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Original Article

On R-left cancellative semigroups

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Abstract

Suppose R is a Green's relation on a semigroup *S* and let $\mathrm{R}LC(S)=\{a\in S:\forall x,y\in S, ax=ay\Rightarrow x\,\mathrm{R}\,y\}$. It is obvious that R*LC(S)* is a subsemigroup of *S* if it is nonempty. The purpose of this paper is to study some properties of R*LC(S* .

Keywords: left cancellative, Green's relations

1. Introduction

In 1951, Green defined the equivalence relation R on any semigroup *S* by the rule that, for $a,b\in S$, $a\,\mathrm{R}\,b$ if and only if *a* and *b* generate the same principal right ideal, that is, $aS^1=bS^1$. In this case, we say that *a* and *b* are *R* equivalent, and write $(a,b)\in\mathrm{R}$ or. $a\,\mathrm{R}\,b$ In addition, *R* is a left congruence (that is, $a\,\mathrm{R}\,b$ implies $ca\,\mathrm{R}\,cb$ for all $c\in S$) (Howie, 1995). An element *a* of a semigroup *s* is called an R-left cancellative element if for every $x,y\in S$, $ax=ay$ implies $x\,\mathrm{R}\,y$ and *S* is called an R-left cancellative semigroup if all elements of *S* are *R*-left cancellative. Then R-left cancellative is a generalization of left cancellative. Notice that every right simple semigroup is trivially an R-left cancellative semigroup since it has only one R-class. Hence every group is also an R-left cancellative semigroup. The notion of R-left cancellative for semigroups was introduced by Golchin and Mchammadzadeh (2007). Shyr (1976) studied some properties of a left cancellative subsemigroup of a semigroup and generalized some results on left cancellative semigroups. Let R*LC(S)* be the set of all R-left cancellative elements of *S*, that is,

$$\mathrm{R}\,LC(S)=\{a\in S:\forall x,y\in S, ax=ay\Rightarrow x\,\mathrm{R}\,y\}.$$

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The aim of this paper is to discuss some properties of R*LC(S)* using some of the results obtained by Shyr (1976). Before going further, we begin with examples which illustrate with some semigroups *S* defined by its Cayley tables, that R*LC(S)* can be several types of subsets of *S*.

Example 1. Let *S = {a, b, c, d,}* be a semigroup with the multiplication defined by:

.	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*c*
b	*b*	*c*	*a*	*a*
c	*c*	*a*	*b*	*b*
d	*c*	*a*	*b*	*b*

It easy to verify that

$aS^1=bS^1=cS^1=\{a,b,c\}, dS^1=S$ and $\mathrm{R}\,LC(S)=\varnothing$.

Example 2. Let *S = {a, b, c, d,}* be a semigroup with the multiplication defined by:

.	*a*	*b*	*c*	*d*
a	*a*	*b*	*a*	*b*
b	*b*	*a*	*b*	*a*
c	*a*	*b*	*c*	*d*
d	*b*	*a*	*d*	*c*

Then $aS^1 = bS^1 = \{a,b\}$, $cS^1 = dS^1 = S$ and $\mathrm{R}LC(S) = \{c,d\}$ is a proper subset of *S*.

Example 3. Let *BL(X)* be the semigroup of all one-to-one mappings $\alpha : X \to X$ with the property that $X \setminus X\alpha$ is infinite where *X* is a countably infinite set (Baer and Levi, 1932). Also, authors showed that *BL(X)* is a right simple semigroup which is not a group and hence it is an R-left cancellative semigroup. But *BL(X)* is not left cancellative. The proof is as follows: let *X* be a countably infinite set and let *A* be a subset of *X* such that

$$|X| = |X \setminus A| = |A|.$$

Then we write

$$A = \{a_n : n \in \mathrm{N}\}$$

with $a_i \neq a_j$ if $i \neq j$.

Define $\alpha, \beta, \gamma : X \to X$ by

$$x\alpha = \begin{cases} a_{2n} & \text{if } x = a_n \text{ for some } n \in \mathrm{N}, \\ x & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a_{3n} & \text{if } x = a_{2n+1} \text{ for some } n \in \mathrm{N}, \\ x & \text{otherwise,} \end{cases}$$

and

$$x\gamma = \begin{cases} a_{5n} & \text{if } x = a_{2n+1} \text{ for some } n \in \mathrm{N}, \\ x & \text{otherwise,} \end{cases}$$

Then

$$\alpha, \beta, \gamma \in BL(X).$$

Since

for every $n \in \mathrm{N}$, $a_n\alpha\beta = a_{2n}\beta = a_{2n} = a_{2n}\gamma = a_n\alpha\gamma$

and for every $x \in X \setminus A$, $x\alpha\beta = x\beta = x = x\gamma = x\alpha\gamma$,

we deduce that $\alpha\beta = \alpha\gamma$ but $\beta \neq \gamma$ which imply that α is not left cancellative.

Proposition 1. R*LC(S)* is a subsemigroup of *S* if it is a nonempty set.

Proof: Suppose that $\mathrm{R}LC(S) \neq \varnothing$. Let $a, b \in \mathrm{R}LC(S)$ and $x, y \in S$ be such that $abx = aby$. Since $a \in \mathrm{R}LC(S)$, it follows that $bx \,\mathrm{R}\, by$. Then $bx = byc$ and $by = bxd$ for some $c, d \in S^1$. Since $b \in \mathrm{R}LC(S)$, we deduce that $x \,\mathrm{R}\, yc$ and $y \,\mathrm{R}\, xd$. Hence $x = ycm$ and $y = xdn$ for some $m, n \in S^1$ which imply that $x \,\mathrm{R}\, y$. Therefore $ab \in \mathrm{R}LC(S)$, as required.

Some of the elementary properties of the sub-semigroup R*LC(S* of a semigroup *S* that carry over results appear in the next proposition.

Proposition 2. Let *S* be a semigroup. Then the following statements hold.

(i) If $x \in \mathrm{R}LC(S)$ and $x^2 = x$, then $y \,\mathrm{R}\, xy$ for all $y \in S$.

(ii) If $x \in \mathrm{R}LC(S)$ and $x = xy$ for some $y \in S$, then $y \,\mathrm{R}\, y^2$.

(iii) If $x \in \mathrm{R}LC(S)$ and $x = xy$ for some $a, b \in S$, then $b \in \mathrm{R}LC(S)$.

(iv) If R*LC(S)* has an idempotent, then $[\mathrm{R}LC(S)]^2 = \mathrm{R}LC(S)$ and $S^2 = S$.

(v) If R*LC(S)* contains a right ideal of *S*, then R*LC=(S)*

(vi) Every left identity is contained in R*LC(S)*.

(vii) $S \setminus \mathrm{R}LC(S)$ is a left ideal of *S* if and only if $\mathrm{R}LC(S) \neq S$.

Proof: The proofs of (i), (ii) and (iii) are easy. (iv) Suppose that $x \in \mathrm{R}LC(S)$ and $x^2 = x$. It suffices to verify that $\mathrm{R}LC(S) \subseteq [\mathrm{R}LC(S)]^2$. Let $a \in \mathrm{R}LC(S)$, then by Proposition 1, $xa \in \mathrm{R}LC(S)$. From (i), we obtain that $a \,\mathrm{R}\, xa$. Then $a = xau$ for some $u \in S^1$. If $u = 1$, then $a = xa \in [\mathrm{R}LC(S)]^2$. If $u \neq 1$, then $u \in \mathrm{R}LC(S)$ by (iii). This means that $a \in [\mathrm{R}LC(S)]^2$. Thus $\mathrm{R}LC(S) = [\mathrm{R}LC(S)]^2$, as required. Let $y \in S$, then $y \,\mathrm{R}\, xy$ by (i). Hence $y = xyv$ for some $v \in S^1$. This shows that $S = S^2$ holds. (v) Assume that *A* is a right ideal of *S* such that $A \subseteq \mathrm{R}LC(S)$. Let $x \in S$ and $a \in A$. Then $ax \in A \subseteq \mathrm{R}LC(S)$. By (iii), we obtain $x \in \mathrm{R}LC(S)$. This yields property (v). (vii) We have in fact established that $S \setminus \mathrm{R}LC(S)$ is a left ideal of *S* implies $S \setminus \mathrm{R}LC(S) \neq \varnothing$. Thus $\mathrm{R}LC(S) \neq S$. Conversely, suppose that $\mathrm{R}LC(S) \neq S$. Then $S \setminus \mathrm{R}LC(S) \neq \varnothing$. Let $a \in S$ and $b \in S \setminus \mathrm{R}LC(S)$. If $ab \in \mathrm{R}LC(S)$, then $b \in \mathrm{R}LC(S)$ by (iii) which is a contradiction. We deduce that $ab \notin \mathrm{R}LC(S)$, and therefore $ab \in S \setminus \mathrm{R}LC(S)$.

Theorem 3. If a semigroup *S* has a right identity 1 and $\mathrm{R}LC(S) \neq \varnothing$, then 1 is the identity of *S*.

Proof: Suppose that *S* has 1 as its a right identity and $\mathrm{R}LC(S) \neq \varnothing$. Let $x \in \mathrm{R}LC(S)$. Then $X = X^1$. By Proposition 2 (iii), we obtain $1 \in \mathrm{R}LC(S)$. Let $y \in S$, then by Proposition 2 (i), we have $y \,\mathrm{R}\, 1y$. Then $y = 1yb$ for some $b \in S^1$. Hence $1y = 1(1yb) = 1(yb) = y$. This proves that 1 is the identity of *S*.

It is natural to ask if the analogous of Theorem 3 holds for a left identity in a semigroup. The following example is the answer.

Example 4. Let *S = {a, b, c, d,}* be a semigroup with the multiplication defined by:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	c	c
d	a	b	c	d

Evidently, d is a left identity of S and $d \in \mathrm{R}LC(S)$ but d is not a right identity of S.

Theorem 4. Let G_e be a subgroup of a semigroup S having e as its identity. Then either

$G_e \subseteq \mathrm{R}LC(S)$ or

$G_e \subseteq S \setminus \mathrm{R}LC(S)$.

Proof: Suppose that G_e is not contained in R*LC(S)*. Then we are assured $x \in G_e \setminus \mathrm{R}LC(S)$. Since $x \notin \mathrm{R}LC(S)$, there exist $a, b \in S$ such that $xa = xb$ but $(a, b) \notin \mathrm{R}$. Also, we note here that $ea = x^{-1}xa = x^{-1}xb = eb$. But then whatever the choice of $g \in G_e$, $ga = gea = geb = gb$. Since $(a, b) \notin \mathrm{R}$, it then follows that $g \notin \mathrm{R}LC(S)$ for all $g \in G_e$. This proves that $G_e \subseteq S \setminus \mathrm{R}LC(S)$, as required.

Theorem 5. Let R*LC(S)* be a subsemigroup of a semigroup S with $\mathrm{R}LC(S) \neq S$. If $aS = S$ for all $a \in \mathrm{R}LC(S)$, then $S \setminus \mathrm{R}LC(S)$ is an ideal of S.

Proof: Suppose that $aS = S$ for all $a \in \mathrm{R}LC(S)$. By virtue of Proposition 2(vii), $S \setminus \mathrm{R}LC(S)$ is a left ideal of S. It remains to show that $S \setminus \mathrm{R}LC(S)$ is a right ideal of S. Let $a \in S \setminus \mathrm{R}LC(S)$ and $s \in S$. Then there exist elements $x, y \in S$ satisfying $ax = ay$ and $(x, y) \notin \mathrm{R}$. Suppose that $as \in \mathrm{R}LC(S)$. By Proposition 2 (iii), $s \in \mathrm{R}LC(S)$. By assumption, we have $sx' = x$ and $sy' = y$ for some $x', y' \in S$. Since the relation R is a left compatible, we deduce that $(x', y') \notin \mathrm{R}$. We now have $asy' = ay = ax = asx'$ and $(x', y') \notin \mathrm{R}$ which imply $as \notin \mathrm{R}LC(S)$. This shows that $S \setminus \mathrm{R}LC(S)$ is a right ideal of S. Therefore the theorem is completely proved.

Theorem 6. Let S be an R-left cancellative semigroup and I a right ideal of S. If I is a commutative semigroup, then $s_1s_2 \mathrm{R} s_2s_1$ for all $s_1, s_2 \in S$, and hence R is a congruence on S.

Proof: Suppose that I is a commutative semigroup. Let $s_1, s_2 \in S$ and $t_1, t_2 \in I$. Since I a right ideal of S, we have $t_1s_2, t_2s_1 \in I$. Hence

$$\begin{aligned}(t_1t_2)(s_1s_2) &= (t_1(t_2s_1))s_2\\ &= (t_2s_1)(t_1s_2)\\ &= ((t_1s_2)t_2)s_1\\ &= (t_2(t_1s_2))s_1\\ &= (t_2t_1)(s_2s_1)\\ &= (t_1t_2)(s_2s_1)\end{aligned}$$

Since S is R-left cancellative, it then follows that $s_1s_2 \mathrm{R} s_2s_1$.

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