a conservative method, the area of steel calculated using STM is less than that required according to IS 456: 2000 recommendations.
- From the STM, it is seen that the force in the bottom member is constant through out its length. This means that arch action is predominant in a deep beam. Further, this implies that the reinforcement provided at the mid span should be extended up to the support. Thus, STM model helps to understand the behaviour of the structural elements and will be extremely useful for detailing the member.

Thus deep beams can be designed using STM more rationally.

Notations

A_{si} : Area of crack control reinforcement
A_{st} : Area of tension steel reinforcement
C : Compressive force
D : Depth of beam
f_{ck} : Characteristic compressive strength of concrete
f_{cn} : Permissible stress in nodes
f_{cs} : Permissible stress in struts
f_y : Yield stress of steel
h : Height of truss
L : Effective span
L_B : Length of bearing plate
M_U : Factored moment
S_i : Spacing of crack control reinforcement
T : Tensile force
W_b : Width of beam
W_S : Width of strut
W_T : Width of tie
W_U : Factored load
Z : Lever arm

Table 3. Area of steel for different loading conditions

Type of loading	Strut and tie Model		IS 456 design recommendations
	Force in the tie T (kN)	Area of steel (mm^2) $A_{st} = T/(0.87f_y)$	Area of steel (mm^2)
Central point load	812.27	2249.74	——
Two point load	522.65	1447.58	——
UDL	405.35	1122.7	1298.3

q : Strut inclination
g_s : Partial (material) safety factor for steel
f_i : Angle made by crack control reinforcement with the axis of the strut
b_s : Stress reduction factor for strut
b_n : Stress reduction factor for node

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