Songklanakarin J. Sci. Technol.
39 (1), 11-18, Jan. - Feb. 2017

http://www.sjst.psu.ac.th

Original Article

A modified hyperbolic secant distribution

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Received: 17 December 2015; Revised: 4 March 2016; Accepted: 24 March 2016

Abstract

The aims of this paper are to introduce and validate a new distribution, which is related to the hyperbolic function. The differential equation is applied to obtain the survival and probabilistic functions. The moment generating function is provided by using the integral transform. The proposed distribution is applied to real data sets. It is demonstrated that it can be used as an alternative model in various disciplines such as electronics, financial, weather, and arrival times.

Keywords: hyperbolic function, hyperbolic secant distribution, secant function

1. Introduction

The hyperbolic function is an important mathematical function in relation to trigonometric functions, exponential functions, and complex numbers. The hyperbolic secant is a part of a set of hyperbolic functions, which is defined as $\operatorname{sech}(x) = \dfrac{2}{(e^{x} + e^{-x})}$. In 1934, Baten introduced a probability distribution related to the hyperbolic secant function, which is called the hyperbolic secant distribution (HSD) (Baten, 1934). This work was then expanded by Talacko (1956), who proposed the distribution for financial return models. Consequently, the HSD provides an optimal fit and exhibits more leptokurtosis than both the normal and logistic distributions. However, it is limited in its utility; it cannot take on various shapes due to lack of flexibility in its parameters, thus natural behaviors cannot be sufficiently explained with this model.

Many branches of natural sciences emphasize the study of phenomena such as the spreading of disease or growth of a population by looking at rates of change. Differential equation is an important technique used to solve these types of a problem. Differential equation is a useful technique in explaining statistical properties and a vital tool in proving or disproving statistic-based issues. This model can be applied to a survival analysis, lifetime data analysis, and reliability analysis. This paper introduces techniques used to generate a new distribution that is constructed by modifying the hyperbolic functions into a more flexible model using differential equations.

We implement the technique with set $(S(t))$ be a state at time t, defined $(S(t_0) = S(t))$,..., $(S(t_n) = S(t + \Delta t))$ at $t_0 = t, t_1 = t + \tau, t_2 = t + 2\tau, \ldots, t_n = t + n\tau$ where n belongs to $\{1,2,3,\ldots,T\}$, $\tau = T/n$, and $\Delta S(t_k) = S(t_k) - S(t_{k-1})$. The process can be expressed $S(t_0) = S(t), S(t_1) - S(t_0) = \Delta S_1$, $S(t_2) - S(t_1) = \Delta S_2, \ldots, S(t_k) - S(t_{k-1}) = \Delta S_k$.

Under the assumption that τ is very small, we obtain $S(t_k) - S(t_{k-1}) = \Delta S_k = dS(t)$.

Some properties of the survival function, $S(\cdot)$, are right continuous in t. Nevertheless, for continuous survival time T, $S(\cdot)$ is continuous with non-increasing function, ($S(t_0) = 1$ and $\lim_{n\to\infty} S(t_n) = 0$).

The rest of the paper is organized as follows. In Section 2, a new family of distributions is proposed from the hyperbolic functions using differential equations to develop this new probability function. Also we will apply real data sets to the proposed model in Section 3 before starting our conclusions in Section 4.

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2. A New Distribution

This section presents an alternative way in which the new probability function could be developed. We apply the differential equation to derive a probability density function by setting up the first order differential equation, and solve it to get a survival function. We then take the derivative of the survival function. Consequently, a probability density function associated with survival function will be obtained.

Definition 1:

Let T be a random variable on a probability space (Ω, F, P) with a probability density function $(f(t;\theta))$, a distribution function $(F(t;\theta))$ and survival function $(S(t;\theta) = 1 - F(t;\theta))$ and $(f(t;\theta))$ defined by $(F^{'}(t;\theta))$ or $-(S^{'}(t;\theta))$

Proposition 1:

Let T be a random variable on a probability space (Ω, F, P) with probability density function $f(t;k,a)$ and $T \in [0,\infty)$, which produces the equation below:

$$f(t) = -\frac{e^{a+t}k}{(e^{2a}+e^{2t})\pi} + \frac{e^{a-t}(2\pi - k\log[2e^{2a}] + k\log[e^{2a}+e^{2t}])}{2\pi} \quad (1)$$

where a is an initial value and

$$0 \le k \le -\frac{2\pi}{e^{a}\left[(2a-1)+\log(2)-\log(1-e^{2a})\right]}.$$

Proof:

We propose the new distribution by finding the solution of the following differential equation

$$S'(t) + S(t) = \frac{kI_a}{\pi(e^{(t-a)} + e^{-(t-a)})}, \qquad S(a) = 1, \quad (2)$$

when

$$I_a = \begin{cases} 1 & a \in [0,\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Taking the first derivative, we will obtain the probability functions as seen in Eq. (1). As for the solution function of the differential equation, we set $U(t)$ as an arbitrary function, then multiplying Eq. (2) by $U(t)$ we obtain

$$U(t)S'(t) + U(t)S(t) = \frac{U(t)kI_a}{\pi(e^{(t-a)} + e^{-(t-a)})},$$

by derivative product rule, it is then set to the following form

$$(U(t)S(t))' = \frac{U(t)kI_a}{\pi(e^{(t-a)} + e^{-(t-a)})}, \quad (3)$$

where $U(t) = e^{\int dt}$.

Therefore, the result of differential equation Eq. (2) is

$$S(t) = \frac{1}{U(t)}\left[\int \frac{U(t)kI_a}{\pi(e^{(t-a)} + e^{-(t-a)})}dt + C\right],$$

which is the survival function.

In this case $U(t) = e^{t}$ and the solution of the differential equation with initial value problem $S(a) = 1$ is

$$S(t) = \frac{e^{(a-t)}(2\pi - k\log(2e^{2a}) + k\log(2e^{2a} + 2e^{2t}))}{2\pi}.$$

Now, taking the first derivative, the pdf of T is

$$f(t) = \frac{e^{a-t}(e^{(a-t)}(2\pi - k\log(2e^{2a}) + k\log(2e^{2a} + 2e^{2t})))}{2\pi} - \frac{e^{a+t}k}{(e^{2a}+e^{2t})\pi}.$$

Corollary 1:

Let T be a random variable with the pdf $f(t;b)$, which could be expressed as

$$f(t;b) = \frac{2e^{t-b}}{(1+e^{2t-2b})\pi}, \quad (4)$$

with the location parameter b and initial value $c \le b$.

Proof:

Similar to Eq. (1), $f(t) = \frac{2e^{t-b}}{(1+e^{2t-2b})\pi}$ obtained by solving the differential equation expressed as

$$S^{'}(t) = \frac{-2}{\pi(e^{(t-b)} + e^{-(t-b)})}, \qquad S(c) = 1, \quad (5)$$

by integrating both sides, we obtain the survival function, which is the result from solving Eq.(5)

$$S(t) = \frac{\pi + 2\arctan(e^{c-b}) - 2\arctan(e^{t-b})}{\pi},$$

then by taking first derivative, the pdf is obtained in the form

$$f(t;b) = \frac{2e^{t-b}}{(1+e^{2t-2b})\pi},$$

where $T \in (-\infty,\infty)$.

It is also possible to present the general differential equation as

$$S^{'}(t) + AS(t) = \frac{AkI_a}{\pi(e^{(t-a)} + e^{-(t-a)})} - \frac{2(1-A)}{\pi(e^{(t-b)} + e^{-(t-b)})}, \quad (6)$$

by imposing $A = 0$ or 1. The Eq.(6) can be reduced to Eq. (2) and Eq.(5), respectively. Following this logic, $S_1(a) = 1$ and $S_2(c) = 1$ are therefore the initial values of Eq. (2) and Eq. (5), respectively.

Moreover, the pdf derived from the survival function is a probability function, which is sastified the following properties;

1. Setting $S(a)=1$ and $S(c)=1$ are initial value functions
2. The survival function is monotonic decresing
3. $\lim_{t\to\infty} S(t)=0$
4. If the pdf correspond to parameter space, then it will be greater than zero

2.1 Some properties of the new distribution

There are many methods to solve the differential equations. An important method uses the Laplace transform, which is related to the moment generating functions.

The Laplace transform of Eq. (1) is given by

$$L(s)=\int_a^\infty e^{-st} f(t)dt\ ;\ \ s>0$$

$$=\int_a^\infty e^{-st}\left(-\frac{e^{a+t}k}{(e^{2a}+e^{2t})\pi}+\frac{e^{a-t}(2\pi-k\log[2e^{2a}]+k\log[e^{2a}+e^{2t}])}{2\pi}\right)d(t)$$

$$=\ [-\int_a^\infty\left(\frac{e^{-a-(s-1)t}k}{(1+e^{2t-2a})\pi}\right)+\left(\int_a^\infty\frac{e^{a-(s+1)t}2\pi}{2\pi}\right)-\left(\int_a^\infty\frac{ke^{a-(s+1)t}\log[2e^{2a}]}{2\pi}\right)+\left(\int_a^\infty\frac{ke^{a-(s+1)t}\log[e^{2a}+e^{2t}]}{2\pi}\right)]dt$$

$$=-\int_a^\infty\left(\frac{e^{-a-(s-1)t}k}{(1+e^{2t-2a})\pi}\right)dt+\left(\frac{e^{a-(s+1)t}}{-(s+1)}\right)\Big|_a^\infty-\left(\frac{(\log[2e^{2a}]k)}{2\pi}\frac{e^{a-(s+1)t}}{-(s+1)}\right)\Big|_a^\infty+\left(\int_a^\infty\frac{ke^{a-(s+1)t}\log[e^{2a}+e^{2t}]}{2\pi}\right)dt$$

$$=-\int_a^\infty\left(\frac{e^{-a-(s-1)t}k}{(1+e^{2t-2a})\pi}\right)dt+\left(\frac{e^{-sa}}{(s+1)}\right)-\left(\frac{(\log[2e^{2a}]k)}{2\pi}\frac{e^{-sa}}{(s+1)}\right)+\left(\int_a^\infty\frac{ke^{a-(s+1)t}\log[e^{2a}+e^{2t}]}{2\pi}\right)dt. \tag{7}$$

Focusing on the last term and using the by part integration technique, we have

$$\left(\int\frac{ke^{a-(s+1)t}\log[e^{2a}+e^{2t}]}{2\pi}\right)dt=\left[\frac{k\log[e^{2a}+e^{2t}]}{2\pi}\cdot\frac{e^{a-(s+1)t}}{-(s+1)}+\int\left(\frac{e^{a-(s+1)t}}{(s+1)}\frac{k}{(1+e^{2a-2t})\pi}\right)dt\right], \tag{8}$$

substitue the last term of Eq. (7) with Eq. (8), becomes

$$L(s)=-\int_a^\infty\left(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi}\right)dt+\left(\frac{e^{-sa}}{(s+1)}\right)-\left(\frac{(\log[2e^{2a}]k)}{2\pi}\frac{e^{-sa}}{(s+1)}\right)$$

$$+\left[\frac{k\log[e^{2a}+e^{2t}]}{2\pi}\cdot\frac{e^{a-(s+1)t}}{-(s+1)}\right]\Big|_a^\infty+\int_a^\infty\left(\frac{e^{a-(s+1)t}}{(s+1)}\frac{k}{(1+e^{2a-2t})\pi}\right)dt$$

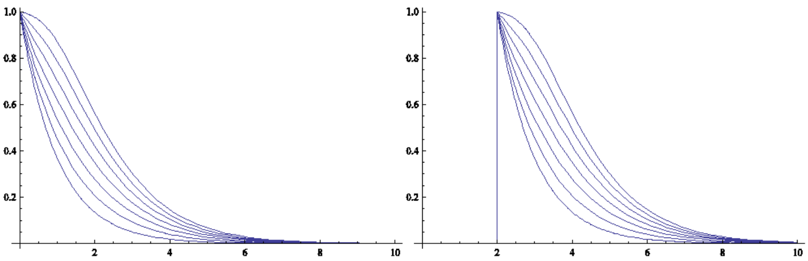
$$=\left(\frac{1}{(s+1)}-1\right)\cdot\int_a^\infty\left(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi}\right)dt+\left(\frac{e^{-sa}}{(s+1)}\right)-\left(\frac{(\log[2e^{2a}]k)}{2\pi}\frac{e^{-sa}}{(s+1)}\right)+\left(\frac{e^{-sa}k\log[2e^{2a}]}{((s+1))2\pi}\right). \tag{9}$$

The convergence of the integral part of Eq. (9) will be shown in Appendix. Under the condition of $0\le a\le t$ with a power series, we rearrange Eq. (9) to obtain the following Laplace transformation

$$L(S)=\left[\frac{k}{\pi}\left(\frac{1}{(s+1)}-1\right)\int_a^\infty\left(e^{a-(s+1)t}\sum_{n=0}^{\infty}((-1)^n e^{2(a-t)n})\right)\right]dt+\left(\frac{e^{-sa}}{(s+1)}\right).$$

In addition, the moment generating function (mgf) of t is given by $M(t)=E(e^{st})=L(-s)$, for all moments if the Laplace transform exists (as shown in Appendix), and where the rth moment is $E(t^r)=(-1)^r L^r(0)$. In particular,

The moment generating function of Eq. (4) is given by



(a) parameters $a = 0$ **(b) parameters $a = 2$**

Figure 1. Some plots of the survival function of the new distribution where k = 0, 1, 2, …, 6 from the bottom line respectively

$$
\begin{aligned}
M(s) &= \int_{-\infty}^{\infty} e^{st} f(t)dt \\
&= \int_{-\infty}^{\infty} e^{st} \frac{2}{(e^{-(t-b)} + e^{(t-b)})\pi} dt \\
&= \int_{-\infty}^{b} \frac{2e^{(s+1)t-b}}{(1-(-1e^{2(t-b)}))\pi} dt + \int_{b}^{\infty} \frac{2e^{(s-1)t+b}}{(1-(-1e^{-2(t-b)}))\pi} dt \ ; s > 0 \\
&= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n e^{sb}}{(s+1+2n)}
\end{aligned}
\tag{10}
$$

In the case of Eq. (4) where symmetric properties define the set of domain as $(-\infty, \infty)$, then $M(s)$ can be expressed as Eq.(10)

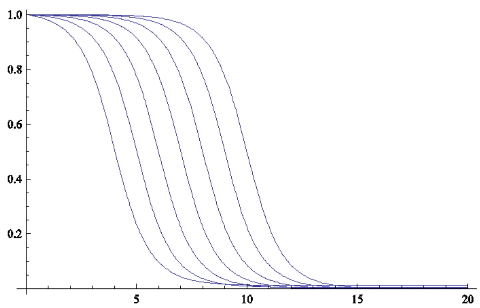
Some plots of the survival and reliability functions related to the proposed distribution are presented in Figure 1.

The survival function of Eq. (4) are exhibited in Figure 2.

In survival analysis, the hazard function is defined as $h(t) = \dfrac{f(t)}{S(t)}$ where $f(t)$ and $S(t)$ are the pdf and survival function, respectively. The hazard function of the proposed distribution (Eq. (1)) can be written as

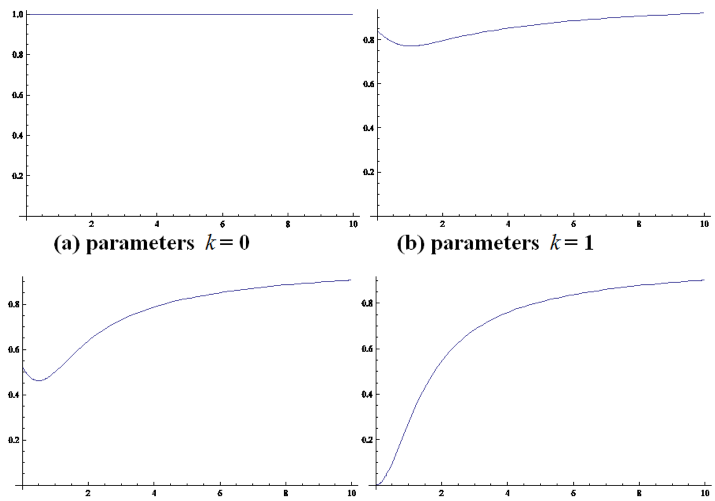
$$h(t) = \frac{-2e^{2t}k}{(e^{2a} + e^{2t})(2\pi - k\log[2e^{2a}] + k\log[e^{2a} + e^{2t}])} + 1.$$

Some shapes of the hazard function of the proposed distribution (Eq. (1)) with some parameters are shown in Figure 3



(a) parameters b = 0;a = 4, 5, 6, …, 11

Figure 2. Some plots of the survival function for the new distribution with some parameter values of Eq. (4)



(a) parameters $k = 0$ **(b) parameters $k = 1$**

(c) parameters $k = 3$ **(d) parameters $k = 2\pi$**

Figure 3. Some plots of hazard functions for the proposed distribution with parameter a = 0 of Eq.(1)

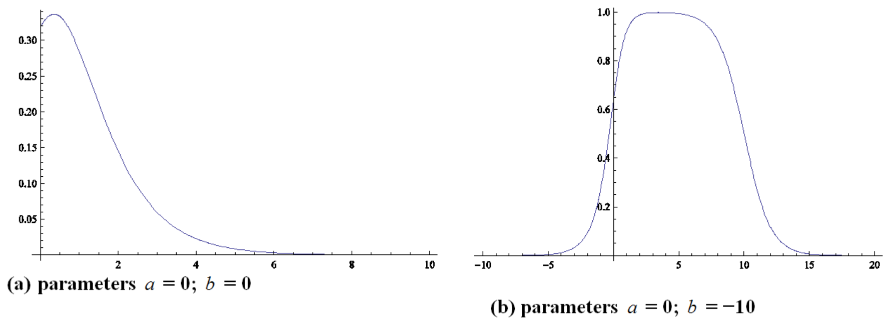


Figure 4. Some plots of the hazard functions for the proposed distribution with some parameter values of Eq.(4)

The hazard function of Eq. (4) can be written as

$$h(t) = \frac{2e^{t-\mathrm{b}}}{(1+e^{2(t-\mathrm{b})})(\pi + 2\arctan(e^{(c-\mathrm{b})})) - 2\arctan(e^{(t-\mathrm{b})})}),$$

where several function with some parameters are illustrated in Figure 4.

Figure 5 shows some shapes of the pdf based on some selected parameter values. It illustrates that the new proposed distribution consists of various shapes.

The symmetric behaviors of Eq. (4) are illustrated in Figure 6.

3. Applications

Four data sets are fitted with the proposed distribution. The first example deals with the rate of change of failure times of electronic devices reported by Domma (2014). The second example is the rate of change of soybean prices at Chicago Board of Trade (CBOT) from June 09 - June 14 fitted with the proposed distribution. The third and fourth data sets are US July precipitation (Top 8 soybeans production state) from 1988-2015 and the inter-arrival times (10 minutes) data set for cars (Law, 2015), respectively.

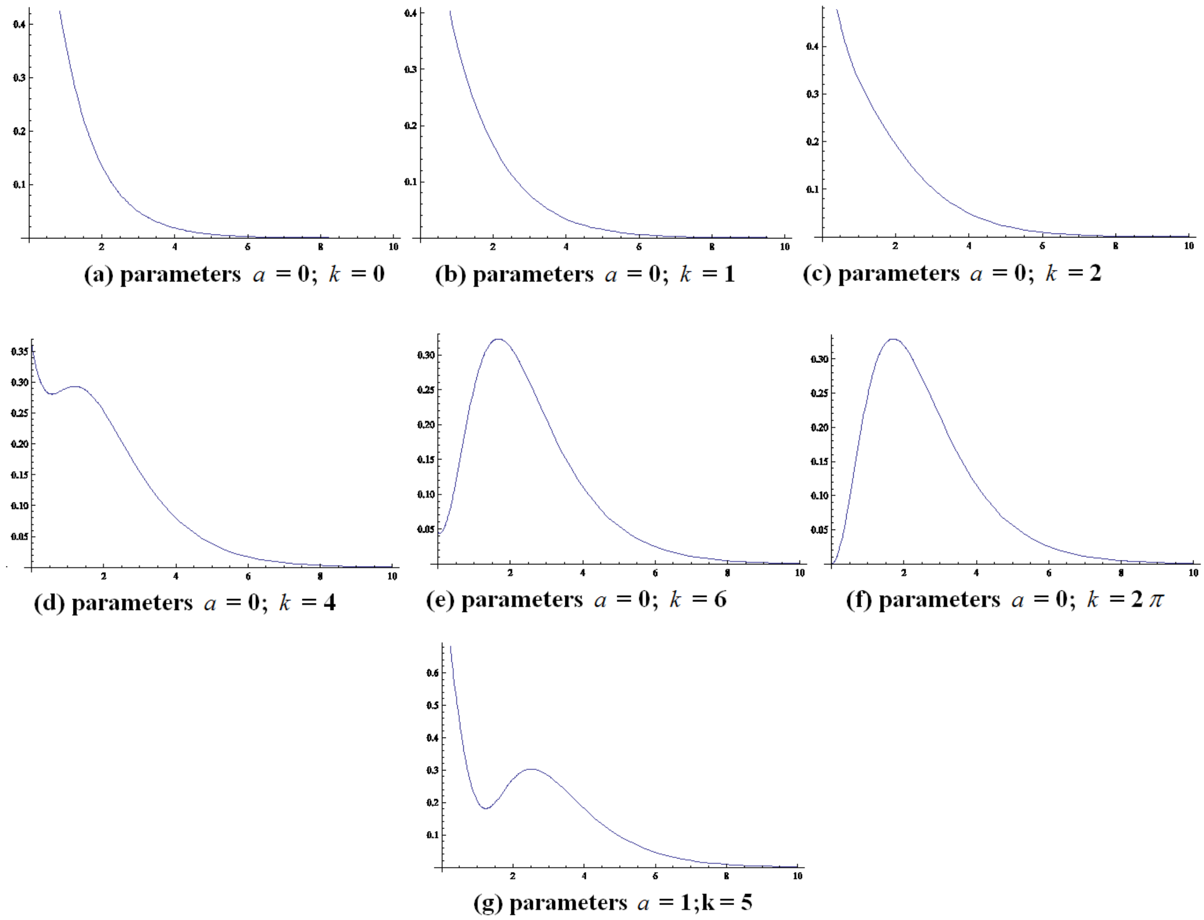


Figure 5. Some pdf plots of new distribution with some parameter values of Eq. (1)

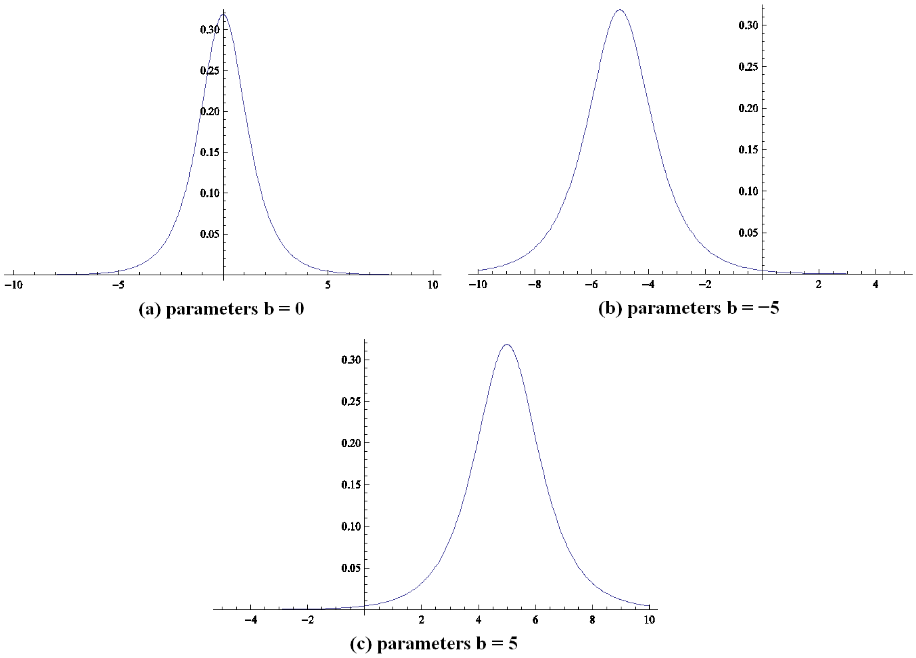


Figure 6. Some pdf plots of the new distribution with some parameter values of Eq. (4)

The estimated parameters could be carried out by maximum likelihood estimation (MLE). In this study, we use bbmle (Bolker & R Team, 2014) package of R programming language (R Core Team, 2014) to obtain the parameter estimates.

The fitting distributions for the four examples are verified as shown in Figure 7. In addition, the estimated parameters using MLE are shown in Table 1.

4. Conclusions

The aim of this paper is to propose an alternative method to generate a new survival function. We have done so using the relationship between differential equation and hyperbolic function, and obtained the solution of the method as non-increasing function; we obtain the survival and the probability density functions from these techniques. Moreover,

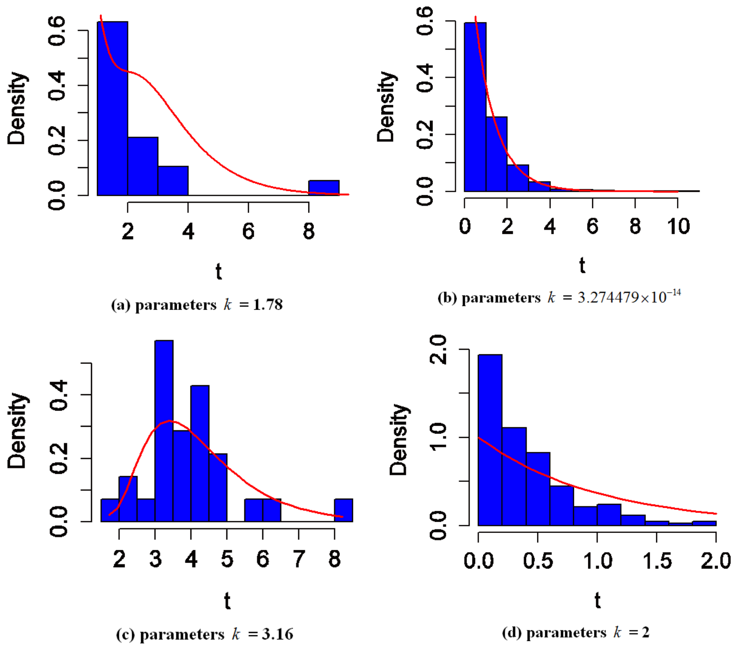


Figure 7. Some fitted distributions with $a = \min(t)$

Table 1. The estimated parameters using MLE

Data set	Estimated parameters	
	a	*k*
1. The rate of change of failure times of electronic devices	0	1.78
2. The rate of change of soybean prices at Chicago Board of Trade (CBOT) from June 09 - June 14	0	3.274479×10^{-14}
3. July precipitation from 1988-2015	0	3.16
4. The inter-arrival times (10 minutes) data set for cars	0	2

we organize the general form of the model Eq.(6), which produces two survival and the probabilistic functions. The various graphical styles of the result can be used to demonstrate its flexibility in analyzing behavior in real data.

Acknowledgements

The authors would like to thank to Department of Statistics, Faculty of Science and the Graduate School of Kasetsart University. Also we thank to Charoen Pokphand Foods (CPF) for supporting the first author.

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APPENDIX

A convergence will be proved

$$(\frac{1}{(s+1)}-1)\cdot\int_a^\infty(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi})dt+(\frac{e^{-sa}}{(s+1)})-(\frac{(\log[2e^{2a}]k)}{2\pi}\frac{e^{-sa}}{(s+1)}).$$

The integral is improper integral, we use the comparison test to prove convergence by the following theorem

A comparison test theorem

Suppose that f and g are continuous functions with $0\le f(t)\le g(\mathrm{t})$ for $t\ge a$ then

1. If $\int_a^\infty \mathrm{g(t)}dt$ is convergent, then $\int_a^\infty f(\mathrm{t})dt$ is convergent.
2. If $\int_a^\infty \mathrm{g(t)}dt$ is divergent, then $\int_a^\infty f(\mathrm{t})dt$ is divergent.

Espectially, this improper integral as shown

$$\int_a^\infty(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi})dt.$$

We set $\int_a^\infty g(t)dt=\int_a^\infty e^{a-(s+1)t}k,\ \int_a^\infty f(t)dt=\int_a^\infty(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi})dt,$ whereas $g(t)$ is greater than $f(t)$ for all $t\ge a, s>0,$ and

$$\int_a^\infty g(t)dt=\int_a^\infty e^{a-(s+1)t}k=\frac{e^a k}{(s+1)}[e^{-(s+1)a}]\ ;\ t\ge a, s>0.$$

It is verified that $\int_a^\infty \mathrm{g(t)}dt$ converge to $\frac{e^a k}{(s+1)}[e^{-(s+1)a}]$, thererfore, $\int_a^\infty f(t)dt=\int_a^\infty(\frac{e^{a-(s+1)t}k}{(1+e^{2a-2t})\pi})dt$ is convergent too.