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Review Article

On fuzzy quasi-prime ideals in near left almost rings

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Abstract

In this investigation we studied fuzzy quasi-prime, weakly fuzzy quasi-prime, fuzzy completely prime and weakly fuzzy completely prime ideals in nLA-rings. Some characterizations of fuzzy quasi-prime and weakly fuzzy quasi-prime ideals were obtained. Moreover, we investigated relationships between fuzzy completely prime (weakly fuzzy completely prime) and fuzzy quasi-prime (weakly fuzzy quasi-prime) ideals in nLA-rings.

Keywords: nLA-ring, fuzzy quasi-prime, fuzzy completely prime, weakly fuzzy quasi-prime, weakly fuzzy completely prime

1. Introduction

Let N be a non-empty set. A fuzzy subset of N is, by definition, an arbitrary mapping $f:N\to[0,1]$, where $[0,1]$ is the usual interval of real numbers. In 1965, Zadeh (Zadeh, 1965) introduced the concept of fuzzy subset. Abou-Zaid (Abou Zaid, 1991) introduced the notion of a fuzzy subnear-ring, and studied fuzzy left (right) ideals of a near-ring, and gave some properties of fuzzy prime ideals of a near-ring. Birkenmeier and Heatherly (Birkenmeier & Heatherly, 1990) showed that 3-prime (3-semiprime) ideals in an LSD or RSD near-ring are also completely prime (completely semi-prime). In (Birkenmeier & Heatherly, 1989), these authors proved that 3-prime ideals in a medial near-ring are also completely prime.

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Yusuf in (Yusuf, 2006) introduced the concept of a left almost ring (LA-ring). A non-empty set R with two binary operations "$+$" and "$\cdot$" is called a left almost ring, if $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup and distributing "$\cdot$" over "$+$" holds. Further in (Shah, Ali, & Rehman, 2011a) Shah and Rehman generalize the notion of a commutative semigroup rings into LA-rings, and also generalized the notion of an LA-ring into a near left almost ring. A near left almost ring N is a LA-group under "$+$", an LA-semigroup under "$\cdot$" and has left distributive property of "$\cdot$" over "$+$". Shah *et al*. (2011b) asserted that a commutative ring $(R,+,\cdot)$ we can always generate an LA-ring $(R,\oplus,\cdot)$ by de-fining, for $a,b\in R, a\oplus b=b-a$ and $a\cdot b$ as in the ring. In 2017, the authors introduced the concept of fuzzy subsets in nLA-rings.

In this study we followed the lines adopted in (Abou Zaid, 1991; Shah *et al.*, 2011) and established the notion of fuzzy subsets of nLA-rings. Specifically, we characterize the fuzzy quasi-prime, fuzzy completely prime and weakly fuzzy quasi-prime ideals in nLA-rings. Moreover, we investigated relationships between fuzzy completely prime and weakly fuzzy quasi-prime ideals in nLA-rings.

2. Preliminaries

This section lists some basic definitions and concepts of nLA-rings that are also used with fuzzy subsets in this article. They are as follows:

Definition 2.1 (Kazim & Naseeruddin, 1978) A groupoid S is called a left almost semigroup (simply an LA-semigroup) if it satisfies the left invertive law: $(ab)c=(cb)a,$ for all $a,b,c\in S.$

Example 2.2 (Mushtaq & Yusuf, 1978) Define a mapping $Z\times Z\to Z$ by $a\cdot b=b-a,$ for all $a,b\in Z$ where " — " is a usual subtraction of integers. Then $(Z,\cdot)$ is an LA-semigroup.

Definition 2.3 (Mushtaq & Kamran, 1996) An LA-semigroup $(G,+)$ is called a left almost group (simply an LA-group), if there exists left identity $0\in G$ (that is $0+a=a,$ for all $a\in G$), and for all $a\in G$ there exists $-a\in G$ such that $a+(-a)=0=-a+a.$

Definition 2.4 (Shah *et al.*, 2011b) Let $(N,+)$ be an LA-group. Then N is said to be a near left almost ring (or simply an nLA-ring), if there exists a mapping $N\times N\mapsto N$ (the image of (x,y) is denoted by xy) satisfying the following conditions;

1. $x(y+z)=xy+xz;$
2. $(xy)z=(zy)x,$ for all $x,y,z\in N.$

Example 2.5 (Shah *et al.*, 2011b) Let $(F,+,\cdot)$ be a field. Then $(F,\oplus,\cdot)$ is an nLA-ring on defining the binary operations as: for $x,y\in F, x\oplus y=y-x$ and

$$x\cdot y=\begin{cases}0 & ;x=0 \text{ or } y=0\\ yx^{-1} & ;\text{otherwise.}\end{cases}$$

Let N be an nLA-ring. If S is a non-empty subset of N and S is itself an nLA-ring under the binary operation induced by $N,$ then S is called an nLA-subring of N (Shah *et al.*, 2011b). An LA-subring I of N is called a left ideal of N if $NI\subseteq I$ and I is called a right ideal of N if for all $m,n\in N$ and $i\in I$ such that $(i+m)n-mn\in I$ and I is called an ideal of N if I is both a left and a right ideal of N (Shah *et al.*, 2011b). A left ideal P of an nLA-ring N is said to be quasi-prime ideal of N if and only if $AB\subseteq P$ implies either $A\subseteq P$ or $B\subseteq P$ for any left ideals A,B of N and left ideal P is called weakly quasi-prime if $\{0\}\neq AB\subseteq P$ implies either $A\subseteq P$ or $B\subseteq P$ for any left ideals A,B of N. A left ideal P is called completely quasi-prime if $a,b\in N, ab\in P$ implies either $a\in P$ or $b\in P$ and left ideal P is called weakly completely quasi-prime if $a,b\in N, 0\neq ab\in P$ implies either $a\in P$ or $b\in P.$ It can be easily seen that a completely quasi-prime (weakly completely quasi-prime) ideal of an nLA-ring with left identity N is quasi-prime (weakly quasi-prime).

A function f from N to the unit interval $[0,1]$ is a fuzzy subset of N (Zadeh, 1965). A nLA-ring N itself is a fuzzy subset of N such that $N(x)=1,$ for all $x\in N,$ denoted also by $N.$ Let f and g be two fuzzy subsets of N (Yairayong, 2017). Then the inclusion relation $f\subseteq g$ is defined $f(x)\leq g(x),$ for all $x\in N.$ $f\cap g$ and $f\cup g$ are fuzzy subsets of N defined by

$$(f\cap g)(x)=min\{f(x),g(x)\},(f\cup g)(x)=max\{f(x),g(x)\}$$

for all $x\in N.$ More generally, if $\{f_\alpha:\alpha\in\beta\}$ is a family of fuzzy subsets of $N,$ then $\bigcap_{\alpha\in\beta} f_\alpha$ and $\bigcup_{\alpha\in\beta} f_\alpha$ are defined as follows:

$$\left(\bigcap_{\alpha\in\beta} f_\alpha\right)(x)=\bigcap_{\alpha\in\beta} f_\alpha(x)=inf\{f_\alpha(x):\alpha\in\beta\},$$

$$\left(\bigcup_{\alpha\in\beta} f_\alpha\right)(x)=\bigcup_{\alpha\in\beta} f_\alpha(x)=sup\{f_\alpha(x):\alpha\in\beta\}$$

and will be the intersection and union of the family $\{f_\alpha : \alpha \in \beta\}$ of fuzzy subset of N (Yairayong, 2017). The product $f \circ g$ (Yairayong, 2017) is defined as follows;

$$(f \circ g)(x) = \begin{cases} \bigcup_{x=yz} min\{f(y), g(z)\} & ; \exists y, z \in N, \text{such that } x = yz \\ 0 & ; \text{otherwise} \end{cases}$$

A fuzzy subset f of N is called a fuzzy nLA-subring of N if $f(x-y) \geq min\{f(x), f(y)\}$ and $f(xy) \geq min\{f(x), f(y)\}$, for all $x, y \in N$ (Yairayong, 2017). A fuzzy nLA-subring f of an nLA-ring N is called a fuzzy left ideal of N if $f(xy) \geq f(y)$ for all $x, y \in N$. A fuzzy right ideal of N is a fuzzy nLA-subring f of N such that $f((x+y)z - yz) \geq f(x)$, for all $x, y, z \in N$. A fuzzy ideal of N is a fuzzy nLA-subring f of N such that $f(xy) \geq f(y)$ and $f((x+y)z - yz) \geq f(x)$, for all $x, y, z \in N$ (Yairayong, 2017).

Lemma 2.6 (Yairayong, 2017) Let N be an nLA-ring. If f, g, h are fuzzy subsets of N, then $(f \circ g) \circ h = (h \circ g) \circ f$.

Lemma 2.7 (Yairayong, 2017) For any fuzzy subsets f, g, h and k of an nLA-ring with left identity N, the following statements are true.

1. $f \circ (g \circ h) = g \circ (f \circ h)$.
2. $(f \circ g) \circ (h \circ k) = (k \circ h) \circ (g \circ f)$.
3. $(f \circ g) \circ (h \circ k) = (f \circ h) \circ (g \circ k)$.
4. $N \circ N = N$.

Lemma 2.8 (Yairayong, 2017) Let f be a fuzzy subset of an nLA-ring N. Then the following properties hold.

1. $f(0) \geq f(x)$ for all $x \in N$.
2. f is a fuzzy nLA-subring of N if and only if $f \circ f \subseteq f$ and $f(x-y) \geq min\{f(x), f(y)\}$ for all $x, y \in N$.
3. f is a fuzzy left ideal of N if and only if $N \circ f \subseteq f$.

Lemma 2.9 (Yairayong, 2017) Let I be a non empty subset of an nLA-ring $N, t \in (0,1]$ and let tf_I be a fuzzy set of N such that

$$tf_I(x) = \begin{cases} t & ; x \in I \\ 0 & ; \text{otherwise.} \end{cases}$$

Then the following properties hold.

1. I is an nLA-subring of N if and only if tf_I is a fuzzy nLA-subring of N.
2. I is a left ideal (right ideal, ideal) of N if and only if tf_I is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of N.

Definition 2.10 Let N be an nLA-ring, $x \in N$ and $t \in (0,1]$. A fuzzy point x_t of N is defined by the rule that

$$x_t(y) = \begin{cases} t & ; x = y \\ 0 & ; \text{otherwise.} \end{cases}$$

It is accepted that x_t is a mapping from N into $[0,1]$ then a fuzzy point of N is a fuzzy subset of N. For any fuzzy subset f of N, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel. Clearly, $tf_x = x_t$.

Lemma 2.11 Let A, B be any non empty subset of an nLA-ring N. Then for any $t \in (0,1]$ the following statements are true.

1. $tf_A \circ tf_B = tf_{AB}$.
2. $tf_A = \bigcup_{a \in A} a_t$.

Proof. Straightforward.

3. Fuzzy quasi-prime ideals of nLA-rings

The following theorems seem to play important roles in the study fuzzy completely prime and fuzzy quasi-prime ideals in nLA-rings; they will be used frequently, and normally we shall make no reference to these definitions.

Definition 3.1 A fuzzy subset f of an nLA-ring of N is called fuzzy completely prime if $max\{f(x), f(y)\} \geq f(x-y)$ and $max\{f(x), f(y)\} \geq f(xy)$, where $x, y \in N$.

Example 3.2 Let $N = \{0, e, a\}$ be a set with two binary operations as follows:

$+$	0	e	a
0	0	e	a
e	a	0	e
a	e	a	0

$\cdot$	0	e	a
0	0	0	0
e	0	e	a
a	0	a	e

By Definition 2.4, N is an nLA-ring. We define a fuzzy subset $f : N \longrightarrow [0,1]$ by $f(x) = 0$, for all $x \in N$. By Definition 3.1, f is a fuzzy completely prime ideal of N.

Theorem 3.3 Let N be an nLA-ring. Then f is a fuzzy nLA-subring of N if and only if $1-f$ is fuzzy completely prime of N.

Proof. $(\Rightarrow)$ Clearly, $f(xy) \geq min\{f(x), f(y)\}$ and $f(x-y) \geq min\{f(x), f(y)\}$, for all $x, y \in N$. Then $1-f(xy) \leq 1-min\{f(x), f(y)\}$ and $1-f(x-y) \leq 1-min\{f(x), f(y)\}$, for all $x, y \in N$. If $f(x) \leq f(y)$, then $1-f(x) \geq 1-f(y)$. Consider

$$max\{1-f(x),1-f(y)\} = 1-f(x) \geq 1-f(xy)$$

and

$$max\{1-f(x),1-f(y)\} = 1-f(x) \geq 1-f(x-y).$$

By Definition 3.1, $1-f$ is fuzzy completely prime subset of N. If $f(x)>f(y)$, then

$$max\{1-f(x),1-f(y)\} = 1-f(y) \geq 1-f(xy)$$

and

$$max\{1-f(x),1-f(y)\} = 1-f(y) \geq 1-f(x-y).$$

Hence $1-f$ is fuzzy completely prime subset of N.

($\Leftarrow$) Assume that $1-f$ is fuzzy completely prime ideal of N. Then $max\{1-f(x),1-f(y)\}\geq 1-f(x-y)$ and $max\{1-f(x),1-f(y)\}\geq 1-f(xy)$ for all $x,y\in N$. Clearly, $1-max\{1-f(x),1-f(y)\}\leq 1-(1-f(x-y))=f(x-y)$ and $1-max\{1-f(x),1-f(y)\}\leq 1-(1-f(xy))=f(xy)$, for all $x,y\in N$. If $1-f(x)\leq 1-f(y)$, then $f(x)=1-(1-f(x))\geq 1-(1-f(y))=f(y)$. For all $x,y\in N$,

$$f(xy) \geq 1-max\{1-f(x),1-f(y)\} = min\{f(x),f(y)\}$$

and

$$f(x-y) \geq 1-max\{1-f(x),1-f(y)\} = min\{f(x),f(y)\}$$

If $1-f(x)>1-f(y)$, then $f(x)=1-(1-f(x))<1-(1-f(y))=f(y)$. This implies that

$$f(xy) \geq 1-max\{1-f(x),1-f(y)\} = min\{f(x),f(y)\}$$

and

$$f(x-y) \geq 1-max\{1-f(x),1-f(y)\} = min\{f(x),f(y)\}$$

for all $x,y\in N$. Consequently f is a fuzzy nLA-subring of N.

Definition 3.4 A fuzzy subset f of an nLA-ring of N is called weakly fuzzy completely prime if for each $x,y\in N$ with $xy\neq 0$, it holds that $max\{f(x),f(y)\}\geq f(x-y)$ and $max\{f(x),f(y)\}\geq f(xy)$.

Remark. If f is a fuzzy completely prime subset of N, then f is a weakly fuzzy completely prime subset of N.

Example 3.5 Let $N=\{0,e,a,b,c,d\}$ be a set with two binary operations as follows:

+	0	e	a	b	c	d
0	0	e	a	b	c	d
e	d	0	e	a	b	c
a	c	d	0	e	a	b
b	b	c	d	0	e	a
c	a	b	c	d	0	e
d	e	a	b	c	d	0

·	0	e	a	b	c	d
0	0	0	0	0	0	0
e	0	e	a	b	c	d
a	0	a	c	0	a	c
b	0	b	0	b	0	b
c	0	c	a	0	c	a
d	0	d	c	b	a	e

By Definition 2.4, N is an nLA-ring. We define a fuzzy subset $f: N \longrightarrow [0,1]$ by

$$f(x)=\begin{cases}1 & ;x=0\\ 0 & ;\text{otherwise.}\end{cases}$$

It is easy to see that f is a weakly fuzzy completely prime subset of N. But f is not a fuzzy completely prime subset of N, since $max\{f(c), f(b)\}=max\{0,0\}=0,$ while $f(c\cdot b)=f(0)=1.$

Corollary 3.6 Let f be a fuzzy left ideal of an nLA-ring N. Then f is fuzzy completely prime (weakly fuzzy completely prime) subset of N if and only if $max\{f(x), f(y)\}=f(xy)$ and $max\{f(x), f(y)\}\geq f(x-y),$ for all $x, y\in N(xy\neq 0).$

Proof. Obvious.

Theorem 3.7 If f_i are fuzzy completely prime (weakly fuzzy completely prime) ideals of an nLA-ring N, then $\bigcup_{i\in I} f_i$ is fuzzy completely prime (weakly fuzzy completely prime) subset of N.

Proof. For all $i\in I$ and $x, y\in N$, $f_i(x-y)\leq max\{f_i(x), f_i(y)\},$ for all $i\in I$. Clearly, $f_i(x-y)\leq max\left\{\bigcup_{i\in I} f_i(x), \bigcup_{i\in I} f_i(y)\right\}.$ Therefore $\bigcup_{i\in I} f_i(x-y)\leq max\left\{\bigcup_{i\in I} f_i(x), \bigcup_{i\in I} f_i(y)\right\}.$ Similarly, $\bigcup_{i\in I} f_i(xy)\leq max\left\{\bigcup_{i\in I} f_i(x), \bigcup_{i\in I} f_i(y)\right\}.$ Consequently $\bigcup_{i\in I} f_i$ is a fuzzy completely prime subset of N.

Theorem 3.8 Let P be a left ideal of an nLA-ring N. Then P is a completely prime (weakly completely prime) ideal of N if and only if tf_P is a fuzzy completely prime (weakly fuzzy completely prime) subset of N.

Proof. $(\Rightarrow)$ By Lemma 2.11, tf_P is a fuzzy left ideal of N. Let $x, y \in N$. If $xy \notin P$, then $tf_P(xy) = 0 \leq max\{tf_P(x), tf_P(y)\}$. If $xy \in P$, then $x \in P$ or $y \in P$. Thus $tf_P(x) = t$ or $tf_P(y) = t$. Thus $tf_P(xy) = t = max\{tf_P(x), tf_P(y)\}$. Therefore f_P is a fuzzy completely prime subset of N.

$(\Leftarrow)$ Let $x, y \in N$ such that $xy \in P$. Then $tf_P(xy) = t$. Since tf_P is a fuzzy completely prime ideal of N, we have $tf_P(xy) \leq max\{tf_P(x), tf_P(y)\}$. Clearly, $tf_P(x) = t$ or $tf_P(y) = 1$. Thus $x \in P$ or $y \in P$. Therefore P is a completely prime ideal of N.

Definition 3.9 Let N be an nLA-ring and $t \in (0,1]$. A fuzzy left ideal f of N is said to be a fuzzy quasi-prime if $tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B \subseteq f$ and for the left ideals A and B in N.

Example 3.10 Let $N = \{0, e, a, b, c\}$ be a set with two binary operations as follows:

+	0	e	a	b	c
0	0	e	a	b	c
e	e	0	e	a	b
a	a	e	0	e	a
b	b	a	e	0	e
c	c	b	a	e	0

·	0	e	a	b	c
0	0	0	0	0	0
e	0	e	a	b	c
a	0	c	e	c	a
b	0	a	c	e	b
c	0	c	b	a	e

By Definition 2.4, N is an nLA-ring. We define a fuzzy subset $f : N \longrightarrow [0,1]$ by $f(x) = 0$, for all $x \in N$. By Definition 3.9, f is a fuzzy quasi-prime ideal of N.

Example 3.11 Let $N = \{0, e, a, b, c, d, x, y\}$ be a set with two binary operations as follows:

+	0	e	a	b	c	d	x	y
0	0	e	a	b	c	d	x	y
e	a	0	b	e	x	c	y	d
a	e	b	0	a	d	y	c	x
b	b	a	e	0	y	x	d	c
c	c	d	x	y	0	e	a	b
d	x	c	y	d	a	0	b	e
x	d	y	c	x	e	b	0	a
y	y	x	d	c	b	a	e	0

$\cdot$	0	e	a	b	c	d	x	y
0	0	0	0	0	0	0	0	0
e	0	c	c	0	0	c	c	0
a	0	c	c	0	0	c	c	0
b	0	0	0	0	0	0	0	0
c	0	b	b	0	0	b	b	0
d	0	y	y	0	0	y	y	0
x	0	y	y	0	0	y	y	0
y	0	b	b	0	0	b	b	0

By Definition 2.4, N is an nLA-ring. We define a fuzzy subset $f: N \longrightarrow [0,1]$ by

$$f(x) = \begin{cases} 1 & ; x \in \{0\} \\ 0.9 & ; x \in \{c\} \\ 0 & ; \text{otherwise.} \end{cases}$$

By Theorem 3.3, $1-f$ is a fuzzy completely prime subset of N. But $1-f$ is not a fuzzy quasi-prime ideal of N, since $1-f(e \cdot e) = 1-f(c) = 1-0.9 = 0.1$ while $1-f(e) = 1-0 = 1$.

Definition 3.12 Let N be an nLA-ring and $t \in (0,1]$. A fuzzy left ideal f of N is said to be a weakly fuzzy quasi-prime if $tg_0 \neq tg_A \circ tg_B \subseteq f$ implies $tg_A \subseteq f$ or $tg_B \subseteq f$ and for the left ideals A and B in N.

Remark. If f is a fuzzy quasi-prime ideal of N, then f is a weakly fuzzy quasi-prime ideal of N.

Theorem 3.13 Let g be a fuzzy left ideal of an nLA-ring with left identity N. Then the following conditions are equivalent:

1. g is a fuzzy quasi-prime ideal of N.
2. For any $x, y \in N$ and $t \in (0,1]$ if $x_t \circ (N \circ y_t) \subseteq g$, then $x_t \in g$ or $y_t \in g$.
3. For any $x, y \in N$ and $t \in (0,1]$ if $tf_x \circ tf_y \subseteq g$, then $x_t \in g$ or $y_t \in g$.
4. If A and B are left ideals of N such that $tf_A \circ tf_B \subseteq g$, then $tf_A \subseteq g$ or $tf_B \subseteq g$.

Proof. $(1 \Rightarrow 2)$ Let $x, y \in N$ and $t \in (0,1]$ such that $x_t \circ (N \circ y_t) \subseteq g$. By Lemma 2.6, 2.7 and 2.11, it follows that

$$\begin{aligned} tf_{(xe)N} \circ tf_{(ye)N} &= (tf_{(xe)} \circ N) \circ (tf_{ye} \circ N) \\ &= ((tf_x \circ tf_e) \circ (tf_y \circ tf_e)) \circ (N \circ N) \\ &= ((tf_e \circ tf_e) \circ (tf_y \circ tf_x)) \circ (N \circ N) \\ &= (tf_{ee} \circ (tf_y \circ tf_x)) \circ (N \circ N) \\ &= N \circ (tf_x \circ tf_x) \\ &= x_t \circ (N \circ y_t) \\ &\subseteq g. \end{aligned}$$

Thus by hypothesis, it follows that $x_t = tf_x = tf_{(ee)x} = tf_{(xe)e} \subseteq tf_{(xe)N} \subseteq g$ or $y_t = tf_y = tf_{(ee)y} = tf_{(ye)e} \subseteq tf_{(ye)N} \subseteq g$. Consequently $x_t \in g$ or $y_t \in g$.

$(2 \Rightarrow 3)$ Let $x, y \in N, t \in (0,1]$ such that $tf_x \circ tf_y \subseteq g$. By Lemmas 2.7 and 2.8,

$$\begin{aligned} x_t \circ (N \circ y_t) &\subseteq tf_x \circ (N \circ tf_y) \\ &\subseteq N \circ g \\ &\subseteq g. \end{aligned}$$

Thus, by hypothesis $x_t \in g$ or $y_t \in g$.

$(3 \Rightarrow 4)$ Let A and B be two left ideals of N. Then, by Lemma 2.9, we get tf_A and tf_B are fuzzy left ideals of N. Suppose that $tf_A \circ tf_B \subseteq g$ and $tf_B \not\subset g$. Then there exists $y \in B$ such that $y_t \notin g$. For any $x \in A$ by Lemma 2.11,

$$\begin{aligned} tf_x \circ tf_y &= tf_{xy} \\ &= tf_A \circ tf_B \\ &\subseteq g. \end{aligned}$$

Since $y_t \notin g$, we have $tf_x \subseteq g$. Clearly, $x_t \in g$. By Lemma 2.11, $tf_A = \bigcup_{x \in A} x_t \subseteq g$.

$(4 \Rightarrow 1)$ Obvious.

Corollary 3.14 Let g be a fuzzy left ideal of an nLA-ring with left identity N. Then the following conditions are equivalent:

1. g is a weakly fuzzy quasi-prime ideal of N.
2. For any $x, y \in N$ and $t \in (0,1]$ if $0_t \neq x_t \circ (N \circ y_t) \subseteq g$, then $x_t \in g$ or $y_t \in g$.
3. For any $x, y \in N$ and $t \in (0,1]$ if $0_t \neq tf_x \circ tf_y \subseteq g$, then $x_t \in g$ or $y_t \in g$.
4. If A and B are left ideals of N such that $0_t \neq tf_A \circ tf_B \subseteq g$, then $tf_A \subseteq g$ or $tf_B \subseteq g$.

Proof. Similar to the proof of Theorem 3.13.

Corollary 3.15 Let g be a fuzzy left ideal of an nLA-ring with left identity N. Then the following statements are equivalent:

1. g is a fuzzy quasi-prime (weakly fuzzy quasi-prime) ideal of N.
2. For any $x, y \in N$ and $t \in (0,1]$ if $x_t \circ y_t \in g \left(0_t \neq x_t \circ y_t \in g\right)$, then $x_t \in g$ or $y_t \in g$.

Proof. Straightforward by Theorem 3.13.

Example 3.16 Let $N = \{0, e, a, b, c, d, x, y\}$ be a set with two binary operations as follows:

$+$	0	e	a	b	d	5	x	y
0	0	e	a	b	c	d	6	y
e	a	0	b	e	x	c	y	d
a	e	b	0	a	d	y	c	x
b	b	a	e	0	y	x	d	c
c	c	d	x	y	0	e	a	b
d	x	c	y	d	a	0	b	e
x	d	y	c	x	e	b	0	a
y	y	x	d	c	b	a	e	0

$\cdot$	0	e	a	b	c	d	x	y
0	0	0	0	0	0	0	0	0
e	0	c	c	0	0	c	c	0
a	0	c	c	0	0	c	c	0
b	0	0	0	0	0	0	0	0
c	0	b	b	0	0	b	b	0
d	0	y	y	0	0	y	y	0
x	0	y	y	0	0	y	y	0
y	0	b	b	0	0	b	b	0

By Definition 2.4, N is an nLA-ring. We define a fuzzy subset $f: N \longrightarrow [0,1]$ by

$$f(x)=\begin{cases}1 & ; x\in\{0\}\\ 0.5 & ; x\in\{c\}\\ 0 & ; \text{otherwise.}\end{cases}$$

By Definition 3.12, f is a weakly fuzzy quasi-prime ideal of N. But f is not a fuzzy quasi-prime ideal of N, since $d_1 \circ b_1 \in f$ while $d_1 \notin f$ and $b_1 \notin f$.

Theorem 3.17 If f is a fuzzy quasi-prime ideal of an nLA-ring with left identity N, then $inf\{f(a^2(Nb^2))\} = max\{f(a^2), f(b^2)\}$ for all $a,b \in N$.

Proof. Suppose that $inf\{f(a^2(Nb^2))\} \neq max\{f(a^2), f(b^2)\}$. Since f is fuzzy left ideal of N, we have $f(a^2(rb^2)) \geq f(rb^2) \geq f(b^2)$ and $f(a^2(rb^2)) = f((rb^2)a^2) \geq f(a^2)$, for all $r \in N$. Clearly, $inf\{f(a^2(Nb^2))\} > max\{f(a^2), f(b^2)\}$. Let $inf\{f(a^2(Nb^2))\} = t$. By Lemma 2. 9, $tg_{(a^2e)N}$ and $tg_{(b^2e)N}$ are two fuzzy left ideals of N. If $tg_{(a^2e)N} \circ tg_{(b^2e)N}(x) = t$, then $t = \bigcup_{x=yz} min\{tg_{(a^2e)N}(y), tg_{(b^2e)N}(z)\}$. Then there exist $u \in (a^2e)N$ and $v \in (b^2e)N$ such that $uv = x$. Clearly, $u = (a^2e)m, v = (b^2e)n$ for some $m,n \in N$. Consider

$$\begin{aligned} f(x) &= f(uv) \\ &= f(((a^2e)m)((b^2e)n)) \\ &= f((nm)((b^2e)(a^2e))) \\ &\geq f((b^2e)(a^2e)) \\ &= f((b^2a^2)(ee)) \\ &\geq inf\{f(a^2(Nb^2))\} \\ &= t. \end{aligned}$$

Thus $tg_{(a^2e)N} \circ tg_{(b^2e)N} \subseteq f$. By Definition 3. 9, $tg_{(a^2e)N} \subseteq f$ or $tg_{(b^2e)N} \subseteq f$. This implies that $tg_{(a^2e)N}(a^2) = tg_{(a^2e)N}((a^2e)e) = t$ or $tg_{(b^2e)N}(b^2) = tg_{(b^2e)N}((b^2e)e) = t$. But from $t \leq max\{f(a^2), f(b^2)\} < inf\{f(a^2(Nb^2))\} = t$. This is a contradiction. Hence $inf\{f(a^2(Nb^2))\} = max\{f(a^2), f(b^2)\}$, where $a,b \in N$.

Theorem 3.18 If f is a fuzzy quasi-prime ideal of an nLA-ring with left identity N, then $max\{f(x), f(y)\} \geq f(xy),$ where $x, y \in N.$

Proof. Let $x, y \in N.$ If $f(xy) > max\{f(x), f(y)\},$ then there exists $t \in (0,1)$ such that $f(xy) > t > max\{f(x), f(y)\}.$ By Lemma 2.7,

$$x_t \circ (N \circ y_t) = N \circ (x_t \circ y_t) \subseteq N \circ f$$

for all $x, y \in N.$ By Theorem 3.13, $x_t \in P$ or $y_t \in P.$ This is a contradiction. Consequently $max\{f(x), f(y)\} \geq P(xy),$ where $x, y \in N.$

Theorem 3.19 Let f be a fuzzy left ideal of an nLA-ring with left identity $N.$ Then f is a fuzzy quasi-prime (weakly fuzzy quasi-prime) ideal of N if and only if f is a fuzzy completely prime (weakly fuzzy completely prime) subset of $N.$

Proof. Straightforward by Theorem 3.18.

4. Conclusions

Many new classes of nLA-rings have been discovered recently. All these have attracted detailed investigations of these newly discovered classes. This article investigated the fuzzy quasi-prime, weakly fuzzy quasi-prime, fuzzy completely prime and weakly fuzzy completely prime ideals in nLA-rings. Some characterizations of fuzzy quasi-prime and weakly fuzzy quasi-prime ideals were obtained. Moreover, we investigated the relationships between fuzzy completely prime and fuzzy quasi-prime ideals in an nLA-ring. Finally, we obtained necessary and sufficient conditions for a fuzzy completely prime subset to be a fuzzy quasi-prime ideal in an nLA-ring.

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