Songklanakarin J. Sci. Technol.
40 (2), 402-408, Mar. - Apr. 2018

http://www.sjst.psu.ac.th

Original Article

New bounds on Poisson approximation to the distribution of a sum of negative binomial random variables

Kanint Teerapabolarn[1, 2*]

[1] *Department of Mathematics, Faculty of Science, Burapha University, Mueang, Chonburi, 20131 Thailand*

[2] *Centre of Excellence in Mathematics, Commission on Higher Education, Ratchathewi, Bangkok, 10400 Thailand*

Received:1 November 2016; Accepted: 11 January 2017

Abstract

The Stein-Chen method is usedto give new bounds, non-uniform bounds, for the distances between the distribution of a sum of independent negative binomial random variables and a Poisson distribution with mean, $\sum_{i=1}^{n}\frac{r_i q_i}{p_i}$ where r_i and $p_i = 1 - q_i$ are parameters of each negative binomial distribution. Results of this study are superior than those presented in Teerapabolarn (2014) and Hung and Giang (2016).

Keywords: negative binomial distribution, Poisson approximation, non-uniform bound, Stein-Chen method

1. Introduction

In probability theory and statistics, the negative binomial distribution with parameters $r \in \mathbb{R}^+$ and $p \in (0,1)$ is an important discrete distribution with a long history as same as the binomial distribution. When $r \in \mathbb{N}$, it is called the Pascal distribution with parameters $r \in \mathbb{N}$ and $p \in (0,1)$, and when $r = 1$, it is called the geometric distribution with parameter p. Note that, the negative binomial distribution can be considered as a mixture of a Poisson distribution with a gamma mixing distribution (Karlis & Xekalaki, 2005). In addition, some research topics related to Poisson approximation pointed out that the Poisson distribution with mean $\frac{rq}{p}$ or rq is a good approximation of the negative binomial distribution with parameters r and p when $q = 1 - p$ is small, which can be found in Vervaat (1969), Romanowska (1977), Gerber (1984), Roos (2003) and Teerapabolarn (2012). However, our interest is approximating thedistribution of a sum of n independent negative binomial random variables by a Poisson distribution, which is the main context of this study.

Let $S_n = \sum_{i=1}^{n} X_i$, where $X_1, \ldots, X_n$ are independent random variables following the negative binomial distributions, each with the probability mass function $p_{X_i}(x) = \frac{\Gamma(r_i + x)}{\Gamma(r_i)x!} q_i^x p_i^{r_i}$, $x \in \mathbb{N} \cup \{0\}$. Let Z_λ denote a Poisson random variable with mean λ $(\lambda > 0)$. From the conclusion mentioned above and for each $i \in \{1, \ldots, n\}$, it follows that if q_i is small, then the negative binomial distribution with parameters r_i and p_i is approximated by a Poisson distribution with mean $\frac{r_i q_i}{p_i}$

*Corresponding author
Email address: kanint@buu.ac.th

or r_iq_i. Additionally, we know that the distribution of a sum of n independent Poisson random variables, each with mean , λ_i is the Poisson distribution with mean $\lambda=\sum_{i=1}^{n}\lambda_i$,thus it is appropriate to approximate the distribution of S_n by a Poisson distribution with mean $\lambda=E(S_n)=\sum_{i=1}^{n}\frac{r_iq_i}{p_i}$ or $\lambda=\sum_{i=1}^{n}r_iq_i$ when all q_i are small. In the past few years, some authors haveto give uniform and non-uniform bounds on Poisson approximation to the distribution of S_n with both Poisson means $\lambda=\sum_{i=1}^{n}\frac{r_iq_i}{p_i}$ and $\lambda=\sum_{i=1}^{n}r_iq_i$ as follows.

For the Poisson mean $\lambda=\sum_{i=1}^{n}r_iq_i$, Vellaisamy and Upadhye (2009) used the method of exponents to give a uniform bound

$$d_A(S_n,Z_\lambda)\le\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}\min\left\{1,\frac{1}{\sqrt{2\lambda e}}\right\} \tag{1.1}$$

for $A\subseteq \mathrm{N}\cup\{0\}$, where $d_A(S_n,Z_\lambda)=|P(S_n\in A)-P(Z_\lambda\in A)|$ is the distance between the distribution of S_n and a Poisson distribution with mean λ .In the case of cumulative probability approximation, Teerapabolarn (2017b) used the Stein-Chen method to give uniform and non-uniform bounds for the ratio between the cumulative distribution function of S_n, $P(S_n\le x_0)$, and the Poisson cumulative distribution function, $P(Z_\lambda\le x_0)$, in the form

$$1-\frac{e^\lambda-\lambda-1}{\lambda^2}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}\le\sup_{x_0\ge0}\left\{\frac{P(S_n\le x_0)}{P(Z_\lambda\le x_0)}\right\}\le1 \tag{1.2}$$

for $x_0\in\mathrm{N}\cup\{0\}$ and

$$1-\frac{\varphi(x_0)}{x_0+1}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}\le\frac{P(S_n\le x_0)}{P(Z_\lambda\le x_0)}\le1, \tag{1.3}$$

where $\varphi(x_0)=\begin{cases}\frac{e^\lambda-\lambda-1}{\lambda^2} & \text{if } x_0=0,\\ \frac{1-P(Z_\lambda\le x_0)}{p_\lambda(x_0+1)} & \text{if } x_0\ge1.\end{cases}$ and

$p_\lambda(x_0+1)=\frac{e^{-\lambda}\lambda^{x_0+1}}{(x_0+1)!}$. For $r_i\in\mathrm{N}$, Hung and Giang (2016) used the Stein-Chen method to give two non-uniform bounds in the following forms:

$$\frac{e^\lambda-1}{\lambda}\sum_{i=1}^{n}\min\left\{\alpha_i,\frac{\beta_i-\alpha_i}{x_0+1}\right\}\le P(S_n\le x_0)-P(Z_\lambda\le x_0)\le0 \tag{1.4}$$

For $x_0\in\mathrm{N}\cup\{0\}$ and

$$d_{K_{x_0}}(S_n,Z_\lambda)\le\frac{P(Z_\lambda\le x_0)(1-P(Z_\lambda\le x_0))}{p_\lambda(x_0+1)}\sum_{i=1}^{n}\min\left\{\alpha_i,\frac{\beta_i-\alpha_i}{x_0+1}\right\}, \tag{1.5}$$

where,$d_{K_{x_0}}(S_n,Z_\lambda)=|P(S_n\le x_0)-P(Z_\lambda\le x_0)|$ $\alpha_i=1-p_i^{r_i}-r_iq_ip_i^{r_i}$ and $\beta_i=r_i(p_i^{-r_i}-1-r_iq_ip_i^{r_i})$. In the case of pointwise approximation, Teerapabolarn (20167a) used the Stein-Chen method to give a uniform bound in the form

$$d_{x_0}(S_n,Z_\lambda)\le\begin{cases}\frac{1-e^{-\lambda}(1+\lambda)}{\lambda}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i} & \text{, if } x_0=1,\\ \max\left\{\frac{1-P(Z_\lambda\le x_0-1)}{x_0+1},\frac{P(Z_\lambda\le x_0-1)}{x_0}\right\}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}, & \text{if } x_0\ge2\end{cases} \tag{1.6}$$

for $x_0\in\mathrm{N}$,where $d_{x_0}(S_n,Z_\lambda)=|P(S_n=x_0)-P(Z_\lambda=x_0)|$

For the Poisson mean $\lambda=\sum_{i=1}^{n}\frac{r_iq_i}{p_i}$, Teerapabolarn (2014) used the Stein–Chen method and w-functions to give a uniform bound in the form

$$d_A(S_n,Z_\lambda)\le\frac{1-e^{-\lambda}}{\lambda}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i^2} \tag{1.7}$$

for $A\subseteq\mathrm{N}\cup\{0\}$. For $r_i\in\mathrm{N}$, Hung and Giang (2016) used the Stein-Chen method to give a uniform bound as follows:

$$d_A(S_n,Z_\lambda)\le\sum_{i=1}^{n}\min\left\{\frac{1-e^{-\lambda}}{\lambda}\frac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\frac{q_i}{p_i}, \tag{1.8}$$

and they also gave a non-uniform bound for cumulative probability approximation in the form

$$d_{K_{x_0}}(S_n,Z_\lambda)\le\frac{e^\lambda-1}{\lambda}\sum_{i=1}^{n}\min\left\{\frac{r_iq_i}{(x_0+1)p_i},1-p_i^{r_i}\right\}\frac{q_i}{p_i} \tag{1.9}$$

for $x_0\in\mathrm{N}\cup\{0\}$. In the case of pointwise approximation, Teerapabolarn (2015a) used the same tools as in Teerapabolarn (2014) to give a non-uniform bound in the form

$$d_{x_0}(S_n,Z_\lambda)\le\min\left\{\frac{1-e^{-\lambda}}{\lambda},\frac{1}{x_0}\right\}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i^2} \tag{1.10}$$

for $x_0\in\mathrm{N}$.

We observe that the bound in (1.8) is worse than that in (1.7) because it cannot be applied to the case $r_i>0$ and $r_i\notin\mathrm{N}$, even though it may be sharper than that in (1.7). Furthermore, both bounds in (1.7) and (1.8) do not change along $A\subseteq\mathrm{N}\cup\{0\}$, which may be inappropriate for measuring the accuracy of the approximation.Notice that, the bound in (1.9) cannot be applied in the case $r_i>0$ and $r_i\notin\mathrm{N}$. In this paper, we aim to determine new bounds, non-uniform bounds,with respect to the bounds in (1.7)-(1.9).

2. Method

In 1972, Stein introduced a power full method for the normal approximation, which is called Stein's method. Later, Chen (1975) developed and applied Stein's method to the Poisson approximation, which is called the Stein-Chen method. Stein's equation for Poisson distribution with mean $\lambda > 0$, for given *h*, is of the form

$$h(x) - P_\lambda(h) = \lambda f(x+1) - xf(x), \tag{2.1}$$

where $P_\lambda(h) = e^{-\lambda}\sum_{k=0}^{\infty} h(k)\frac{\lambda^k}{k!}$ and f and h are bounded real valued functions defined on $\mathbb{N}\cup\{0\}$. For $A \subseteq \mathbb{N}\cup\{0\}$, let $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 \text{ , if } x \in A, \\ 0 \text{ , if } x \notin A. \end{cases} \tag{2.2}$$

Following Barbour *et al*. (1992), the solution f_A of (2.1) can be expressed as

$$f_A(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[P_\lambda(h_{A\cap C_{x-1}}) - P_\lambda(h_A)P_\lambda(h_{C_{x-1}})] & \text{if } x \ge 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.3}$$

where $x \in \mathbb{N}$ and $C_{x-1} = \{0,...,x-1\}$. Similarly, for $A = C_{x_0}$ and $x_0 \in \mathbb{N}\cup\{0\}$, $f_{C_{x_0}}$ is of the form

$$f_{C_{x_0}}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[P_\lambda(h_{C_{x-1}})P_\lambda(1-h_{C_{x_0}})] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[P_\lambda(h_{C_{x_0}})P_\lambda(1-h_{C_{x-1}})] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.4}$$

Let $\Delta f_A(x) = f_A(x+1) - f_A(x)$ and $\Delta f_{C_{x_0}}(x) = f_{C_{x_0}}(x+1) - f_{C_{x_0}}(x)$, for giving the desired results, we also need the following lemma.

Lemma 2.1 *Let* $x \in \mathbb{N}$, $x_A^* = \min\{x \mid x \in A\}$ and $x_A^\Theta = \max\{x \mid C_x \subseteq A\}$, then we have the following:

1). For Δf_A and $A \subseteq \mathbb{N}\cup\{0\}$,

$$|\Delta f_A(x)| \le \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_A}\right\}, \tag{2.5}$$

where $\frac{1}{x_A}$ is taken to be 1 when $x_A = 0$ and for $x_A > 0$, it is given by

$$\frac{1}{x_A} = \begin{cases} \frac{1}{x_A^\Theta} & if\ \ 0 \in A, \\ \frac{1}{x_A^*-1} & if\ \ 0 \notin A, \end{cases}$$

and

$$|\Delta f_A(x)| \le \tfrac{1}{x}. \tag{2.6}$$

2). For $\Delta f_{C_{x_0}}$ and $x_0 \in \mathbb{N}$,

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_0}, \frac{e^{\lambda}-1}{(x_0+1)\lambda}\right\} \tag{2.7}$$

and

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \tfrac{1}{x}. \tag{2.8}$$

Proof. 1) The inequality (2.5) follows directly from Teerapabolarn (2015b) and inequality (2.6) follows from Barbour *et al.* (1992).

2). For $A = C_{x_0}$, we have $x_A^\Theta = \max\{x \mid C_x \subseteq A\} = x_0$ and $\frac{1}{x_A} = \frac{1}{x_0}$, thus (2.5) becomes

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_0}\right\}. \tag{2.9}$$

Teerapabolarn (2007) showed that

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \frac{e^{\lambda}-1}{(x_0+1)\lambda}. \tag{2.10}$$

Combining the bounds in (2.9) and (2.10), the bound in (2.7) is obtained, and finally, the bound in (2.8) can be obtained from the bound in (2.6).

3. Main Results

The main point of this study is to determine new bounds, non-uniform bounds, for two distances $d_A(S_n, Z_\lambda)$ and $d_{K_{x_0}}(S_n, Z_\lambda)$. The following theorem gives one desired result.

Theorem 3.1 Let $A \subseteq \mathbb{N}\cup\{0\}$ and $\lambda = \sum_{i=1}^{n}\frac{r_i q_i}{p_i}$, then we have the following inequality.

$$d_A(S_n, Z_\lambda) \le \sum_{i=1}^{n}\min\left\{\min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_A}\right\}\frac{r_i q_i}{p_i}, 1-p_i^{r_i}\right\}\frac{q_i}{p_i}. \tag{3.1}$$

Proof. Substituting *x* and *h* by S_n and h_A respectively, and we take expectation to (2.1), yields

$$
\begin{aligned}
d_A(S_n, Z_\lambda) &= \left| E\left[\lambda f(S_n+1) - S_n f(S_n)\right] \right| \\
&= \left| E\left[\sum_{i=1}^{n} \tfrac{r_i q_i}{p_i} f(S_n+1) - \sum_{i=1}^{n} X_i f(S_n) \right] \right| \\
&= \left| \sum_{i=1}^{n} E\left[\tfrac{r_i q_i}{p_i} f(S_n+1) - X_i f(S_n) \right] \right|, \qquad (3.2)
\end{aligned}
$$

where $f = f_A$ is defined in (2.3). For $i = 1, \ldots, n,$ let $S_{n,i} = S_n - X_i$, then we obtain

$$
\begin{aligned}
&E\left[\tfrac{r_i q_i}{p_i} f(S_n+1) - X_i f(S_n) \right] \\
&= E\left[\tfrac{r_i q_i}{p_i} f(S_{n,i}+X_i+1) - X_i f(S_{n,i}+X_i) \right] \\
&= E\left\{ E\left[\left(\tfrac{r_i q_i}{p_i} f(S_{n,i}+X_i+1) - X_i f(S_{n,i}+X_i) \right) \Big| X_i \right] \right\} \\
&= \sum_{x=0}^{\infty} E\left[\left(\tfrac{r_i q_i}{p_i} f(S_{n,i}+X_i+1) - X_i f(S_{n,i}+X_i) \right) \Big| X_i = x \right] p_{X_i}(x) \\
&= E\left[\tfrac{r_i q_i}{p_i} f(S_{n,i}+1) \right] p_{X_i}(0) + E\left[\tfrac{r_i q_i}{p_i} f(S_{n,i}+2) - f(S_{n,i}+1) \right] p_{X_i}(1) \\
&+ E\left[\tfrac{r_i q_i}{p_i} f(S_{n,i}+3) - 2f(S_{n,i}+2) \right] p_{X_i}(2) + E\left[\tfrac{r_i q_i}{p_i} f(S_{n,i}+4) - 3f(S_{n,i}+3) \right] p_{X_i}(3) + \cdots \\
&= r_i q_i p_i^{r_i-1} E\left[f(S_{n,i}+1) \right] + r_i^2 q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] - r_i q_i p_i^{r_i} E\left[f(S_{n,i}+1) \right] \\
&+ \tfrac{r_i^2 (r_i+1) q_i^3 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+3) \right] - r_i (r_i+1) q_i^2 p_i^{r_i} E\left[f(S_{n,i}+2) \right] \\
&+ \tfrac{r_i^2 (r_i+1)(r_i+2) q_i^4 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+4) \right] - \tfrac{r_i (r_i+1)(r_i+2) q_i^3 p_i^{r_i}}{2} E\left[f(S_{n,i}+3) \right] + \cdots \\
&= r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+1) \right] + r_i^2 q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] - r_i q_i^2 p_i^{r_i} E\left[f(S_{n,i}+2) \right] \\
&+ \tfrac{r_i^2 (r_i+1) q_i^4 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+3) \right] - r_i (r_i+1) q_i^3 p_i^{r_i} E\left[f(S_{n,i}+3) \right] \\
&+ \tfrac{r_i^2 (r_i+1)(r_i+2) q_i^5 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+4) \right] - \tfrac{r_i (r_i+1)(r_i+2) q_i^4 p_i^{r_i}}{2} E\left[f(S_{n,i}+4) \right] + \cdots \\
&= r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+1) \right] + r_i^2 q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] - r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] \\
&+ r_i q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] + \tfrac{r_i^2 (r_i+1) q_i^4 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+3) \right] - r_i (r_i+1) q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+3) \right] \\
&+ r_i (r_i+1) q_i^4 p_i^{r_i-1} E\left[f(S_{n,i}+3) \right] + \tfrac{r_i^2 (r_i+1)(r_i+2) q_i^5 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+4) \right] - \tfrac{r_i (r_i+1)(r_i+2) q_i^4 p_i^{r_i-1}}{2} \\
&\times E\left[f(S_{n,i}+4) \right] + \tfrac{r_i (r_i+1)(r_i+2) q_i^5 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+4) \right] + \cdots \qquad (\text{by } -1 + q_i = -p_i) \\
&= r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+1) \right] - r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] \\
&+ r_i (r_i+1) q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+2) \right] - r_i (r_i+1) q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+3) \right] \\
&+ \tfrac{r_i (r_i+1)(r_i+2) q_i^4 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+3) \right] - \tfrac{r_i (r_i+1)(r_i+2) q_i^4 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+4) \right] \\
&+ \tfrac{r_i (r_i+1)(r_i+2)(r_i+3) q_i^5 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+4) \right] - \tfrac{r_i (r_i+1)(r_i+2)(r_i+3) q_i^5 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+5) \right] + \cdots \\
&= r_i q_i^2 p_i^{r_i-1} E\left[f(S_{n,i}+1) - f(S_{n,i}+2) \right] + r_i (r_i+1) q_i^3 p_i^{r_i-1} E\left[f(S_{n,i}+2) - f(S_{n,i}+3) \right] \\
&+ \tfrac{r_i (r_i+1)(r_i+2) q_i^4 p_i^{r_i-1}}{2} E\left[f(S_{n,i}+3) - f(S_{n,i}+4) \right] + \tfrac{r_i (r_i+1)(r_i+2)(r_i+3) q_i^5 p_i^{r_i-1}}{3!} E\left[f(S_{n,i}+4) - f(S_{n,i}+5) \right] + \cdots \\
&= \tfrac{q_i}{p_i} \left\{ \tfrac{\Gamma(r_i+1)}{\Gamma(r_i)1!} q_i p_i^{r_i} E\left[f(S_{n,i}+1) - f(S_{n,i}+2) \right] + 2 \tfrac{\Gamma(r_i+2)}{\Gamma(r_i)2!} q_i^2 p_i^{r_i} E\left[f(S_{n,i}+2) - f(S_{n,i}+3) \right] \right.
\end{aligned}
$$

$$+3\tfrac{\Gamma(r_i+3)}{\Gamma(r_i)3!}q_i^3p_i^{r_i}E\left[f(S_{n,i}+3)-f(S_{n,i}+4)\right]+4\tfrac{\Gamma(r_i+4)}{\Gamma(r_i)4!}q_i^4p_i^{r_i}$$
$$\times E\left[f(S_{n,i}+4)-f(S_{n,i}+5)\right]+\cdots\Big\}.$$
$$=\tfrac{q_i}{p_i}\sum_{x=1}^{\infty}xp_{X_i}(x)E\left[f(S_{n,i}+x)-f(S_{n,i}+x+1)\right] \quad (3.3)$$

Putting the result in (3.3) to (3.2), we have that

$$d_A(S_n,Z_\lambda)=\left|\sum_{i=1}^{n}\tfrac{q_i}{p_i}\sum_{x=1}^{\infty}xp_{X_i}(x)E\left[f(S_{n,i}+x)-f(S_{n,i}+x+1)\right]\right|$$
$$\le\sum_{i=1}^{n}\tfrac{q_i}{p_i}\sum_{x=1}^{\infty}xp_{X_i}(x)E\left|f(S_{n,i}+x+1)-f(S_{n,i}+x)\right|$$
$$=\sum_{i=1}^{n}\tfrac{q_i}{p_i}\sum_{x=1}^{\infty}xp_{X_i}(x)E\left|\Delta f(S_{n,i}+x)\right|$$
$$=\sum_{i=1}^{n}\tfrac{q_i}{p_i}\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\left|\Delta f(j+x)\right|P(S_{n,i}=j)\cdot \quad (3.4)$$

Because, by (2.5),

$$\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\left|\Delta f(j+x)\right|P(S_{n,i}=j)$$
$$\le\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}P(S_{n,i}=j)$$
$$=\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}E(X_i)$$
$$=\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}\tfrac{r_iq_i}{p_i} \quad (3.5)$$

and, by (2.6),

$$\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\left|\Delta f(j+x)\right|P(S_{n,i}=j)$$
$$\le\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\tfrac{1}{j+x}P(S_{n,i}=j)$$
$$\le\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\tfrac{1}{x}P(S_{n,i}=j)$$
$$=\sum_{x=1}^{\infty}p_{X_i}(x)$$
$$=1-p_i^{r_i}, \quad (3.6)$$

thus from (3.5) and (3.6), we obtain

$$\sum_{x=1}^{\infty}xp_{X_i}(x)\sum_{j=0}^{\infty}\left|\Delta f(j+x)\right|P(S_{n,i}=j)$$
$$\le\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}. \quad (3.7)$$

Substituting the bound in (3.7) to (3.4), it follows that

$$d_A(S_n,Z_\lambda)\le\sum_{i=1}^{n}\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i},$$

This gives the Theorem 3.1.

For cumulative probability approximation, it is noted that in the case $x_0=0$, we can compute the exact probability of $S_n=0$, that is, $P(S_n=0)=\prod_{i=1}^{n}p_i^{r_i}$. So, in this case, a new non-uniform bound for $d_{K_{x_0}}(S_n,Z_\lambda)$, when $x_0\in\mathrm{N}$, is as follows.

Theorem 3.2 Let $x_0\in\mathrm{N}$ and $\lambda=\sum_{i=1}^{n}\tfrac{r_iq_i}{p_i}$, then the following inequality holds:

$$d_{K_{x_0}}(S_n,Z_\lambda)\le\sum_{i=1}^{n}\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_0},\tfrac{e^{\lambda}-1}{(x_0+1)\lambda}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i}. \quad (3.8)$$

Proof. Using the same arguments detailed as in the proof of Theorem 3.1 together with Lemma 2.1(2), the result in (3.8) is obtained.

Remark 1) By comparing the bounds in (1.7), (1.8) and (3.1), it is seen that

$$\sum_{i=1}^{n}\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i}\le\tfrac{1-e^{-\lambda}}{\lambda}\sum_{i=1}^{n}\tfrac{r_iq_i^2}{p_i^2} \text{ and}$$

$$\sum_{i=1}^{n}\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_A}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i}\le\sum_{i=1}^{n}\min\left\{\tfrac{1-e^{-\lambda}}{\lambda}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i}$$

and the bound in (3.1) can be applied for all cases of r_i, which is wider than the bound in (1.8). Therefore, the result in (3.1) is better than those presented in (1.7) and (1.8). Similarly, the result in (3.8) is also better than that presented in (1.9).

2) If we combine the results in (3.1) and (1.10), then a new non-uniform bound for $d_{x_0}(S_n,Z_\lambda)$, when $x_0\in\mathrm{N}$, is of the form

$$d_{x_0}(S_n,Z_\lambda)\le\sum_{i=1}^{n}\min\left\{\min\left\{\tfrac{1-e^{-\lambda}}{\lambda},\tfrac{1}{x_0}\right\}\tfrac{r_iq_i}{p_i},1-p_i^{r_i}\right\}\tfrac{q_i}{p_i}. \quad (3.9)$$

It is a slightly improvement of (1.10).

For approximating the distribution of a negative binomial random variable X with parameters $r\in\mathrm{R}^+$ and by

$p \in (0,1)$ a Poisson distribution with mean $\lambda = \frac{rq}{p}$, we can apply the results Theorems 3.1 and 3.2 and (3.9) to give new results as follows.

Corollary 3.1 For $\lambda = \frac{rq}{p}$, then we have the following.

1) For $A \subseteq \mathbb{N} \cup \{0\}$,

$$d_A(X, Z_\lambda) \le \min\left\{\frac{\lambda}{x_A}, 1-p^r\right\}\frac{q}{p} \tag{3.10}$$

2) For $x_0 \in \mathbb{N}$,

$$d_{K_{x_0}}(X, Z_\lambda) \le \min\left\{\frac{\lambda}{x_0}, \frac{e^\lambda - 1}{x_0+1}, 1-p^r\right\}\frac{q}{p} \tag{3.11}$$

and

$$d_{x_0}(X, Z_\lambda) \le \min\left\{\frac{\lambda}{x_0}, 1-p^r\right\}\frac{q}{p} \tag{3.12}$$

Proof. Because all results in (3.10)-(3.12) can be obtained by using similar method it suffices to show the result in (3.10). Applying (3.1), we have

$$d_A(X, Z_\lambda) \le \min\left\{\min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_A}\right\}\frac{rq}{p}, 1-p^r\right\}\frac{q}{p}$$
$$= \min\left\{1-e^{-\lambda}, \frac{\lambda}{x_A}, 1-p^r\right\}\frac{q}{p}. \tag{3.13}$$

Because, by Taylor's expansion,
$e^{\frac{1-p}{p}} = 1 + \frac{1-p}{p} + \frac{[(1-p)/p]^2}{2!} + \cdots = \frac{1}{p} + \frac{[(1-p)/p]^2}{2!} + \cdots > \frac{1}{p}$, we have $p > e^{-\frac{1-p}{p}} = e^{-\frac{q}{p}}$ and $p^r > e^{-\frac{rq}{p}}$, which implies that $1-p^r < 1-e^{-\frac{rq}{p}} = 1-e^{-\lambda}$. Therefore, the inequality (3.13) reduce to

$$d_A(X, Z_\lambda) \le \min\left\{\frac{\lambda}{x_A}, 1-p^r\right\}\frac{q}{p}.$$

From which, the result in (3.10) is proved.

If $r_1 = r_2 = \cdots = r_n = 1$, then $\lambda = \sum_{i=1}^{n} \frac{q_i}{p_i}$ and the results in Theorems 3.1 and 3.2 and (3.9) become to be the results in the Poisson approximation for a sum of independent geometric random variables, which present in the following corollary.

Corollary 3.2 If $r_1 = r_2 = \cdots = r_n = 1$ and $\lambda = \sum_{i=1}^{n} \frac{q_i}{p_i}$, then we have the following.

1) For $A \subseteq \mathbb{N} \cup \{0\}$,

$$d_A(S_n, Z_\lambda) \le \sum_{i=1}^{n} \min\left\{\min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_A}\right\}\frac{1}{p_i}, 1\right\}\frac{q_i^2}{p_i}. \tag{3.14}$$

2) For $x_0 \in \mathbb{N}$,

$$d_{K_{x_0}}(S_n, Z_\lambda) \le \sum_{i=1}^{n} \min\left\{\min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_0}, \frac{e^\lambda - 1}{(x_0+1)\lambda}\right\}\frac{1}{p_i}, 1\right\}\frac{q_i^2}{p_i} \tag{3.15}$$

and

$$d_{x_0}(S_n, Z_\lambda) \le \sum_{i=1}^{n} \min\left\{\min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_0}\right\}\frac{1}{p_i}, 1\right\}\frac{q_i^2}{p_i}. \tag{3.16}$$

4. Conclusions

The new bounds, non-uniform bounds, in this study were obtained by using the Stein-Chen method. Each bound can be used to approximate the error of the distance between the distribution of a sum of independent negative binomial distribution and a Poisson distribution with mean $\lambda = E(S_n) = \sum_{i=1}^{n} \frac{r_i q_i}{p_i}$ as well when all q_i are small.

Furthermore, by comparing the results in this study and the results in Teerapabolarn (2014) and Hung and Giang (2016), it can be concluded that the results in this studyaresuperiorthan those presented in Teerapabolarn (2014) and Hung and Giang (2016).

References

Barbour, A. D., Holst, L., & Janson, S. (1992). *Poisson approximation (Oxford studies in probability 2)*. Oxford, England: Clarendon Press.

Chen, L. H. Y. (1975). Poisson approximation for dependent trials. *Annals of Probability, 3*, 534-545.

Gerber, H. U. (1984). Error bounds for the compound Poisson approximation. *Insurance Mathematics Economics, 3*, 191-194.

Hung, T. L., & Giang, L. T. (2016). On bounds in Poisson approximation for distributions of independent negative binomial distributed random variables. *SpringerPlus, 5*, 1-12.

Karlis, D., & Xekalaki, E. (2005). Mixed Poisson Distributions. *International Statistical Review, 73*, 35-58.

Romanowska, M. (1977). A note on the upper bound for the distance in total variation between the binomial and the Poisson distributions. *Statistica Neerlandica, 31*, 127–130.

Roos, B. (2003). Improvements in the Poisson approximation of mixed Poisson distributions. *Journal of Statistical Planning and Inference, 113,* 467–483.

Stein, C. M. (1972). A bound for the error in normal approximation to the distribution of a sum of dependentrandom variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability 2*, 583-602.

Teerapabolarn, K. (2012). The least upper bound on the Poisson-negative binomial relative error .*Communications in Statistics-Theory and Methods, 41,* 1833-1838.

Teerapabolarn, K. (2014). Poisson approximation for independent negative binomial random variables.*International Journal of Pure and Applied Mathematics, 93,* 779-781.

Teerapabolarn, K. (2015a). Pointwise Poisson approximation for independent negative binomial random variables.*Global Journal of Pure and Applied Mathematics, 11*, 1979-1982.

Teerapabolarn, K. (2015b). New non-uniform bounds on Poisson approximation for dependent Bernoulli Trials. *Bulletin of the Malaysian Mathematical Sciences Society, 38*, 231-248.

Teerapabolarn, K. (2017a). A non-uniform bound on Poisson approximation for a sum of negative binomial random variables. *Songklanakarin Journal of Science and Technology, 39*(3), 355-358.

Teerapabolarn, K. (2017b). Poisson approximation for a sum of negative binomial random variables. *Bulletin of the Malaysian Mathematical Science Society, 40*(2), 931-939.

Vellaisamy, P., & Upadhye, N. S. (2009). Compound negative binomial approximations for sums of random variables. *Probability and Mathematical Statistics, 29*, 205-226.

Vervaat, W. (1969). Upper bound for distance in total variation between the binomial or negative binomial and the Poisson distribution. *Statistica Neerlandica, 23*, 79-86.