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Original Article

On properties of generalized bipolar fuzzy semigroups

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Abstract

In this paper, we introduce a generalization of a bipolar fuzzy subsemigroup, namely a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup. The notions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (right, bi-) ideals are discussed. Some necessary and sufficient conditions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (right, bi-) ideals are obtained. Furthermore, any regular semigroup is characterized in terms of generalized bipolar fuzzy semigroups.

Keywords: generalized bipolar fuzzy semigroup, $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup, fuzzy left (right, bi-) ideal, regular semigroup

1. Introduction

Mostly bipolarity separates positive and negative information. The bipolar information can be seen in many situations, such as in decision making problems, organization problems, economic problems, evaluation, risk management, environmental and social impact assessment etc. Therefore bipolar information affects the effectiveness and efficiency of problem solving. Thus the concept of bipolar fuzzy sets is of high interest in mathematics.

Zadeh (1965) introduced the concept of a fuzzy set that can be applied in many areas including mathematics, statistics, computer science, electrical instruments, industrial operations, business, engineering, social decisions, etc. Fuzzy sets are used to mathematically deal with imprecise and uncertain conditions. Rosenfeld (1971) applied fuzzy sets to fuzzy subgroups. Then, fuzzy sets were used in the theory of semigroups in 1979. Kuroki (1981) initiated fuzzy semigroups based on the notion of fuzzy ideals of semigroups and introduced some properties of fuzzy ideals and fuzzy bi-ideals of semigroups. Davvaz and Khan (2011) studied and characterized regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals. Siripitukdet and Ruanon (2013) introduced a fuzzy interior ideal with threshold $(s,t]$ and gave some interesting properties of fuzzy interior ideals with threshold $(s,t]$. Jun and Song (2017) introduced $(\tilde{\alpha}, \tilde{\beta})$-fuzzy left (right, bi-) ideals of semigroups.

The fundamental concepts of bipolar fuzzy sets were introduced by Zhang (1994). He innovated the bipolar fuzzy set for bipolar fuzzy logic, which has been widely applied to solve many real-world problems. After that, the notions of bipolar fuzziness and interval-based bipolar fuzzy logic have been generalized to real-valued bipolar fuzzy logic by Zhang (1998). Lee (2000) studied the notion of bipolar valued fuzzy sets, which are an extension of fuzzy sets. The concepts of bipolar fuzzy subalgebra and ideals in BCK/BCI-algebras have been presented by Lee (2009). Jun and Park (2009) proposed bipolar fuzzy regularity, regular subalgebra, filter, and closed quasi filter in BCH-algebras. Kim *et al*. (2011) studied the notions of bipolar fuzzy subsemigroups, bipolar fuzzy left (right) ideals and bipolar fuzzy bi-ideals. They provided some necessary and sufficient conditions for a bipolar fuzzy subsemigroup and a bipolar fuzzy left (right, bi-) ideal of semigroups. Yaqoob (2012) investigated some

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properties of bipolar-valued fuzzy left (right, bi-, interior) ideals of LA-subsemigroups. In the same year, the notion of bipolar fuzzy sets in Γ-semigroups was presented by Majumder (2012). Ban *et al*. (2012) proposed bipolar fuzzy ideals with operators in semigroups by using a set Ω and gave the notion of bipolar valued fuzzy subsemigroups by using bipolar Ω-fuzzy subsemigroups. Faisal *et al*. (2012) introduced the concepts of bipolar-valued fuzzification of ordered AG-groupoids and investigated some properties of bipolar-valued fuzzy ideals of intra-regular ordered AG-groupoids. Min and Shenggang (2014) introduced the concept of bipolar fuzzy h-ideals. They obtained relations of bipolar fuzzy h-ideals and h-ideals of hemirings. Yaqoob *et al*. (2016) studied bipolar (λ,δ)-fuzzy sets in Γ-semihypergroups. Faruk *et al.* (2016) studied the group structure on bipolar soft sets and investigated some of its properties.

In this paper, generalizations of bipolar fuzzy semigroups are introduced. Definitions and properties of $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-bipolar fuzzy left (right, bi-) ideals are provided. Some necessary and sufficient conditions of $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-bipolar fuzzy left (right, bi-) ideals are obtained. Finally, we characterize a regular semigroup in terms of generalized bipolar fuzzy semigroups.

2. Preliminaries

In this section, we give definitions that are used in this paper along with examples. By a *subsemigroup* of a semigroup S we mean a non-empty subset A of S such that $A^2 \subseteq A$ and by a *left (right) ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A(AS \subseteq A)$. By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of a semigroup S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. A semigroup S is called *regular* if for all $a \in S$ there exists $x \in S$ such that $a = axa$.

The following theorems are provided in the next section.

Theorem 2.1 (Mordeson, Malik, & Kuroki, 2003) For a semigroup S, the following statements are equivalent.

(i) S is regular.

(ii) $R \cap L = RL$ for every right ideal R and every left ideal L of S.

Definition 2.2 Let X be a set, a fuzzy subset (or fuzzy set) f on X is mapping $f: X \to [0,1]$ where [0,1] is the usual interval of real numbers.

The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy sets on S defined by

$$(f \wedge g)(x) = f(x) \wedge g(x),$$
$$(f \vee g)(x) = f(x) \vee g(x)$$

for all $x \in S$.

A product of two fuzzy sets f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz} \{f(y) \wedge g(z)\}, & \text{if } x = yz \text{ for some } y,z \in S; \\ 0, & \text{if otherwise.} \end{cases}$$

Definition 2.3 (Kim *et al.*, 2011) Let S be a non-empty set. A *bipolar fuzzy set* (*BF*-set) f on S is an object having a form

$$f := \{(x, f_p(x), f_n(x)) : x \in S\},$$

where $f_p: S \to [0,1]$ and $f_n: S \to [-1,0]$.

Definition 2.4 A *BF*-set $f = (S; f_p, f_n)$ and $\alpha \in [0,1], \beta \in [-1,0]$, the sets

$$P(f;\alpha) := \{x \in S | f_p(x) \geq \alpha\}$$

and

$$N(f;\beta) := \{x \in S | f_n(x) \leq \beta\}$$

are called a positive α –cut and negative β-cut of f, respectively. The set $C(f;(\alpha,\beta)) := P(f;\alpha) \cap N(f;\beta)$ is called the bipolar (α,β)-cut of f.

Definition 2.5 (Kim *et al.*, 2011) A *BF*-set $f = (S; f_p, f_n)$ on a semigroup S is called *BF-subsemigroup* on S if it satisfies the following conditions:

1. $f_p(xy) \geq f_p(x) \wedge f_p(y)$,
2. $f_n(xy) \leq f_n(x) \vee f_n(y)$

for all $x, y \in S$.

Definition 2.6 (Kim *et al.*, 2011) A *BF*-set $f = (S; f_p, f_n)$ on a semigroup S is called *BF-left (right) ideal* on S if it satisfies the following conditions:

1. $f_p(xy) \geq f_p(y)$ ($f_p(xy) \geq f_p(x)$,)
2. $f_n(xy) \leq f_n(y)$ ($f_n(xy) \leq f_n(x)$)

for all $x, y \in S$.

Definition 2.7 (Kim *et al.*, 2011) A *BF*-set $f = (S; f_p, f_n)$ on a semigroup S is called *BF-bi-ideal* on S if it is a *BF*-subsemigroup on S that satisfies the following conditions:

1. $f_p(xay) \geq f_p(x) \wedge f_p(y)$,
2. $f_n(xay) \leq f_n(x) \vee f_n(y)$

for all $x, a, y \in S$.

We shall give the generalization on *BF*-subsemigroup which is defined by Kim *et al.* (2011).

Definition 2.8 A *BF*-set $f = (S; f_p, f_n)$ on a semigroup S is called a $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-*BF-subsemigroup* on S where $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$ if it satisfies the following conditions:

1. $f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$,
2. $f_n(xy) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

for all $x, y \in S$.

Note that every *BF*-subsemigroup is a $(0,1,-1,0)$-*BF*-subsemigroup.

The following examples show that $f = (S; f_p, f_n)$ is a $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-*BF*-subsemigroup on S but $f = (S; f_p, f_n)$ is not a *BF*-subsemigroup on S.

Example 2.9 The set $S = \{2, 3, 4, \ldots\}$ is a semigroup under usual multiplication and let $f = (S; f_p, f_n)$ be a *BF*-set on S defined as follows:

$$f_p(x) \coloneqq (2x+1)^{-1} \text{ and } f_n(x) \coloneqq -(2x+1)^{-1}$$

for all $x \in S$.

Let $x, y \in S$. Then

$$f_p(xy) = (2xy+1)^{-1} < (2x+1)^{-1} = f_p(x)$$

and

$$f_p(xy) = (2xy+1)^{-1} < (2y+1)^{-1} = f_p(y),$$

and so $f_p(xy) < f_p(x) \wedge f_p(y)$.

Therefore $f = (S; f_p, f_n)$ is not a *BF*-subsemigroup on S.

Let $\alpha_2 \in [0,1]$, $\beta_1 \in [-1,0], \alpha_1 = \frac{1}{5}$ and $\beta_2 = -\frac{1}{5}$. Then for all $x, y \in S$, $f_p(xy) \vee \frac{1}{5} \geq (2x+1)^{-1} \wedge (2y+1)^{-1} \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$
and

$$f_n(xy) \wedge -\frac{1}{5} \leq -(2x+1)^{-1} \vee -(2y+1)^{-1} \leq f_n(x) \vee f_n(y) \vee \beta_1.$$

Hence $f = (S; f_p, f_n)$ is a $(\frac{1}{5}, \alpha_2, \beta_1, -\frac{1}{5})$-*BF*-subsemigroup on S.
Note that $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup on S for all $\alpha_1 \geq \frac{1}{5}$ and $\beta_2 \leq -\frac{1}{5}$.

Definition 2.10 A *BF*-set $f = (S; f_p, f_n)$ on a semigroup S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-left (right) ideal* on S where $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$ if it satisfies the following conditions:

1. $f_p(xy) \vee \alpha_1 \geq f_p(y) \wedge \alpha_2$ $(f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge \alpha_2)$,
2. $f_n(xy) \wedge \beta_2 \leq f_n(y) \vee \beta_1$ $(f_n(xy) \wedge \beta_2 \leq f_n(x) \vee \beta_1)$

for all $x, y \in S$.

A *BF*- set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-ideal* on S, $(\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0])$, if it is both a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left ideal and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*- right ideal on S.

Definition 2.11 A $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup $f = (S; f_p, f_n)$ on a semigroup S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-bi-ideal* on S that satisfies the following conditions:

1. $f_p(xay) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$,
2. $f_n(xay) \wedge \beta_2 \leq f_n(x) \wedge f_n(y) \vee \beta_1$

for all $x, a, y \in S$.

Example 2.12 Let $S = \{a, b, c, d\}$ be a semigroup having this Cayley table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a *BF*-set $f = (S; f_p, f_n)$ on S as follows:

$\circ$	a	b	c	d
f_p	0.5	0.6	0.8	0.9
f_n	-0.6	-0.4	-0.3	-0.3

It is easy to show that $f = (S; f_p, f_n)$ is a $(0.6, 0.7; -0.4, -0.3)$-*BF*-bi-ideal on S. But we see that f is not a $(0.6, 0.7; -0.4, -0.3)$-*BF*-ideal on S since $f_p(cd) \vee 0.6 = f_p(a) \vee 0.6 = 0.5 \vee 0.6 = 0.6 < 0.7 = f_p(c) \wedge 0.7$; and f is not a *BF*-bi-ideal on S since $f_p(bab) = f_p(a) = 0.5 < 0.6 = f_p(b)$.

Remark 2.7 (i) Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-ideal of a semigroup S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup of S.
(ii) Every *BF*-left (right, bi-) ideal is $(0,1,-1,0)$-*BF*-left (right, bi-) ideal.
(iii) Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-bi-ideal of a semi-group S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup of S.

3. Regular Semigroups

In this section, we introduce a product of *BF*-sets and characterize a regular semigroup by generalized bipolar fuzzy subsemigroups.

Let $f = (S: f_p, f_n)$ and $g = (S: g_p, g_n)$ be two *BF*-sets on a semigroup S and $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$. Define two fuzzy sets $f_p^{(\alpha_1, \alpha_2)}$ and $f_n^{(\beta_1, \beta_2)}$ on S as follows:

$$f_p^{(\alpha_1, \alpha_2)}(x) = \big(f_p(x) \wedge \alpha_1\big) \vee \alpha_2,$$
$$f_n^{(\beta_1, \beta_2)}(x) = (f_n(x) \vee \beta_2) \wedge \beta_1$$

for all $x \in S$.

Define two operations $\overset{(\alpha_1, \alpha_2)}{\wedge}$ and $\underset{(\beta_1, \beta_2)}{\vee}$ on S as follows:

$$\left(f_p \overset{(\alpha_1, \alpha_2)}{\wedge} g_p\right)(x) = \Big(\big(f_p \wedge g_p\big)(x) \wedge \alpha_1\Big) \vee \alpha_2,$$
$$\left(f_n \underset{(\beta_1, \beta_2)}{\vee} g_n\right)(x) = \big((f_n \vee g_n)(x) \vee \beta_2\big) \wedge \beta_1$$

for all $x \in S$, and define products $f_p \overset{(\alpha_1, \alpha_2)}{\circ} g_p$ and $f_n \underset{(\beta_1, \beta_2)}{\circ} g_n$ as follows: For all $x \in S$,

$$\left(f_p \overset{(\alpha_1, \alpha_2)}{\circ} g_p\right)(x) = \Big(\big(f_p \bar{\circ} g_p\big)(x) \wedge \alpha_1\Big) \vee \alpha_2,$$
$$\left(f_n \underset{(\beta_1, \beta_2)}{\circ} g_n\right)(x) = \Big(\big(f_n \underline{\circ} g_n\big)(x) \vee \beta_2\Big) \wedge \beta_1,$$

Where

$$\big(f_p \bar{\circ} g_p\big)(x) = \begin{cases} \bigvee\limits_{x=yz} \{f_p(y) \wedge g_p(z)\} & \text{if } \ x = yz \ \text{ for some } \ y, z \in S; \\ 0 & \text{if } \ \text{otherwise}, \end{cases}$$
$$\big(f_n \underline{\circ} g_n\big)(x) = \begin{cases} \bigwedge\limits_{y=yz} \{f_n(y) \vee g_n(z)\} & \text{if } \ x = yz \ \text{ for some } \ y, z \in S; \\ 0 & \text{if } \ \text{otherwise}. \end{cases}$$

Set

$$f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g := (S; f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p, f_n \underset{(\beta_1,\beta_2)}{\circ} g_n).$$

Then it is a *BF*-set.

Note that
(i) $f_p^{(1,0)}(x) = f_p(x)$,
(ii) $f_n^{(0,-1)}(x) = f_n(x)$,
(iii) $f = (S; f_p, f_n) = (S; f_p^{(1,0)}, f_n^{(0,-1)})$,
(iv) $(f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g)_p = f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p$ and $(f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g)_n = f_n \underset{(\beta_1,\beta_2)}{\circ} g_n$.

Theorem 3.1 Let $f = (S; f_p, f_n)$ be a *BF*-set on a semigroup S. Then f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup on S if and only if $f_p \overset{(\alpha_2, \alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2, \alpha_1)}$ and $f_n \underset{(\beta_2, \beta_1)}{\circ} f_n \geq f_n^{(\beta_2, \beta_1)}$.

Proof. ($\Rightarrow$) It is true for the trivial case. Consider, for $x \in S$,

$$\left(f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p\right)(x) = \left((f_p \bar{\circ} f_p)(x) \wedge \alpha_2\right) \vee \alpha_1$$
$$= \left(\bigvee_{x=yz} \{f_p(y) \wedge f_p(z)\} \wedge \alpha_2\right) \vee \alpha_1$$
$$= \left(\bigvee_{x=yz} \{f_p(y) \wedge f_p(z) \wedge \alpha_2\} \wedge \alpha_2\right) \vee \alpha_1$$
$$\leq \left(\bigvee_{x=yz} \{f_p(yz) \vee \alpha_1\} \wedge \alpha_2\right) \vee \alpha_1$$
$$= \left(\bigvee_{x=yz} \{f_p(x) \vee \alpha_1\} \wedge \alpha_2\right) \vee \alpha_1$$
$$= \left((f_p(x) \vee \alpha_1) \wedge \alpha_2\right) \vee \alpha_1$$
$$= (f_p(x) \wedge \alpha_2) \vee \alpha_1$$
$$= f_p^{(\alpha_2,\ \alpha_1)}(x).$$

Hence, $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\ \alpha_1)}$.

Similarly, we can show that $f_n \underset{(\beta_2,\ \beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\ \beta_1)}$.

($\Leftarrow$) Conversely, let $x, y \in S$. Then

$$f_p(xy) \vee \alpha_1 \geq \left(f_p(xy) \wedge \alpha_2\right) \vee \alpha_1$$
$$\geq \left(f_p(xy) \wedge \alpha_2\right) \vee \alpha_1$$
$$= f_p^{(\alpha_2, \alpha_1)}(xy)$$
$$\geq \left(f_p \overset{(\alpha_2, \alpha_1)}{\circ} f_p\right)(xy)$$
$$= \left((f_p \bar{\circ} f_p)(xy) \wedge \alpha_2\right) \vee \alpha_1$$
$$= \left(\bigvee_{xy=ab} \{f_p(a) \wedge f_p(b)\} \wedge \alpha_2\right) \vee \alpha_1$$
$$\geq \left(\left(f_p(x) \wedge f_p(y)\right) \wedge \alpha_2\right) \vee \alpha_1$$
$$\geq f_p(x) \wedge f_p(y) \wedge \alpha_2.$$

Similarly, we can show that $f_n(xy) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$ for all $x, y \in S$. Therefore f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-subsemigroup on S for all $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$.

Let S be a semigroup and $\emptyset \neq I \subseteq S$. A positive characteristic function and a negative characteristic function are respectively defined by

$$C_I^p: S \to [0,1], x \mapsto C_I^p(x) := \begin{cases} 1, & x \in I; \\ 0, & x \notin I, \end{cases}$$

and

$$C_I^n: S \to [-1,0], x \mapsto C_I^n(x) := \begin{cases} -1, & x \in I; \\ 0, & x \notin I. \end{cases}$$

Remark 3.2

(i) For the sake of simplicity, we shall use the symbol $C_I = (S; C_I^p, C_I^n)$ for the *BF*- set. That is, $C_I = (S; C_I^p, C_I^n) = (S; (C_I)_p, (C_I)_n)$. We call a *bipolar characteristic function*.

(ii) If $I = S$, then $C_S = \left(S; C_S^p, C_S^n\right)$. In this case, we shall denote $\mathcal{S} = \left(S, \mathcal{S}_p, \mathcal{S}_n\right)$. Hence $\mathcal{S}_p(x) = 1$ and $\mathcal{S}_n(x) = -1$ for all $x \in S$.

In the following theorem, some necessary and sufficient conditions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (right) ideals are obtained.

Theorem 3.3 Let $f = (S; f_p, f_n)$ be a *BF*-set on a semigroup *S*. Then the following statements are equivalent.
(i) f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left (right) ideal on S.
(ii) $\mathcal{S}_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\ \alpha_1)} (f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} \mathcal{S}_p \leq f_p^{(\alpha_2,\ \alpha_1)})$ and $\mathcal{S}_n \underset{(\beta_2,\ \beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\ \beta_1)} (f_n \underset{(\beta_2,\ \beta_1)}{\circ} \mathcal{S}_n \geq f_n^{(\beta_2,\ \beta_1)})$.

Proof. ($\Rightarrow$) It is true for the trivial case. Consider, for $x \in S$,

$$
\begin{aligned}
(\mathcal{S}_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p)(x) &= ((\mathcal{S}_p \,\bar{\circ}\, f_p)(x) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{\mathcal{S}_p(y) \wedge f_p(z)\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{1 \wedge f_p(z) \wedge \alpha_2\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{f_p(z) \wedge \alpha_2\} \wedge \alpha_2) \vee \alpha_1 \\
&\leq (\bigvee_{x=yz} \{f_p(yz) \vee \alpha_1\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{f_p(x) \vee \alpha_1\} \wedge \alpha_2\}) \vee \alpha_1 \\
&= ((f_p(x) \vee \alpha_1) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(x) \wedge \alpha_2) \vee \alpha_1 \\
&= f_p^{(\alpha_2,\ \alpha_1)}(x).
\end{aligned}
$$

Hence, $\mathcal{S}_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\ \alpha_1)}$.
Similarly, we can show that $\mathcal{S}_n \underset{(\beta_2,\ \beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\ \beta_1)}$.
($\Leftarrow$) Conversely, let $x, y \in S$. Then

$$
\begin{aligned}
f_p(xy) \vee \alpha_1 &\geq \left(f_p(xy) \wedge \alpha_2\right) \vee \alpha_1 \\
&= f_p^{(\alpha_2,\ \alpha_1)}(xy) \\
&\geq \left(\mathcal{S}_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p\right)(xy) \\
&= \left((\mathcal{S}_p \,\bar{\circ}\, f_p)(xy) \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(\bigvee_{xy=ab} \left\{\mathcal{S}_p(a) \wedge f_p(b)\right\} \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(\bigvee_{xy=ab} \left\{1 \wedge f_p(b)\right\} \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(\bigvee_{xy=ab} \left\{f_p(b)\right\} \wedge \alpha_2\right) \vee \alpha_1 \\
&\geq \left(f_p(y) \wedge \alpha_2\right) \vee \alpha_1 \\
&\geq f_p(y) \wedge \alpha_2.
\end{aligned}
$$

Similarly, we can show that $f_n(xy) \wedge \beta_2 \leq f_n(y) \vee \beta_1$ for all $x, y \in S$. Therefore f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left ideal on S for all $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$.

In the following theorem, some necessary and sufficient conditions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy bi-ideals are obtained.

Theorem 3.4 Let $f = (S; f_p, f_n)$ be a *BF*-set on a semigroup *S*. Then the following statements are equivalent.
(i) f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-bi-ideal on a semigroup S.
(ii) $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\ \alpha_1)}$ and $f_n \underset{(\beta_2,\ \beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\ \beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\ \beta_1)}$.

Proof. The proof is similar to the proof of Theorem 3.3.
The following theorem will be used in Theorem 3.5.

Theorem 3.4. Let S be a semigroup. Then I is a left (right, bi-) ideal of S if and only if the bipolar characteristic function $C_I = (S; C_I^n, C_I^p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left (right, bi-) ideal on *S* for all $\alpha_1, \alpha_2 \in [-1,0], \beta_1, \beta_2 \in [0,1]$.

Proof. The proof is straightforward.

Theorem 3.5 If $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left ideal and $g = (S; g_p, g_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-right ideal on a semigroup *S*, then $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\ \beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\ \beta_1)}{\vee} g_n$.

Proof. ($\Rightarrow$)We omit the trivial case. Consider,

$$
\begin{aligned}
&(f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p)(x) \\
&= ((f_p \,\bar{\circ}\, g_p)(x) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{f_p(y) \wedge g_p(z)\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{f_p(y) \wedge g_p(z) \wedge \alpha_2 \wedge \alpha_2\}) \wedge \alpha_2 \vee \alpha_1 \\
&\leq (\bigvee_{x=yz} \{(f_p(yz) \vee \alpha_1) \wedge (g_p(yz) \vee \alpha_1)\} \wedge \alpha_2) \vee \alpha_1 \\
&= \ (\bigvee_{x=yz} \{(f_p(yz) \wedge g_p(yz)) \vee \alpha_1\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{x=yz} \{(f_p(x) \wedge g_p(x)) \vee \alpha_1\} \wedge \alpha_2) \vee \alpha_1 \\
&\leq ((f_p(x) \wedge g_p(x) \vee \alpha_1) \wedge \alpha_2) \vee \alpha_1 \\
&= (((f_p \wedge g_p)(x) \vee \alpha_1) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p \vee g_p)(x) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p)(x)
\end{aligned}
$$

for all $x \in S$.

Hence, $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} f_p \leq f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p$. Similarly, we can show that $f_n \underset{(\beta_2,\ \beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\ \beta_1)}{\vee} g_n$.

The following lemma is clearly true.

Lemma 3.6 Let A and B be non-empty subsets of a semigroup S. Then the following holds.
(i) $(C_A)_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} (C_B)_p = (C_{A \cap B})_p^{(\alpha_2,\ \alpha_1)}$.
(ii) $(C_A)_n \overset{(\beta_2,\ \beta_1)}{\vee} (C_B)_n = (C_{A \cup B})_p^{(\beta_2,\ \beta_1)}$.
(iii) $(C_A)_p \overset{(\alpha_2,\ \alpha_1)}{\circ} (C_B)_p = (C_{AB})_p^{(\alpha_2,\ \alpha_1)}$.
(iv) $(C_A)_n \overset{(\beta_2,\ \beta_1)}{\circ} (C_B)_n = (C_{AB})_p^{(\beta_2,\ \beta_1)}$.

Next, we characterize a regular semigroup by generalizations of *BF*-subsemigroups.

Theorem 3.7 For a semigroup S, the following conditions are equivalent.
(i) S is regular,
(ii) $f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\ \beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\ \beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-right ideal $f = (S; f_p, f_n)$ on S and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left ideal $g = (S; g_p, g_n)$ on S.

Proof. ($\Rightarrow$) Let f be $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-right ideal f and let g be $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF*-left ideal on S. Then by Theorem 3.5, we have $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\ \beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\ \beta_1)}{\vee} g_n$. Let $a \in S$. Then there exists $x \in S$ such that $a = axa$. This is true for the trivial case. Consider,

$$\begin{aligned}
\left(f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p\right)(a) &= \left((f_p \bar{\circ} g_p)(a) \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(\bigvee_{a=axa} \{f_p(ax) \wedge g_p(a)\} \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(\bigvee_{a=axa} \{f_p(ax) \wedge g_p(a) \wedge \alpha_2\}\right) \vee \alpha_1 \\
&= \bigvee_{a=axa} \{(f_p(ax) \vee \alpha_1) \wedge (g_p(a) \wedge \alpha_2) \vee \alpha_1\} \\
&= \bigvee_{a=axa} \{(f_p(ax) \vee \alpha_1 \vee \alpha_1) \wedge (g_p(a) \wedge \alpha_2) \vee \alpha_1\} \\
&\geq \bigvee_{a=axa} \left\{\left((f_p(a) \wedge \alpha_2) \vee \alpha_1\right) \wedge (g_p(a) \wedge \alpha_2) \vee \alpha_1\right\} \\
&= \bigvee_{a=axa} \{f_p(a) \wedge g_p(a) \wedge \alpha_2\} \vee \alpha_1 \\
&\geq (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= \left((f_p \wedge g_p)(a) \wedge \alpha_2\right) \vee \alpha_1 \\
&= \left(f_p \overset{(\alpha_1, \alpha_2)}{\wedge} g_p\right)(a).
\end{aligned}$$

Thus, $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p$. Hence, $f_p \overset{(\alpha_2,\ \alpha_1)}{\circ} g_p = f_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} g_p$.

Similarly, we can show that $f_n \underset{(\beta_2,\ \beta_1)}{\circ} g_n = f_n \underset{(\beta_2,\ \beta_1)}{\vee} g_n$.

Conversely, let R be a right ideal and let L be a left ideal of S. Then, by Theorem 3.4 and Lemma 3.6, we have

$$\begin{aligned}
(C_{RL})_p^{(\alpha_2,\ \alpha_1)} &= (C_R)_p \overset{(\alpha_2,\ \alpha_1)}{\circ} (C_L)_p \\
&= (C_R)_p \overset{(\alpha_2,\ \alpha_1)}{\wedge} (C_L)_p \\
&= (C_{R\cap L})_p^{(\alpha_2,\ \alpha_1)}.
\end{aligned}$$

Thus, $R \cap L = RL$. Therefor it follows from Theorem 2.1 that S is regular.

4. Conclusions

This work presented generalizations of bipolar fuzzy sets. We have introduced the concept of generalized bipolar fuzzy sets and applied it to semigroup theory. A bipolar fuzzy set has both positive and negative components. It has been frequently applied to real-world problems in economics, environmental issues, evaluations, risk management, decision making problems, etc. Therefore, we established generalized bipolar fuzzy sets on semigroups, which provides algebraic structure to relevant problems.

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References

Ban, H. Y., Kim, M. J., & Park, Y. J. (2012). Bipolar fuzzy ideals with operators in semigroups. *Annals of Fuzzy Mathematics and Informatics*, *4*(2), 253-265.

Davvaz, B., & Khan, A. (2011). Characterizations of regular ordered semigroups in terms of (α,β)-fuzzy generalized bi-ideals. *Information Sciences*, *181*, 1759-1770.

Faruk, K., Imtiaz, A., & Aman, U. (2012). Some results in bipolar-valued fuzzy ordered AG-groupoids. *Discussions Mathematicae - General Algebra and Applications*, *32*, 55-76.

Faruk, K., Imtiaz, A., & Aman, U. (2016). Bipolar soft group. *Journal of Intelligent & Fuzzy Systems*, *31*(1), 651-662.

Feng, K., & Corsini, P. (2012). (λ,μ)-fuzzy ideals of ordered semigroups. *Annals of Fuzzy Mathematics and Informatics*, *4*(1), 123-129.

Feng, Y., Duan, H., & Zeng, Q. (2010). (λ,μ)-Fuzzy sublattices and (λ,μ)-fuzzy subhyperlattices. *Fuzzy Information and Engineering*, *78*, 17-26.

Jun, Y. B., & Park, C. H. (2009). Filters of BCH-algebras based on bipolar-valued fuzzy sets. *International Mathematical Forum*, *4*, 631-643.

Jun, Y. B., & Kavikumar, J. (2011). Bipolar fuzzy finite state machines. *Bulletin of the Malaysian Mathematical Sciences Society*, *34*, 181-188.

Jun, Y. B., & Song, S. Z. (2017). Generalized fuzzy ideals in semigroups. *Thai Journal of Mathematics*, *5*(1), 227 –236.

Kim, C. S., Kang, J. G., & Kang, J. M. (2011). Ideal theory of semigroups based on the bipolar valued fuzzy set theory. *Annals of Fuzzy Mathematics and Informatics*, *2*(2), 193-206.

Kuroki, N. (1979). Fuzzy bi-ideal in semigroups. *Commentarii mathematici Universitatis Sancti Pauli, 27*, 17-21.

Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroup. *Fuzzy Sets and Systems*, *5*(2), 203-215.

Lee, K. J. (2009). Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. *Bulletin of the Malaysian Mathematical Sciences Society*, *32*, 361-373.

Lee, K. M. (2000). Bipolar-valued fuzzy sets and their operations. *Proceeding International Conference on Intelligent Technologies, Bangkok, Thailand*, 307-312.

Majumder, S. K. (2012). Bipolar valued fuzzy sets in Γ –semigroups. *Mathematica Aeterna*, *2*(30), 203–213.

Min, Z., & Shenggang, L. (2014). Applications of Bipolar Fuzzy Theory to Hemirings. *International Journal of Innovative Computing, Information and Control*, *10*(2), 767-781.

Mordeson, J. N., Malik, D. S., & Kuroki, N. (2003). Fuzzy Congruences on Semigroups. *Fuzzy Semigroups, 2003*, 131-168.

Rosenfeld, A. (1971). Fuzzy group. *Journal of Mathematical Analysis and Application*, *35*(3), 512-517.

Siripitukdet, M., & Ruanon, A. (2013). Fuzzy interior ideals with threshold (*s*,*t*]. *Thai Journal of Mathematics, 11*(2), 371-382.

Yaqoob, N. (2012). Bipolar-valued fuzzy ideals in LA-semigroups. *Journal of Advanced Studies in Topology, 3*(1), 60-71.

Yaqoob, N., Aslam, M., Rehman, I., & Khalaf, M. M. (2016). New types of bipolar fuzzy sets in Γ-semihypergroups. *Songklanakarin Journal of Science and Technology*, *38*(2), 119-127.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338-353.

Zhang, W. R. (1994). Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference,* 305-309.

Zhang, W. R. (1998). Bipolar fuzzy sets. *1998 IEEE International Conference on Fuzzy Systems Proceedings*, 835-840.