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Original Article

Confidence intervals for the difference and the ratio of Lognormal means with bounded parameters

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Abstract

In this paper, confidence intervals for the difference between lognormal means and the ratio of lognormal means with restricted parameter spaces are proposed. The modified confidence intervals perform well both for the coverage probability and the expected length. We show these results via Monte Carlo simulation.

Keywords: generalized confidence interval, simple confidence interval for lognormal means, simple confidence interval for ratio of lognormal means, Monte Carlo simulation

1. Introduction

In many practical applications in various areas, such as engineering, science and social science, it is known that there exist bounds on the values of unknown parameters. For example, in engineering process control: the life time of machines and the values of some measurements for controlling machines are bounded, in sciences: the weight or height of subjects or the blood pressures of patients are bounded. Also in social sciences, the retirement ages of public servants is bounded. Statistical inference for bounded parameters is well known; see e.g. Mandelkern (2002), Feldman and Cousins (1998), Roe and Woodroofe (2003) Wang (2008), Wang (2006), Wang (2007), Giampaoli and Singer (2004) Niwitpong (2013a, 2013b, 2013c). When interval estimation is considered in a situation where the parameter to be estimated is bounded, it has been argued that the classical Neyman procedure for setting confidence intervals is unsatisfactory. This is due to the fact that the information regarding the restriction is simply ignored. It is, therefore, of significant interest to construct confidence intervals for the parameters that include the additional information on parameter values being bounded to enhance the accuracy of the interval estimation. Recent papers by Mandelkern (2002), Wang (2006), Wang (2007), Wang (2008), Giampaoli and Singer (2004) are concerned mainly with the normal means but in practice, many applications shown that data from lognormal distribution are fitted to the problems in environment, biology, health sciences and physical sciences see e.g. Niwitpong (2013a, 2013b, 2013c), Kumar and Tipathi (2004), Chen and Zhou (2006), Krisnamoorthy and Mathew (2003), Kumar and Gibbons (2004), Thipathi *et al.* (2009). Therefore in this paper, we extend the work of Niwitpong (2013a) to propose new confidence intervals for the difference between lognormal means and the ratio of lognormal means when the population means are bounded by using the approach presented by Zou et al. (2009) and Donner and Zou (2010). The proposed intervals are evaluated in terms of coverage probability and expected length via Monte Carlo simulation.

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2. Lognormal Distribution and the Parameters of Interest

Let $X_i = \left(X_{i1}, X_{i2}, \ldots, X_{in_i}\right), i = 1,2$ be a random variable having lognormal distribution, $X_i \sim LN\left(\mu_i, \sigma_i^2\right)$ and μ_i, σ_i^2 are respectively the mean and variance of $Y_i =$

$\ln(X_i) \sim N\left(\mu_i, \sigma_i^2\right)$. The probability density function of the lognormal distribution is

$$f\left(X_i, \mu_i, \sigma_i^2\right) = \begin{cases} \frac{1}{X_i \sigma_i \sqrt{2\pi}} \exp\left(-\frac{\left(\ln\left(X_i\right) - \mu_i\right)^2}{2\sigma_i^2}\right) & ; \textit{ for } X_i > 0 \\ 0 & ; \textit{ otherwise}. \end{cases}$$

The mean and the variance of X_i are respectively

$$E\left(X_i\right) = \exp\left(\mu_i + \sigma_i^2 / 2\right) = \delta_i \,,$$

$$Var\left(X_i\right) = \exp\left(2\mu_i + \sigma_i^2\right)\left(\exp\left(\sigma_i^2\right) - 1\right).$$

The parameters of interest of this paper are respectively $\delta_1 - \delta_2$ and $\frac{\delta_1}{\delta_2}$ when the parameter means are bounded: $a_i < \mu_i < b_i, a_i < b_i$ where a_i, b_i are constants.

3. Confidence Interval for Lognormal Means

Zhou and Gao (1997) proposed the confidence interval for lognormal means which is $\exp\left(CI_{cox}\right)$, where

$$CI_{cox} = \left[\bar{Y}_1 + \frac{S_y^2}{2} - Z_{1-\alpha/2}\sqrt{\left(\frac{S_y^2}{n_1} + \frac{S_y^4}{2(n_1-1)}\right)}, \bar{Y}_1 + \frac{S_y^2}{2} + Z_{1-\alpha/2}\sqrt{\left(\frac{S_y^2}{n_1} + \frac{S_y^4}{2(n_1-1)}\right)}\right] = [l_{cox}, u_{cox}]$$

and

$$S_y^2 = (n_1 - 1)^{-1} \sum_{j=1}^{n_1} \left(Y_{1j} - \bar{Y}_1\right)^2,$$

$Z_{1-\alpha/2}$ is $(1-\alpha/2)100^{th}$ percentile of standard normal distribution.

Krishnamoorthy and Mathew (2003) also proposed the confidence interval for the lognormal mean using the generalized confidence interval proposed by Weerahandi (1993) and this confidence interval is given by $\exp\left(CI_{gci}\right)$ where

$$CI_{gci} = \left[T(\alpha/2), T(1-\alpha/2)\right] = [l_{gci}, u_{gci}]$$

when

$$T(Y_1, y_1, \mu_1, \sigma_1^2) = \bar{y}_1 - \frac{Z}{U/\sqrt{n_1 - 1}} \frac{s_y}{\sqrt{n_1}} + \frac{1}{2} \frac{s_y^2}{U^2/(n_1-1)}, U^2 = (n_1-1)S_y^2/\sigma_1^2 \sim \chi_{n-1}^2,$$

s_y is a standard deviation of the observed Y_1.

and $T(\alpha/2)$ is the $(\alpha/2*100)^{th}$ percentile of $T(\bullet)$.

In this case, the generalized test variable, $T(Y_1, y_1, \delta_1)$, is a function of (Y_1, y_1, δ_1) and is required to satisfy the following conditions:

A1. For a fixed y_1, the probability distribution of $T(Y_1, y_1, \delta_1)$ is free of unknown parameters.

A2. The observed value of $T(Y_1, y_1, \delta_1)$ at $Y_1 = y_1$ is simply δ_1.

A3. For the given value of y_1 and δ_1 the distribution of $T(Y_1, y_1, \delta_1)$ is monotone distribution in δ_1

Krishnamoorthy and Mathew (2003) found that the confidence interval $\exp\left(CI_{gci}\right)$ has a good coverage probability compared to the confidence interval $\exp\left(CI_{cox}\right)$.

Zou *et al.* (2009) proposed a simple confidence interval for the sum of parameters which is $\theta_1 + \theta_2$ and the confidence interval for $\theta_1 + \theta_2$ is

$$CI_0 = ((\hat{\theta}_1 + \hat{\theta}_2) - Z_{(1-\alpha/2)}\sqrt{\text{var}(\hat{\theta}_1) + \text{var}(\hat{\theta}_2)}, (\hat{\theta}_1 + \hat{\theta}_2) + Z_{(1-\alpha/2)}\sqrt{\text{var}(\hat{\theta}_1) + \text{var}(\hat{\theta}_2)})$$

$\hat{\theta}_1, \hat{\theta}_2$ are estimators of θ_1, θ_2, in this case, $\hat{\theta}_1 = \bar{Y}$, $\hat{\theta}_2 = S_y^2 / 2$

Suppose $CI_{lu} = \left[L, U\right]$

is the confidence interval for θ_1, θ_2, Zou *et al.* (2009) found that this confidence interval is

$$L = \hat{\theta}_1 + \hat{\theta}_2 - \sqrt{\left(\hat{\theta}_1 - l_1\right)^2 + \left(\hat{\theta}_2 - l_2\right)^2}\,,$$

$$U = \hat{\theta}_1 + \hat{\theta}_2 + \sqrt{\left(u_1 - \hat{\theta}_1\right)^2 + \left(u_2 - \hat{\theta}_2\right)^2}\,, \quad (1)$$

For $\theta_1 = \mu, \theta_2 = \sigma^2/2$, the confidence intervals for θ_1, θ_2 are

$$(l_1, u_1) = \left(\bar{Y}_1 - t_{1-\alpha/2} S_y / \sqrt{n_1}, \bar{Y}_1 + t_{1-\alpha/2} S_y / \sqrt{n_1}\right),$$

$$(l_2, u_2) = \left(\frac{(n_1-1)S_y^2}{2\chi_{1-\alpha/2}^2}, \frac{(n_1-1)S_y^2}{2\chi_{\alpha/2}^2}\right), \quad (2)$$

Puting (2) into (1), we finally have the required confidence interval for the lognormal mean and we denote this confidence intervals as $CI_{rov.}$

4. Confidence Interval for the Difference between Lognormal Means

From confidence intervals CI_{gci}, CI_{rov}, we now construct the confidence interval for using the single confidence interval for each of

4.1 The generalized confidence interval for the difference between lognormal means

Using the same idea as in section 3, we now proposed the generalized confidence interval for the difference between lognormal means which is

$$CI_{GCID} = \left[T_1(\alpha/2), T_1(1-\alpha/2)\right] = [l_{gci-d}, u_{gci-d}]$$

when

$$T_1(Y_1,Y_2,y_1,y_2,\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)=\exp(\Delta_1)-\exp(\Delta_2),$$

$$\Delta_i=\bar{y}_i-\frac{Z}{U_i/\sqrt{n_i-1}}\frac{s_{y_i}}{\sqrt{n_i}}+\frac{1}{2}\frac{s_{y_i}^2}{U_i^2/(n_i-1)},$$

$$U_i^2=(n_i-1)S_{y_i}^2/\sigma_i^2\sim\chi_{n_i-1}^2.$$

4.2 The simple confidence interval for the difference between lognormal means

Using the same idea as in section 3, we now proposed the simple confidence interval for the difference between lognormal means, based on Zou *et al.* (2009) which is

$$CI_{ROVD}=[L_r,U_r]$$

where (L_r,U_r) is

$$L_r=\hat{\delta}_1-\hat{\delta}_2-\sqrt{\left(\hat{\delta}_1-\exp(L_1)\right)^2+\left(\exp(U_2)-\hat{\delta}_2\right)^2},$$

$$U_r=\hat{\delta}_1-\hat{\delta}_2+\sqrt{\left(\exp(U_1)-\hat{\delta}_1\right)^2+\left(\hat{\delta}_2-\exp(L_2)\right)^2}$$ when

$$L_i=\bar{Y}_i+\frac{S_{y_i}^2}{2}-\sqrt{\left(t_{1-\alpha/2,n_1-1}\frac{S_{y_i}}{n_i}\right)^2+\left(\frac{S_{y_i}^2}{2}-\frac{(n_i-1)S_{y_i}^2}{2\chi_{1-\alpha/2,n_i-1}^2}\right)^2}$$

$$U_i=\bar{Y}_i+\frac{S_{y_i}^2}{2}+\sqrt{\left(t_{1-\alpha/2,n_1-1}\frac{S_{y_i}}{n_i}\right)^2+\left(\frac{(n_i-1)S_{y_i}^2}{2\chi_{1-\alpha/2,n_i-1}^2}-\frac{S_{y_i}^2}{2}\right)^2} \quad (3)$$

5. Confidence Interval for the Ratio of Lognormal Means

5.1 The generalized confidence interval for the ratio of lognormal means

From section 3, it is easy to see that the generalized confidence interval for the ratio of lognormal means is

$$CI_{GCIS}=\left[T_p(\alpha/2),T_p(1-\alpha/2)\right]=[l_{gci-p},u_{gci-p}]$$

where

$$T_p(Y_1,Y_2,y_1,y_2,\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)=\frac{\exp(\Delta_1)}{\exp(\Delta_2)},$$

$$\Delta_i=\bar{y}_i-\frac{Z}{U_i/\sqrt{n_i-1}}\frac{s_{y_i}}{\sqrt{n_i}}+\frac{1}{2}\frac{s_{y_i}^2}{U_i^2/(n_i-1)},$$

$$U_i^2=(n_i-1)S_{y_i}^2/\sigma_i^2\sim\chi_{n_i-1}^2.$$

5.2 The simple confidence interval for the ratio of lognormal means

Donner and Zou (2010) proposed the confidence interval for $\varphi=\delta_1/\delta_2$ by setting and setting $\varphi\delta_2-\delta_1=0$ the confidence interval for $\varphi\delta_2-\delta_1=0$ from (L_s,U_s) where

$$L_s=\hat{\delta}_1-\varphi\hat{\delta}_2-\sqrt{\left(\hat{\delta}_1-\exp(L_1)\right)^2+\varphi^2\left(\exp(U_2)-\hat{\delta}_2\right)^2},$$

$$U_s=\hat{\delta}_1-\varphi\hat{\delta}_2-\sqrt{\left(\exp(U_1)-\hat{\delta}_1\right)^2+\varphi^2\left(\hat{\delta}_2-\exp(L_2)\right)^2}.$$

By setting $(L_s,U_s)=(0,0)$, we have the confidence interval for $CI_{rov-p}=(L_z,U_z)$ where

$$L_z=\frac{\hat{\delta}_1\hat{\delta}_2-\sqrt{\left(\hat{\delta}_1\hat{\delta}_2\right)^2-\exp(L_1)\exp(U_2)\left(2\hat{\delta}_1-\exp(L_1)\right)\left(2\hat{\delta}_2-\exp(U_2)\right)}}{\exp(U_2)(2\hat{\delta}_2-\exp(U_2))},$$

$$U_z=\frac{\hat{\delta}_1\hat{\delta}_2-\sqrt{\left(\hat{\delta}_1\hat{\delta}_2\right)^2-\exp(L_2)\exp(U_1)\left(2\hat{\delta}_1-\exp(U_1)\right)\left(2\hat{\delta}_2-\exp(L_2)\right)}}{\exp(L_2)\left(2\hat{\delta}_2-\exp(L_2)\right)} \quad (4)$$

when $(L_i,U_i), i=1,2$ from (3) and the confidence interval for the ratio of lognormal means based on Donner and Zou (2010) is

$$CI_{ROVS}=[L_z,U_z].$$

5.3 The simple confidence interval for the ratio of lognormal means using the separate confidence interval of Zhou and Gua (1997)

From section (3), we have the confidence interval for the single lognormal means which is

$$CI_{cox_i}=\left[\bar{Y}_i+\frac{S_{y_i}^2}{2}-Z_{1-\alpha/2}\sqrt{\left(\frac{S_{y_i}^2}{n_i}+\frac{S_{y_i}^4}{2(n_i-1)}\right)},\bar{Y}_i+\frac{S_{y_i}^2}{2}+Z_{1-\alpha/2}\sqrt{\left(\frac{S_{y_i}^2}{n_i}+\frac{S_{y_i}^4}{2(n_i-1)}\right)}\right]$$

$$=[\mathrm{L}_{ci},U_{ci}]$$

In this case, we set $(\mathrm{L}_{ci},\mathrm{U}_{ci})=(L_i,U_i), i=1,2$ in (4). Hence we have the confidence interval for the ratio of lognormal means, based on Zhou and Gua (1997) which is

$$CI_{COXS}=[L_{cox},U_{cox}]$$

where $[L_{cox},U_{cox}]$ is $[L_z,U_z]$ in (4) but replacing $(\mathrm{L}_{ci},\mathrm{U}_{ci})=(L_i,U_i), i=1,2.$

6. The Confidence Interval for the Difference between Lognormal Means with Restricted Means

Consider, for each bounded population mean, we have

$$a_i < \mu_i < b_i \;\rightarrow a_i^2 < \mu_i^2 < b_i^2$$
$$\rightarrow -b_i^2 < -\mu_i^2 < -a_i^2$$
$$\rightarrow E(Y_i^2) - b_i^2 < E(Y_i^2) - \mu_i^2 < E(Y_i^2) - a_i^2$$
$$\rightarrow \sigma_{b_i}^2 < \sigma_i^2 < \sigma_{a_i}^2 .$$

Hence, the variance of each data set is bounded as well as the difference between means is bounded;

$a_1 - b_2 < \mu_1 - \mu_2 < b_1 - a_2$ and

$$a_i + \frac{\sigma_{b_i}^2}{2} < \mu_i + \frac{\sigma_i^2}{2} < b_i + \frac{\sigma_{a_i}^2}{2} \rightarrow \exp\left(a_i + \frac{\sigma_{b_i}^2}{2}\right)$$

$$< \exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) = \delta_i < \exp\left(b_i + \frac{\sigma_{a_i}^2}{2}\right), \qquad (5)$$

From (5), we have, $e = \gamma_1 - \psi_2 < \delta_1 - \delta_2 < \gamma_2 - \psi_1 = f$,

$\gamma_1 < \delta_1 < \gamma_2, \psi_1 < \delta_2 < \psi_2$

Hence, the difference between means is bounded.

Using Wang (2008), the generalized confidence interval between lognormal means is

$$CI_{GCIDR} = \left[\max\left(e, l_{gci-d}\right), \min\left(f, u_{gci-d}\right)\right]$$

where

$$[l_{gci-d}, u_{gci-d}] = \left[T_1(\alpha/2), T_1(1-\alpha/2)\right].$$

Also, the simple confidence interval between lognormal means with restricted means, based on Zou *et al.* (2009) is

$$CI_{ROVDR} = \left[\max\left(e, L_r\right), \min\left(f, U_r\right)\right]$$

when

$$L_r = \hat{\delta}_1 - \hat{\delta}_2 - \sqrt{\left(\hat{\delta}_1 - \exp(L_1)\right)^2 + \left(\exp(U_2) - \hat{\delta}_2\right)^2},$$
$$U_r = \hat{\delta}_1 - \hat{\delta}_2 + \sqrt{\left(\exp(U_1) - \hat{\delta}_1\right)^2 + \left(\hat{\delta}_2 - \exp(L_2)\right)^2}$$

$$L_i = \overline{Y}_i + \frac{S_{y_i}^2}{2} - \sqrt{\left(t_{1-\alpha/2, n_i-1}\frac{S_{y_i}}{n_i}\right)^2 + \left(\frac{S_{y_i}^2}{2} - \frac{(n_i-1)S_{y_i}^2}{2\chi^2_{1-\alpha/2, n_i-1}}\right)^2}$$
$$U_i = \overline{Y}_i + \frac{S_{y_i}^2}{2} + \sqrt{\left(t_{1-\alpha/2, n_i-1}\frac{S_{y_i}}{n_i}\right)^2 + \left(\frac{(n_i-1)S_{y_i}^2}{2\chi^2_{1-\alpha/2, n_i-1}} - \frac{S_{y_i}^2}{2}\right)^2}.$$

7. Confidence Interval for the Ratio of Lognormal Means with Restricted Means

It is easy to see that, the ratio of means is $\frac{a_1}{b_2} < \frac{\mu_1}{\mu_2} < \frac{b_1}{a_2}$ and setting $p_1 = \frac{\gamma_1}{\psi_2} < \frac{\delta_1}{\delta_2} < \frac{\gamma_2}{\psi_1} = p_2$ when $\gamma_1 < \delta_1 < \gamma_2, \psi_1 < \delta_2 < \psi_2$.

Hence, the ratio of means is bounded.

From section (5.1), the generalized confidence interval for the ratio of lognormal means with restricted means is

$$CI_{GCISR} = \left[\max\left(p_1, l_{gci-p}\right), \min\left(p_2, u_{gci-p}\right)\right]$$

when

$$[l_{gci-p}, u_{gci-p}] = \left[T_p(\alpha/2), T_p(1-\alpha/2)\right].$$

From section (5.2), the simple confidence interval for the ratio of lognormal means with restricted means is

$$CI_{ROVSR} = \left[\max\left(p_1, L_z\right), \min\left(p_2, U_z\right)\right]$$

when

$$\left(L_z, U_z\right) = CI_{rov-p}.$$

From section (5.3), the simple confidence interval for the ratio of lognormal means with restricted means, based on separate confidence interval of Zhou and Gua (1997), is

$$CI_{COXSR} = \left[\max\left(p_1, L_{cox}\right), \min\left(p_2, U_{cox}\right)\right].$$

In the next section, we compare these confidence intervals via coverage probabilities and their average lengths using Monte Carlo simulation

8. Simulation Studies

In this section, we start from the confidence intervals between lognormal means with restricted population means. We employed Monte Carlo simulation to estimate the coverage probability and the average length for each confidence interval: $\text{CI}_{GCID}, \text{CI}_{GCIDR}, \text{CI}_{ROVD}, \text{CI}_{ROVDR}$. We wrote program in R(version 3.0.3) and setting the followings; n_i = $10, 20, 30, 50, 100$ at the level of significance $\alpha = 0.05$ with $10{,}000$ simulation runs. For $\mu_1, \mu_2 \in (-6, 6)$, we set (μ_1, μ_2) $= (-5.9, 5.9), (-5.5, 5.5), (0, 0), (5.5, -5.5), (5.9, -5.9)$. For $\mu_1, \mu_2 \in (0, 6)$, we set $(\mu_1, \mu_2) = (0.1.5.8), (3, 3), (5.8, 0.1)$ and $\sigma_1 = \sigma_2 = 1$. All results are in Tables 8.1-8.4.

From Table 1, there is no difference between coverage probabilities of each confidence interval and these coverage probabilities are close to or above 0.95 for almost cases. The ratio of $\text{CI}_{GCID}, \text{CI}_{GCIDR}$ and $\text{CI}_{ROVD}, \text{CI}_{ROVDR}$ is greater than 1 when parameters μ_1, μ_2 are close to the boundary of which

Table 1. Coverage probability and the ratio of average length widths of confidence intervals $CI_{GCID}, CI_{GCIDR}, CI_{ROVD}, CI_{ROVDR}$ when $\mu_1, \mu_2 \in (-6, 6)$

n_1, n_2	(μ_1, μ_2)	CI_{GCID}, CI_{GCIDR}	$\frac{E(CI_{GCID})}{E(CI_{GCIDR})}$	CI_{ROVD}, CI_{ROVDR}	$\frac{E(CI_{ROVD})}{E(CI_{ROVDR})}$
10	(-5.9,5.9)	0.9576	1.0801	0.9590	1.0830
	(-5.5,5.5)	0.9502	1.0178	0.9561	1.018
	(0,0)	0.9558	1.0000	0.9598	1.0000
	(5.5,-5.5)	0.9592	3.3266	0.9624	3.0872
	(5.9,-5.9)	0.9576	7.6274	0.9594	7.4101
20	(-5.9,5.9)	0.9532	1.1945	0.9574	1.1961
	(-5.5,5.5)	0.9522	1.0162	0.9554	1.0159
	(0,0)	0.9482	1.0000	0.9500	1.0000
	(5.5,-5.5)	0.9474	1.2853	0.9490	1.2705
	(5.9,-5.9)	0.9522	2.9968	0.9544	2.9599
30	(-5.9,5.9)	0.9534	1.2283	0.9556	1.2299
	(-5.5,5.5)	0.9522	1.0075	0.9576	1.0072
	(0,0)	0.9500	1.0000	0.9554	1.0000
	(5.5,-5.5)	0.9510	1.0861	0.9528	1.0827
	(5.9,-5.9)	0.9548	2.3338	0.9586	2.3219
50	(-5.9,5.9)	0.9490	1.2303	0.9540	1.2300
	(-5.5,5.5)	0.9508	1.0011	0.9526	1.0011
	(0,0)	0.9446	1.0000	0.9504	1.0000
	(5.5,-5.5)	0.9524	1.0104	0.9554	1.0097
	(5.9,-5.9)	0.9484	1.8865	0.9520	1.8835
100	(-5.9,5.9)	0.9474	1.1961	0.9526	1.1963
	(-5.5,5.5)	0.9484	1.0000	0.9532	1.0000
	(0,0)	0.9470	1.0000	0.9492	1.000
	(5.5,-5.5)	0.9420	1.0000	0.9474	1.0000
	(5.9,-5.9)	0.9460	1.5058	0.9526	1.5067

show that confidence intervals CI_{GCIDR} and CI_{ROVDR} are shorter than their counterpart confidence intervals whereas they have the same coverage probabilities.

From Table 2, the ratio of confidence intervals CI_{GCID}, $CI_{ROVD}, CI_{GCIDR}, CI_{ROVDR}$ when $\mu_1, \mu_2 \in (-6, 6)$ shown that the average length widths of confidence intervals CI_{ROVD}, CI_{ROVDR} are slightly shorter than the confidence intervals CI_{GCID}, CI_{GCIDR} for small sample sizes, other cases are almost the same.

From Table 3, all coverage probabilities and average length widths of confidence intervals $CI_{GCID}, CI_{GCIDR}, CI_{ROVD}$, CI_{ROVDR} when are not different from Table 1

Table 4 shows that average length widths of each interval of $CI_{GCID}, CI_{ROVD}, CI_{GCIDR}, CI_{ROVDR}$ $\mu_1, \mu_2 \in (0, 6)$ are not different.

The second part of simulation results, the confidence intervals for the ratio of lognormal means with restricted means CI_{COXS}, CI_{COXSR}, CI_{GCID}, $CI_{GCIDR}, CI_{ROVD}, CI_{ROVDR}$ are also considered.

For $\mu_1, \mu_2 \in (-6, 6)$, we set $\mu_1, \mu_2 \in (-5.9, 5.9)$, $(-1, 5), (0, 0), (1, -5), (5.9, -5.9)$. For $\mu_1, \mu_2 \in (0, 6)$, we set $(5.9, 0.5), (5, 1), (3, 3), (0.5, 5.9)$, $\sigma_1 = \sigma_2 = 1$. All results are in Tables 5-8.

From Table 5, we found that coverage probabilities of confidence intervals $CI_{COXS}, CI_{COXSR}, CI_{GCIS}, CI_{GCISR}, CI_{ROVS}$, CI_{ROVSR} are not different when $\mu_1, \mu_2 \in (-6, 6)$. They are close to or above the nominal level of 0.95 for almost cases. The ratio of average lengths of CI_{COXS}, CI_{COXSR} and CI_{GCIS}, CI_{GCISR} is greater than 1 when the parameter values of μ_1, μ_2 are close to the boundary of $(-6, 6)$, for which in this case are (-5.9,5.9) and (5.9,-5.9); as a result confidence intervals CI_{COXSR}, CI_{GCISR} and CI_{ROVSR} are shorter than their counterpart confidence intervals: $CI_{COXS}, CI_{GCIS}, CI_{ROVS}$.

From Table 6, columns 3 and 4, we found that a confidence interval CI_{COXS} is shorter than other confidence intervals CI_{GCIS}, CI_{ROVS} whereas confidence interval CI_{ROVS} is shorter than confidence interval CI_{GCISR} for small sample

Table 2. The ratio of average length widths of confidence intervals $CI_{GCID}, CI_{ROVD}, CI_{GCIDR}, CI_{ROVDR}$ when $\mu_1, \mu_2 \in (-6, 6)$

n_1, n_2	(μ_1, μ_2)	$\frac{E(CI_{GCID})}{E(CI_{ROVD})}$	$\frac{E(CI_{GCIDR})}{E(CI_{ROVDR})}$
10	(-5.9,5.9)	1.0307	1.0335
	(-5.5,5.5)	1.0356	1.0361
	(0,0)	1.0410	1.0410
	(5.5,-5.5)	1.0742	0.9969
	(5.9,-5.9)	1.0253	0.9961
20	(-5.9,5.9)	1.0076	1.0090
	(-5.5,5.5)	1.0099	1.0096
	(0,0)	1.0091	1.0091
	(5.5,-5.5)	1.0103	0.9987
	(5.9,-5.9)	1.0103	0.9978
30	(-5.9,5.9)	1.0050	1.0063
	(-5.5,5.5)	1.0033	1.0030
	(0,0)	1.0024	1.0024
	(5.5,-5.5)	1.0019	0.9987
	(5.9,-5.9)	1.0016	0.9965
50	(-5.9,5.9)	0.9971	0.9968
	(-5.5,5.5)	0.9967	0.9966
	(0,0)	0.9962	0.9962
	(5.5,-5.5)	0.9986	0.9979
	(5.9,-5.9)	0.9985	0.9968
100	(-5.9,5.9)	0.9939	0.9940
	(-5.5,5.5)	0.9931	0.9931
	(0,0)	0.9953	0.9953
	(5.5,-5.5)	0.9932	0.9932
	(5.9,-5.9)	0.9935	0.9940

sizes. In other cases, there are no differences between these confidence intervals.

From Table 7, there are no difference of coverage probabilities between these confidence intervals CI_{COXS}, $CI_{COXSR}, CI_{GCIS}, CI_{GCISR}, CI_{ROVS}, CI_{ROVSR}$ when $\mu_1, \mu_2 \in (0, 6)$.

All coverage probabilities are close to or above the nominal level of 0.95 for almost cases. The ratio of average length widths of confidence intervals CI_{COXS}, CI_{COXSR}, CI_{GCIS}, CI_{GCISR} and CI_{ROVS}, CI_{ROVSR} is greater than 1 when parameter values of μ_1, μ_2 are close to the boundary of $(-6, 6)$ which are (5.9,0.5) and (0.5,5.9). As a result, confidence intervals CI_{COXSR}, CI_{GCISR} and CI_{ROVSR} are shorter than their counterpart confidence intervals: $CI_{COXS}, CI_{GCIS}, CI_{ROVS}$.

Similarly to the results in Tables 4 and 6, Table 8 shows, from columns 3-4, that confidence interval CI_{COXS} is shorter than confidence intervals CI_{GCIS}, CI_{ROVS}. Also confidence interval CI_{ROVS} is shorter than confidence interval CI_{GCIS} for small sample sizes. In addition, the same results appear from columns 6-7, hence confidence interval CI_{COXSR} is shorter than confidence intervals CI_{GCISR}, CI_{ROVSR} and there is no difference between confidence interval CI_{ROVSR} and CI_{GCISR} but confidence interval CI_{ROVSR} is easier to compute than confidence interval CI_{GCISR} which is based on computational approach.

9. Conclusions

In this paper, we propose new confidence intervals for the difference between lognormal means and the ratio of lognormal means with restricted population means. For the confidence interval for the difference between lognormal means, the simple confidence interval, CI_{ROVDR}, based on Zou *et al.* (2009) and Donner and Zou (2010), outperforms other confidence intervals for small sample sizes whereas for other cases one can use both the generalized confidence intervals,

Table 3. Coverage probability and the ratio of average length widths of confidence intervals $CI_{GCID}, CI_{GCIDR}, CI_{ROVD}, CI_{ROVDR}$ when $\mu_1, \mu_2 \in (0,6)$

n_1, n_2	(μ_1, μ_2)	CI_{GCID}, CI_{GCIDR}	$\frac{E(CI_{GCID})}{E(CI_{GCIDR})}$	CI_{ROVD}, CI_{ROVDR}	$\frac{E(CI_{ROVD})}{E(CI_{ROVDR})}$
10	(0.1,5.8)	0.9534	1.0674	0.9570	1.0677
	(3,3)	0.9574	1.0000	0.9616	1.0000
	(5.8,0.1)	0.9540	6.5665	0.9574	6.3839
20	(0.1,5.8)	0.9554	1.1530	0.9582	1.1544
	(3,3)	0.9570	1.0000	0.9590	1.0000
	(5.8,0.1)	0.9522	2.6092	0.9550	2.5736
30	(0.1,5.8)	0.9530	1.1645	0.9556	1.1650
	(3,3)	0.9514	1.0000	0.9542	1.0000
	(5.8,0.1)	0.9496	1.9730	0.9534	1.9624
50	(0.1,5.8)	0.9562	1.1526	0.9592	1.1528
	(3,3)	0.9500	1.0000	0.9550	1.0000
	(5.8,0.1)	0.9474	1.5712	0.9514	1.5696
100	(0.1,5.8)	0.9444	1.1015	0.9494	1.1018
	(3,3)	0.9494	1.0000	0.9538	1.0000
	(5.8,0.1)	0.9434	1.2611	0.9504	1.2610

Table 4. The ratio of average length widths of confidence intervals $CI_{GCID}, CI_{ROVD}, CI_{GCIDR}, CI_{ROVDR}$ when $\mu_1, \mu_2 \in (0,6)$

n_1, n_2	(μ_1, μ_2)	$\frac{E(CI_{GCID})}{E(CI_{ROVD})}$	$\frac{E(CI_{GCIDR})}{E(CI_{ROVDR})}$
10	(0.1,5.8)	1.0355	1.0355
	(3,3)	1.0453	1.0453
	(5.8,0.1)	1.0476	1.0476
20	(0.1,5.8)	1.0054	1.0054
	(3,3)	1.0089	1.0089
	(5.8,0.1)	1.0091	1.0091
30	(0.1,5.8)	1.0020	1.0020
	(3,3)	1.0020	1.0020
	(5.8,0.1)	1.0033	1.0033
50	(0.1,5.8)	0.9981	0.9981
	(3,3)	0.9974	0.9974
	(5.8,0.1)	0.9981	0.9981
100	(0.1,5.8)	0.9941	0.9941
	(3,3)	0.9949	0.9949
	(5.8,0.1)	0.9947	0.9947

CI_{GCIDR}, or confidence interval, CI_{ROVDR}, but confidence interval, CI_{ROVDR}, is easier to use than confidence interval, CI_{GCIDR}, which is based on computational approach. The confidence interval of the ratio of lognormal means, CI_{COXSR}, performs better that others as it provides a shorter expected length and this confidence interval has the coverage probability close to or above the nominal level of 0.95. The results in this paper are also extended to construct the confidence

Table 5. Coverage probability and the ratio of average length widths of confidence intervals $CI_{COXS}, CI_{COXSR}, \mathrm{CI}_{GCID}, \mathrm{CI}_{GCIDR}, \mathrm{CI}_{ROVD}, \mathrm{CI}_{ROVDR}$ when $\mu_1, \mu_2 \in (-6, 6)$

n_1, n_2	(μ_1, μ_2)	CI_{COXS}, CI_{COXSR}	$\frac{E(CI_{COXS})}{E(CI_{COXSR})}$	CI_{GCIS}, CI_{GCISR}	$\frac{E(CI_{GCIS})}{E(CI_{GCISR})}$	CI_{ROVS}, CI_{ROVSR}	$\frac{E(CI_{ROVS})}{E(CI_{ROVSR})}$
10	(-5.9,5.9)	0.9598	1.7252	0.9590	1.7878	0.9634	1.7867
	(-1,5)	0.9556	1.0000	0.9542	1.0000	0.9570	1.0000
	(0,0)	0.9530	1.0000	0.9538	1.0000	0.9564	1.0000
	(1,-5)	0.9586	1.0000	0.9608	1.0000	0.9628	1.0000
	(5.9,-5.9)	0.9540	1.7187	0.9560	1.7793	0.9618	1.7774
20	(-5.9,5.9)	0.9532	1.6194	0.9516	1.6517	0.9550	1.6499
	(-1,5)	0.9498	1.0000	0.9520	1.0000	0.9540	1.0000
	(0,0)	0.9508	1.0000	0.9506	1.0000	0.9522	1.0000
	(1,-5)	0.9526	1.0000	0.9514	10000	0.9540	1.0000
	(5.9,-5.9)	0.9512	1.6138	0.9464	1.6498	0.9534	1.6460
30	(-5.9,5.9)	0.9484	1.5536	0.9444	1.5756	0.9484	1.5751
	(-1,5)	0.9526	1.0000	0.9464	1.0000	0.9540	1.0000
	(0,0)	0.9538	1.0000	0.9532	1.0000	0.9544	1.0000
	(1,-5)	0.9538	1.0000	0.9520	1.0000	0.9546	1.0000
	(5.9,-5.9)	0.9534	1.5492	0.9478	1.5731	0.9532	1.5704
50	(-5.9,5.9)	0.9466	1.4577	0.9436	1.4725	0.9462	1.4721
	(-1,5)	0.9502	1.0000	0.9470	1.0000	0.9502	1.0000
	(0,0)	0.9506	1.0000	0.9482	1.0000	0.9498	1.0000
	(1,-5)	0.9560	1.0000	0.9526	1.0000	0.9560	1.0000
	(5.9,-5.9)	0.9552	1.4472	0.9514	1.4619	0.9554	1.4606
100	(-5.9,5.9)	0.9536	1.3078	0.9516	1.3141	0.9534	1.3145
	(-1,5)	0.9568	1.0000	0.9552	1.0000	0.9560	1.0000
	(0,0)	0.9562	1.0000	0.9526	1.0000	0.9576	1.0000
	(1,-5)	0.9498	1.0000	0.9462	1.0000	0.9498	1.0000
	(5.9,-5.9)	0.9518	1.3100	0.9494	1.3168	0.9524	1.3169

interval for the ratio and the difference between lognormal coefficients of variation with restricted means.

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Table 6. The ratio of average length widths of confidence intervals $CI_{COXS}, CI_{COXSR}, \mathrm{CI}_{GCIS}, \mathrm{CI}_{GCISR}, \mathrm{CI}_{ROVS}, \mathrm{CI}_{ROVSR}$ when $\mu_1, \mu_2 \in (-6, 6)$

n_1, n_2	(μ_1, μ_2) (-6,6)	$\frac{E(CI_{COXS})}{E(CI_{GCIS})}$	$\frac{E(CI_{COXS})}{E(CI_{ROVS})}$	$\frac{E(CI_{GCIS})}{E(CI_{ROVS})}$	$\frac{E(CI_{COXSR})}{E(CI_{GCISR})}$	$\frac{E(CI_{COXSR})}{E(CI_{ROVSR})}$	$\frac{E(CI_{GCISR})}{E(CI_{ROVSR})}$
10	(-5.9,5.9)	0.7562	0.7593	1.0040	0.7836	0.7863	1.0034
	(-1,5)	0.7569	0.7598	1.0038	0.7569	0.7598	1.0038
	(0,0)	0.7580	0.7607	1.0034	0.7580	0.7607	1.0034
	(1,-5)	0.7572	0.7601	1.0037	0.7573	0.7601	1.0037
	(5.9,-5.9)	0.7570	0.7591	1.0028	0.7837	0.7851	1.0017
20	(-5.9,5.9)	0.8944	0.8980	1.0040	0.9122	0.9150	1.0030
	(-1,5)	0.8948	0.8984	1.0039	0.8948	0.8984	1.0039
	(0,0)	0.8963	0.8979	1.0017	0.8963	0.8979	1.0017
	(1,-5)	0.8946	0.8979	1.0036	0.8946	0.8979	1.0036
	(5.9,-5.9)	0.8948	0.8983	1.0038	0.9147	0.9162	1.0015
30	(-5.9,5.9)	0.9380	0.9362	0.9980	0.9513	0.9491	0.9977
	(-1,5)	0.9359	0.9364	1.0005	0.9359	0.9364	1.0005
	(0,0)	0.9358	0.9362	1.0004	0.9358	0.9362	1.0004
	(1,-5)	0.9366	0.9364	0.9997	0.9366	0.9364	0.9997
	(5.9,-5.9)	0.9346	0.9362	1.0016	0.9490	0.9490	0.9999
50	(-5.9,5.9)	0.9666	0.9639	0.9971	0.9764	0.9739	0.9969
	(-1,5)	0.9655	0.9639	0.9983	0.9655	0.9639	0.9983
	(0,0)	0.9666	0.9640	0.9969	0.9669	0.9640	0.9969
	(1,-5)	0.9669	0.9638	0.9967	0.9669	0.9638	0.9967
	(5.9,-5.9)	0.9677	0.9640	0.9962	0.9775	0.9730	0.9953
100	(-5.9,5.9)	0.9892	0.9828	0.9935	0.9940	0.9878	0.9937
	(-1,5)	0.9874	0.9828	0.9953	0.9874	0.9828	0.9953
	(0,0)	0.9899	0.9828	0.9928	0.9899	0.9828	0.9928
	(1,-5)	0.9890	0.9828	0.9937	0.9890	0.9828	0.9937
	(5.9,-5.9)	0.9986	0.9828	0.9941	0.9937	0.9880	0.9942

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Table 7. Coverage probability and the ratio of average length widths of confidence intervals $CI_{COXS}, CI_{COXSR}, \mathrm{CI}_{GCID}, \mathrm{CI}_{GCIDR}, \mathrm{CI}_{ROVD}, \mathrm{CI}_{ROVDR}$ when $\mu_1, \mu_2 \in (0, 6)$

n_1, n_2	(μ_1, μ_2)	CI_{COXS}, CI_{COXSR}	$\frac{E(CI_{COXS})}{E(CI_{COXSR})}$	CI_{GCIS}, CI_{GCISR}	$\frac{E(CI_{GCIS})}{E(CI_{GCISR})}$	CI_{ROVS}, CI_{ROVSR}	$\frac{E(CI_{ROVS})}{E(CI_{ROVSR})}$
10	(5.9,0.5)	0.9570	1.3540	0.9558	1.4615	0.9608	1.4587
	(5,1)	0.9516	1.0144	0.9552	1.0632	0.9588	1.0609
	(3,3)	0.9516	1.0000	0.9518	1.0000	0.9566	1.0000
	(0.5,5.9)	0.9548	1.3627	0.9552	1.4736	0.9610	1.4708
20	(5.9,0.5)	0.9580	1.2084	0.9548	1.2528	0.9584	1.2491
	(5,1)	0.9526	1.0002	0.9492	1.0014	0.9562	1.0011
	(3,3)	0.9500	1.0000	0.9472	1.0000	0.9502	1.0000
	(0.5,5.9)	0.9508	1.2008	0.9518	1.2441	0.9556	1.2410
30	(5.9,0.5)	0.9538	1.1302	0.9512	1.1546	0.9538	1.1519
	(5,1)	0.9550	1.0000	0.9524	1.0000	0.9546	1.0000
	(3,3)	0.9486	1.0000	0.9468	1.0000	0.9496	1.0000
	(0.5,5.9)	0.9508	1.1294	0.9432	1.1538	0.9506	1.1518
50	(5.9,0.5)	0.9544	1.0581	0.9506	1.0673	0.9542	1.0665
	(5,1)	0.9508	1.0000	0.9460	1.0000	0.9522	1.0000
	(3,3)	0.9498	1.0000	0.9478	1.0000	0.9502	1.0000
	(0.5,5.9)	0.9564	1.0580	0.9526	1.0679	0.9572	1.0664
100	(5.9,0.5)	0.9490	1.0078	0.9458	1.0093	0.9500	1.0090
	(5,1)	0.9504	1.0000	0.9486	1.0000	0.9512	1.0000
	(3,3)	0.9512	1.0000	0.9478	1.0000	0.9502	1.0000
	(0.5,5.9)	0.9526	1.0089	0.9458	1.0105	0.9530	1.0101

Table 8. The ratio of average length widths of confidence intervals $CI_{COXS}, CI_{COXSR}, \mathrm{CI}_{GCID}, \mathrm{CI}_{GCIDR}, \mathrm{CI}_{ROVD}, \mathrm{CI}_{ROVDR}$ when $\mu_1, \mu_2 \in (0, 6)$

n_1, n_2	(μ_1, μ_2)	$\frac{E(CI_{COXS})}{E(CI_{GCIS})}$	$\frac{E(CI_{COXS})}{E(CI_{ROVS})}$	$\frac{E(CI_{GCIS})}{E(CI_{ROVS})}$	$\frac{E(CI_{COXSR})}{E(CI_{GCISR})}$	$\frac{E(CI_{COXSR})}{E(CI_{ROVSR})}$	$\frac{E(CI_{GCISR})}{E(CI_{ROVSR})}$
10	(5.9,0.5)	0.7571	0.7599	1.0032	0.8176	0.8187	1.0013
	(5,1)	0.7570	0.7596	1.0034	0.7934	0.7943	1.0012
	(3,3)	0.7587	0.7608	1.0027	0.7587	0.7608	1.0028
	(0.5,5.9)	0.7582	0.7610	1.0037	0.8199	0.8214	1.0017
20	(5.9,0.5)	0.8954	0.8981	1.0029	0.9284	0.9283	0.9999
	(5,1)	0.8944	0.8979	1.0038	0.8955	0.8986	1.0034
	(3,3)	0.8954	0.8983	1.0032	0.8954	0.8983	1.0032
	(0.5,5.9)	0.8962	0.8978	1.0017	0.9286	0.9279	0.9992
30	(5.9,0.5)	0.9358	0.9363	1.0005	0.9560	0.9543	0.9981
	(5,1)	0.9352	0.9363	1.0012	0.9353	0.9364	1.0012
	(3,3)	0.9354	0.9362	1.0008	0.9354	0.9362	1.0008
	(0.5,5.9)	0.9368	0.9364	0.9994	0.9571	0.9550	0.9977
50	(5.9,0.5)	0.9676	0.9639	0.9961	0.9760	0.9716	0.9954
	(5,1)	0.9671	0.9640	0.9968	0.9671	0.9640	0.9968
	(3,3)	0.9672	0.9640	0.9966	0.9672	0.9640	0.9966
	(0.5,5.9)	0.9663	0.9639	0.9975	0.9753	0.9716	0.9962
100	(5.9,0.5)	0.9879	0.9828	0.9948	0.9893	0.9839	0.9945
	(5,1)	0.9880	0.9828	0.9946	0.9880	0.9828	0.9946
	(3,3)	0.9895	0.9828	0.9932	0.9895	0.9828	0.9932
	(0.5,5.9)	0.9884	0.9828	0.9943	0.9900	0.9839	0.9938