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Original Article

Some inequalities for the polar derivative of some classes of polynomials

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Abstract

In this paper, we investigate an upper bound of the polar derivative of a polynomial of degree n

$$p(z) = (z - z_m)^{t_m}(z - z_{m-1})^{t_{m-1}} \cdots (z - z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right)$$

where zeros $z_0, \ldots, z_m$ are in $\{z : |z| < 1\}$ and the remaining $n - (t_m + \cdots + t_0)$ zeros are outside $\{z : |z| < k\}$ where $k \geq 1$. Furthermore, we give a lower bound of this polynomial where zeros $z_0, \ldots, z_m$ are outside $\{z : |z| \leq k\}$ and the remaining $n - (t_m + \cdots + t_0)$ zeros are in $\{z : |z| \leq k\}$ where $k \leq 1$.

Keywords: polar derivative, polynomial, inequality

1. Introduction

Let k be a positive real number. We denote $\{z : |z| < k\}$ and $\{z : |z| \leq k\}$ by $D(0,k)$ and $\overline{D(0,k)}$, respectively.

Consider a polynomial $p(z)$ of degree n. Bernstein (1926) presented the well-known inequality $\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|$. This result is sharp for a polynomial $p(z) = az^n$ where a is a nonzero complex number. Bernstein's inequality is sharp for a special class of polynomials. Not only upper bounds of $\max_{|z|=1}|p'(z)|$ have been studied, but also lower

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bounds.

Lax (1944) proved the conjecture which was posed by Erdős for $p(z)$ having no zero in $D(0,1)$ that $\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|$. This bound was improved by Aziz and Dawood (1988) who proved that $\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\left[\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\right]$. Furthermore, the equality holds for $p(z) = az^n + b$ with $|b| \geq |a|$.

Turán (1939) proved that $\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|$ where $p(z)$ is a polynomial having all its zeros in $\overline{D(0,1)}$. The bounds of Lax (1944) and Turán (1939) are sharp for a polynomial which has all of its zeros on $\{z : |z| = 1\}$.

The improvement of this lower bound was presented by Aziz and Dawood (1988) that $\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\left[\max_{|z|=1}|p(z)| + \min_{|z|=1}|p(z)|\right]$. This new bound is sharp for a polynomial $p(z) = az^n + b$ with $|b| \leq |a|$.

Govil (1991) studied a polynomial $p(z)$ of degree n which has no zero in $D(0,k)$, $k \geq 1$. He proved that

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\left[\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right].$$

Moreover, Govil (1991) also studied a polynomial $p(z)$ of degree n having all its zeros in $\overline{D(0,k)}$, $k \leq 1$, and proved that

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k}\left[\max_{|z|=1}|p(z)| + \frac{1}{k^{n-1}}\min_{|z|=k}|p(z)|\right].$$

Both bounds are sharp and equalities hold for $p(z) = (z+k)^n$.

Aziz and Shah (1997) studied a lower bound of a derivative of a polynomial $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, which has all of its zeros in $D(0,k)$, $k \leq 1$.

Theorem A (Aziz & Shah, 1997) If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having all of its zeros in $D(0,k)$, then for $k \leq 1$

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^{\mu}}\left[\max_{|z|=1}|p(z)| + \frac{1}{k^{n-\mu}}\min_{|z|=k}|p(z)|\right].$$

Equality holds for $p(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$, where n is a multiple of μ.

Although their Theorem stated that $p(z)$ has all of its zeros in $D(0,k)$, $k \leq 1$, it is described in the proof that the result still holds when $p(z)$ has a zero on $\{z : |z| = k\}$. Thus, we restate their theorem as follows.

Theorem [Restate Theorem A] If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $\overline{D(0,k)}$, then for $k \leq 1$

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^{\mu}}\left[\max_{|z|=1}|p(z)| + \frac{1}{k^{n-\mu}}\min_{|z|=k}|p(z)|\right]. \quad (1.1)$$

Somsuwan and Nakprasit (2013) investigated a lower bound of a polynomial of degree n which has a zero outside $\overline{D(0,k)}$ and the remaining zeros in $D(0,k)$, $k \leq 1$. One of their results is as follows.

Theorem 1 (Somsuwan & Nakprasit, 2013) Let $k \le 1$ and $p(z) = (z - z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \le \mu \le n - s,$ be a polynomial of degree n having zero z_0 outside $\overline{D(0,k)}$ and the remaining $n - s$ zeros in $D(0,k)$. Then

$$\max_{|z|=1}|p'(z)| \ge \left[\frac{A}{(1+|z_0|)^s} - \frac{s}{(1+|z_0|)}\right]\max_{|z|=1}|p(z)| + \left[\frac{A}{k^{n-s-\mu}(k+|z_0|)^s}\right], \tag{1.2}$$

where $A = \frac{\left|1-|z_0|\right|^s(n-s)}{(1+k^\mu)}$.

Nakprasit and Somsuwan (2017) investigated an upper bound of a polynomial of degree n which has a zero in $D(0,1)$ and the remaining zeros outside $D(0,k)$, $k \ge 1$. One of their results is as follows.

Theorem 2 (Nakprasit & Somsuwan, 2017) Let $k \ge 1$ and $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z - z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \le \mu \le n - s, 0 \le s \le n - 1.$$

If a zero z_0 is in $D(0,1)$ and the remaining $n - s$ zeros are outside $D(0,k)$, then

$$\max_{|z|=1}|p'(z)| \le \left[\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \frac{A}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|, \tag{1.3}$$

where $A = \frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)}$.

The inequality (1.3) is sharp for a polynomial $p(z) = z^s(z+k)^{n-s}$.

The *polar derivative* of a polynomial $p(z)$ of degree n with respect to a complex number α, denoted by $D_\alpha p(z)$, is defined by $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$. Note that $D_\alpha p(z)$ generalizes the derivative of a polynomial in the sense that $\lim_{\alpha\to\infty}(D_\alpha p(z)/\alpha) = p'(z)$.

The bounds of $D_\alpha p(z)$ have been studied by many researchers. For example, Aziz and Shah (1997, 1998) studied upper bounds of $\max_{|z|=1}|D_\alpha p(z)|$ where $p(z)$ is a polynomial of degree n having no zero in $D(0,k)$, $k \ge 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$. Aziz and Rather (1998), Dewan, Singh, & Mir (2009), and Govil and McTume (2004) studied lower bounds of $\max_{|z|=1}|D_\alpha p(z)|$ where $p(z)$ is a polynomial of degree n having all of its zeros in $\overline{D(0,k)}$, $k \le 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$.

In this paper, we investigate an upper bound of the polar derivative of a polynomial of degree n

$$p(z) = (z - z_m)^{t_m}(z - z_{m-1})^{t_{m-1}} \cdots (z - z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right)$$

where zeros $z_0, \dots, z_m$ are in $D(0,1)$ and the remaining $n - (t_m + \cdots + t_0)$ zeros are outside $D(0,k)$ where $k \ge 1$. Furthermore, we give a lower bound of this polynomial where zeros $z_0, \dots, z_m$ are outside $\overline{D(0,k)}$ and the remaining $n - (t_m + \cdots + t_0)$ zeros are in $\overline{D(0,k)}$ where $k \ge 1$. Consequently, our results generalize the inequalities (1.2) and (1.3).

2. Upper Bound of A Polar Derivative of Polynomials Having at Least One Zero in $D(0,1)$.

In this section, we investigate an upper bound of $\max_{|z|=1}|D_\alpha p(z)|$ where $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p(z)$ is a polynomial of degree n which has some zeros in $D(0,1)$ and the remaining zeros outside $D(0,k)$ where $k \geq 1$.

For a polynomial $p(z)$ of degree n, we define $q(z) = z^n\overline{p(1/\bar{z})}$. Let z be a complex number with $|z| = 1$. It follows from the result of Govil and Rahman (1969) that

$$|p'(z)| + |q'(z)| \leq n \cdot \max_{|z|=1}|p(z)|. \tag{2.1}$$

Furthermore, one can show that

$$|np(z) - zp'(z)| = |q'(z)|. \tag{2.2}$$

Theorem 3 Let $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z - z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \leq \mu \leq n - s, 0 \leq s \leq n - 1.$$

Let $k \geq 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$. If a zero z_0 is in $D(0,1)$ and the remaining $n - s$ zeros are outside $D(0,k)$, then

$$\max_{|z|=1}|D_\alpha p(z)| \leq \left[n + (|\alpha| - 1)\left(\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right)\right]\max_{|z|=1}|p(z)| - \left[\frac{(|\alpha|-1)A}{(k+|z_0|)^s}\right]\min_{|z|=k}|p(z)|, \tag{2.3}$$

where $A = \frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)}$.

Proof: Observe that

$$|D_\alpha p(z)| = |np(z) + (\alpha - z)p'(z)| \leq |np(z) - zp'(z)| + |\alpha||p'(z)| \tag{2.4}$$

Set $q(z) = z^n\overline{p(1/\bar{z})}$. From the relation (2.2), we have $|np(z) - zp'(z)| = |q'(z)|$ for $|z| = 1$.

By substituting this result into (2.4), we obtain for $|z| = 1$ that

$$|D_\alpha p(z)| \leq |q'(z)| + |\alpha||p'(z)| = |q'(z)| + |p'(z)| + (|\alpha| - 1)|p'(z)|.$$

Consequently, $\max_{|z|=1}|D_\alpha p(z)| \leq n \cdot \max_{|z|=1}|p(z)| + (|\alpha| - 1)\max_{|z|=1}|p'(z)|$ from the relation (2.1).

Theorem 2 implies that

$$\max_{|z|=1}|p'(z)| \leq \left[\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right]\max_{|z|=1}|p(z)| - \frac{A}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|,$$

and therefore

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\leq n\max_{|z|=1}|p(z)| + (|\alpha| - 1)\left[\left(\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right)\max_{|z|=1}|p(z)| - \frac{A}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|\right] \\ &\leq \left[n + (|\alpha| - 1)\left(\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right)\right]\max_{|z|=1}|p(z)|\end{aligned}$$

$$-\left[\frac{(|\alpha|-1)A}{(k+|z_0|)^s}\right]\min_{|z|=k}|p(z)|,$$

where $A=\frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^\mu)(1-|z_0|)}$.

Remark 4 Dividing both sides of the inequality (2.3) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the inequality (1.3) in Theorem 2.

In case $z_0=0$, we obtain the following corollary.

Corollary 5 Let $p(z)$ be a polynomial of degree n in the form

$$p(z)=z^s\left(a_0+\sum_{v=\mu}^{n-s}a_vz^v\right), 1\le\mu\le n-s, 0\le s\le n-1.$$

Let $k\ge 1$ and $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$. If all $n-s$ zeros (except a zero at the origin) are outside $D(0,k)$, then

$$\max_{|z|=1}|D_\alpha p(z)|\le\left[\frac{|\alpha|(n+sk^\mu)+(n-s)k^\mu}{1+k^\mu}\right]\max_{|z|=1}|p(z)|-\left[\frac{(|\alpha|-1)(n-s)}{k^s(1+k^\mu)}\right]\min_{|z|=k}|p(z)|. \tag{2.5}$$

Next, we show that the upper bound in (2.3) is sharp for a polynomial $p(z)=z^s(z+k)^{n-s}$ where k is a real number with $|k|\ge 1$.

One can see that $|D_\alpha p(z)|=|\{z^s[(n-s)k+\alpha n]+\alpha skz^{s-1}\}(z+k)^{n-s-1}|$.

Note that $(n-s)k+\alpha n>0$ because $n,k,s\in\mathbb{Z}^+$ and $\alpha\in\mathbb{R}$ with $|\alpha|\ge 1$.

Furthermore, $\max_{|z|=1}|z^s[(n-s)k+\alpha n]+\alpha skz^{s-1}|$ and $\max_{|z|=1}|(z+k)^{n-s-1}|$ are attained at $z=1$.

These results yield that

$$\max_{|z|=1}|D_\alpha p(z)|=\big((n-s)k+\alpha n+\alpha sk\big)(1+k)^{n-s-1}. \tag{2.6}$$

The right side of the inequality (2.3) becomes

$$\left[n+(|\alpha|-1)\left(\frac{s}{(1-0)}+\frac{n-s}{(1+k)(1-0)^s}\right)\right]\max_{|z|=1}|p(z)|-\left[\frac{(|\alpha|-1)(n-s)}{(1+k)(k+0)^s}\right]\min_{|z|=k}|p(z)|$$

$$=\big((n-s)k+\alpha n+\alpha sk\big)(1+k)^{n-s-1},$$

which equals $\max_{|z|=1}|D_\alpha p(z)|$ in (2.6).

This means that the upper bound in Theorem 3 is sharp.

Moreover, this polynomial also makes (2.5) an equality, that is, Corollary 5 is sharp.

Corollary 6 Let $p(z)$ be a polynomial of degree n in the form

$$p(z)=(z-z_1)^{t_1}(z-z_0)^{t_0}\left(a_0+\sum_{v=\mu}^{n-(t_1+t_0)}a_vz^v\right), 1\le\mu\le n-(t_1+t_0),$$

$$0\le t_1+t_0\le n-1.$$

Let $k\ge 1$ and $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$. If zeros z_0 and z_1 are in $D(0,1)$ and the remaining $n-(t_1+t_0)$ zeros are outside $D(0,k)$, then

$$\max_{|z|=1}|D_\alpha p(z)| \le \Big[\frac{t_1(|\alpha|+|z_1|)(1+|z_1|)^{t_1-1}}{(1-|z_1|)^{t_1}}+\frac{(n-t_1)(1+|z_1|)^{t_1}}{(1-|z_1|)^{t_1}}$$

$$+\frac{(|\alpha|-1)(1+|z_1|)^{t_1}}{(1-|z_1|)^{t_1}}\Big(\frac{t_0}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^{t_0}}\Big)\Big]\max_{|z|=1}|p(z)|$$

$$-\Big[\frac{(|\alpha|-1)(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\Big]\min_{|z|=k}|p(z)|,$$

where $A=\frac{(1+|z_0|)^{t_0+1}(n-(t_0+t_1))}{(1+k^\mu)(1-|z_0|)}$.

Proof: Let $p(z)=(z-z_1)^{t_1}p_0(z)$ where $p_0(z)=(z-z_0)^{t_0}\left(a_0+\sum_{v=\mu}^{n-(t_1+t_0)}a_v z^v\right)$ and $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$.

Then $D_\alpha p(z)=(z-z_1)^{t_1}[D_\alpha p_0(z)]+t_1(\alpha-z_1)(z-z_1)^{t_1-1}p_0(z)$,

and $|D_\alpha p(z)|\le|z-z_1|^{t_1}|D_\alpha p_0(z)|+t_1|\alpha-z_1||z-z_1|^{t_1-1}|p_0(z)|$.

Since $|z-z_1|\le|z|+|z_1|=1+|z_1|$ and $|\alpha-z_1|\le|\alpha|+|z_1|$ for $|z|=1$, we get

$$\max_{|z|=1}|D_\alpha p(z)|\le(1+|z_1|)^{t_1}\max_{|z|=1}|D_\alpha p_0(z)|+t_1(|\alpha|+|z_1|)(1+|z_1|)^{t_1-1}\max_{|z|=1}|p_0(z)|.$$

Theorem 3 yields that

$$\max_{|z|=1}|D_\alpha p_0(z)|\le\Big[(n-t_1)+(|\alpha|-1)\Big(\frac{t_0}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^{t_0}}\Big)\Big]\max_{|z|=1}|p_0(z)|-\Big[\frac{(|\alpha|-1)A}{(k+|z_0|)^{t_0}}\Big]\min_{|z|=k}|p_0(z)|,$$

where $A=\frac{(1+|z_0|)^{t_0+1}(n-(t_0+t_1))}{(1+k^\mu)(1-|z_0|)}$.

Therefore,

$$\max_{|z|=1}|D_\alpha p(z)|\le\Big[t_1(|\alpha|+|z_1|)(1+|z_1|)^{t_1-1}+(n-t_1)(1+|z_1|)^{t_1}$$

$$+(|\alpha|-1)(1+|z_1|)^{t_1}\Big(\frac{t_0}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^{t_0}}\Big)\Big]\max_{|z|=1}|p_0(z)|$$

$$-\Big[\frac{(|\alpha|-1)(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}}\Big]\min_{|z|=k}|p_0(z)|. \quad (2.7)$$

On $|z|=1$, we have $|p_0(z)|=\frac{1}{|z-z_1|^{t_1}}\cdot|p(z)|\le\frac{1}{(1-|z_1|)^{t_1}}\cdot|p(z)|$.

Consequently,

$$\max_{|z|=1}|p_0(z)|\le\frac{1}{(1-|z_1|)^{t_1}}\cdot\max_{|z|=1}|p(z)|. \quad (2.8)$$

On $|z|=k$, we have $|p_0(z)|=\frac{1}{|z-z_1|^{t_1}}\cdot|p(z)|\ge\frac{1}{(k+|z_1|)^{t_1}}\cdot|p(z)|$.

Thus,

$$\min_{|z|=k}|p_0(z)|\ge\frac{1}{(k+|z_1|)^{t_1}}\cdot\min_{|z|=k}|p(z)|. \quad (2.9)$$

By substituting *(2.8)* and *(2.9)* in *(2.7)*, we obtain that

$$\max_{|z|=1}|D_\alpha p(z)|\le\Big[\frac{t_1(|\alpha|+|z_1|)(1+|z_1|)^{t_1-1}}{(1-|z_1|)^{t_1}}+\frac{(n-t_1)(1+|z_1|)^{t_1}}{(1-|z_1|)^{t_1}}$$

$$+\frac{(|\alpha|-1)(1+|z_1|)^{t_1}}{(1-|z_1|)^{t_1}}\Big(\frac{t_0}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^{t_0}}\Big)\Big]\max_{|z|=1}|p(z)|$$

$$-\left[\frac{(|\alpha|-1)(1+|z_1|)^{t_1}A}{(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right]\min_{|z|=k}|p(z)|,$$

where $A=\frac{(1+|z_0|)^{t_0+1}(n-(t_0+t_1))}{(1+k^{\mu})(1-|z_0|)}$.

Remark 7 Consider a polynomial of degree n

$$p(z)=(z-z_m)^{t_m}(z-z_{m-1})^{t_{m-1}}\cdots(z-z_0)^{t_0}\left(a_0+\sum_{v=\mu}^{n-(t_m+\cdots+t_0)}a_v z^v\right)$$

where zeros $z_0,\dots,z_m$ are in $D(0,1)$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are outside $D(0,k)$ where $k\geq 1$.

An upper bound of $\max_{|z|=1}|D_\alpha p(z)|$, where $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$, can be obtained by applying Theorem 3 as in the proof of Corollary 6.

Let $p_0(z)=(z-z_0)^{t_0}\left(a_0+\sum_{v=\mu}^{n-(t_m+\cdots+t_0)}a_v z^v\right), p_j(z)=\left(z-z_j\right)^{t_j}p_{j-1}(z)$, for $1\leq j\leq m$, and $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$. Theorem 3 yields an upper bound of $\max_{|z|=1}|D_\alpha p_0(z)|$. Combining this upper bound together with the facts that

$$\max_{|z|=1}|p_0(z)|\leq\frac{1}{(1-|z_1|)^{t_1}}\cdot\max_{|z|=1}|p_1(z)|$$

and

$$\min_{|z|=k}|p_0(z)|\geq\frac{1}{(k+|z_1|)^{t_1}}\cdot\min_{|z|=k}|p_1(z)|,$$

we can obtain an upper bound of $\max_{|z|=1}|D_\alpha p_1(z)|$ as in Corollary 6.

Consequently, an upper bound of $\max_{|z|=1}|D_\alpha p_j(z)|$ for $2\leq j\leq m$ can be obtained by a similar process by using an upper bound of $\max_{|z|=1}|D_\alpha p_{j-1}(z)|$ and the facts that

$$\max_{|z|=1}|p_{j-1}(z)|\leq\frac{1}{(1-|z_j|)^{t_j}}\cdot\max_{|z|=1}|p_j(z)|$$

and

$$\min_{|z|=k}|p_{j-1}(z)|\geq\frac{1}{(k+|z_j|)^{t_j}}\cdot\min_{|z|=k}|p_j(z)|,$$

for $2\leq j\leq m$.

3. Lower Bound of A Polar Derivative of Polynomials Having at Least One Zero Outside $D(0,k)$ where $k\leq 1$.

In this section, we investigate a lower bound of $\max_{|z|=1}|D_\alpha p(z)|$ where $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$ and $p(z)$ is a polynomial of degree n which has some zeros outside $\overline{D(0,k)}$, $k\leq 1$ and other zeros in $\overline{D(0,k)}$ where $k\leq 1$.

Theorem 8 Let $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z - z_0)^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v \right), 1 \le \mu \le n - s, 0 \le s \le n - 1.$$

Let $k \le 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$. If a zero z_0 is outside $\overline{D(0,k)}$ and the remaining $n - s$ zeros are in $\overline{D(0,k)}$, then

$$\max_{|z|=1}|D_\alpha p(z)| \ge \left[\frac{(|\alpha|-1)A}{(1+|z_0|)^s} - \left(n + \frac{s(|\alpha|+1)}{(1+|z_0|)}\right)\right] \max_{|z|=1}|p(z)| + \left[\frac{(|\alpha|-1)A}{k^{n-s-\mu}(k+|z_0|)^s}\right] \min_{|z|=k}|p(z)|,$$

(3.1)

where $A = \frac{|1-|z_0||^s(n-s)}{(1+k^\mu)}$.

Proof: By setting $\phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v$, we can rewrite $p(z) = (z - z_0)^s \phi(z)$.

The derivative of $p(z)$ is $p'(z) = (z - z_0)^s \phi'(z) + \phi(z) s (z - z_0)^{s-1}$ and then

$D_\alpha p(z) = (\alpha - z)(z - z_0)^s \phi'(z) + [n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z).$

The triangle inequality implies that

$$|D_\alpha p(z)| + |[n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z)| \ge |(\alpha - z)(z - z_0)^s \phi'(z)|.$$

One can see that

$$\max_{|z|=1}|D_\alpha p(z)| \ge \max_{|z|=1}|(\alpha - z)(z - z_0)^s \phi'(z)| - \max_{|z|=1}|[n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z)|.$$

For $|z| = 1$, we obtain that $|z - z_0| \le |z| + |z_0| = 1 + |z_0|$.

Since $|z - z_0|^s \ge (|z| - |z_0|)^s = (1 - |z_0|)^s$ for $k < |z_0| < 1$ and $|z - z_0|^s = |z_0 - z|^s \ge (|z_0| - |z|)^s = (|z_0| - 1)^s$ for $|z_0| > 1$, we obtain that $|z - z_0|^s \ge |1 - |z_0||^s$ for $|z_0| > k$.

Consequently,

$$\max_{|z|=1}|D_\alpha p(z)| \ge (|\alpha| - 1)|1 - |z_0||^s \max_{|z|=1}|\phi'(z)| - [n(1 + |z_0|) + s(|\alpha| + 1)](1 + |z_0|)^{s-1} \max_{|z|=1}|\phi(z)|.$$

By applying $\phi(z)$ in the inequality (1.1), we have that

$$\max_{|z|=1}|\phi'(z)| \ge \frac{(n-s)}{(1+k^\mu)}\left[\max_{|z|=1}|\phi(z)| + \frac{1}{k^{n-s-\mu}} \min_{|z|=k}|\phi(z)|\right].$$

This implies that

$$\max_{|z|=1}|D_\alpha p(z)| \ge \left[\frac{(|\alpha|-1)|1-|z_0||^s(n-s)}{(1+k^\mu)} - [n(1 + |z_0|) + s(|\alpha| + 1)](1 + |z_0|)^{s-1}\right] \times \max_{|z|=1}|\phi(z)| + \left[\frac{(|\alpha|-1)|1-|z_0||^s(n-s)}{k^{n-s-\mu}(1+k^\mu)}\right] \min_{|z|=k}|\phi(z)|.$$

Observe that $|\phi(z)| = \frac{1}{|z-z_0|^s} \cdot |p(z)| \ge \frac{1}{(1+|z_0|)^s} \cdot |p(z)|$ for $|z| = 1$.

Thus, $\max_{|z|=1}|\phi(z)| \geq \frac{1}{(1+|z_0|)^s} \cdot \max_{|z|=1}|p(z)|$.

On $|z| = k$, we have that $|\phi(z)| = \frac{1}{|z-z_0|^s} \cdot |p(z)| \geq \frac{1}{(k+|z_0|)^s} \cdot |p(z)|$.

This implies that $\min_{|z|=k}|\phi(z)| \geq \frac{1}{(k+|z_0|)^s} \min_{|z|=k}|p(z)|$.

Therefore,

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \left[\frac{(|\alpha|-1)\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)} - [n(1+|z_0|) + s(|\alpha|+1)](1+|z_0|)^{s-1}\right] \\ &\quad \times \left[\frac{1}{(1+|z_0|)^s} \cdot \max_{|z|=1}|p(z)|\right] + \left[\frac{(|\alpha|-1)\left|1-|z_0|\right|^s (n-s)}{k^{n-s-\mu}(1+k^\mu)}\right]\left[\frac{1}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|\right]. \\ &= \left[\frac{(|\alpha|-1)A}{(1+|z_0|)^s} - \left(n + \frac{s(|\alpha|+1)}{(1+|z_0|)}\right)\right]\max_{|z|=1}|p(z)| \\ &\quad + \left[\frac{(|\alpha|-1)A}{k^{n-s-\mu}(k+|z_0|)^s}\right]\min_{|z|=k}|p(z)|,\end{aligned}$$

where $A = \frac{\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)}$.

Remark 9 (1) Dividing both sides of the inequality (3.1) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the inequality (1.2) in Theorem 1.

(2) In case $p(z)$ has at least one zero on $\{z : |z| = k\}$, we obtain that $\min_{|z|=k}|p(z)| = 0$. Then

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left[\frac{(|\alpha|-1)A}{(1+|z_0|)^s} - \left(n + \frac{s(|\alpha|+1)}{(1+|z_0|)}\right)\right]\max_{|z|=1}|p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)}$.

Corollary 10 Let $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z-z_1)^{t_1}(z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_1+t_0)} a_v z^v\right), 1 \leq \mu \leq n-(t_1+t_0),$$

$$0 \leq t_1 + t_0 \leq n-1.$$

Let $k \leq 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$. If zeros z_0 and z_1 are outside $\overline{D(0,k)}$ and the remaining $n-(t_1+t_0)$ zeros are in $\overline{D(0,k)}$, then

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1} A}{(1+|z_0|)^{t_0}(1+|z_1|)^{t_1}} - \left(\frac{(n-t_1)\left|1-|z_1|\right|^{t_1}}{(1+|z_1|)^{t_1}} + \frac{t_0(|\alpha|+1)\left|1-|z_1|\right|^{t_1}}{(1+|z_0|)(1+|z_1|)^{t_1}}\right) - \frac{t_1(|\alpha|+|z_1|)}{(1+|z_1|)}\right]\max_{|z|=1}|p(z)| \\ &\quad + \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1} A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right]\min_{|z|=k}|p(z)|,\end{aligned}$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{(1+k^\mu)}$.

Proof: Let $p(z) = (z-z_1)^{t_1}p_0(z)$ where $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_1+t_0)} a_v z^v\right)$

and $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$.

Then $D_\alpha p(z) = (z - z_1)^{t_1}[D_\alpha p_0(z)] + t_1(\alpha - z_1)(z - z_1)^{t_1 - 1}p_0(z)$,

and $|D_\alpha p(z)| \geq |z - z_1|^{t_1}|D_\alpha p_0(z)| - t_1|\alpha - z_1||z - z_1|^{t_1-1}|p_0(z)|$.

Since $|z - z_1|^{t_1} \geq (|z| - |z_1|)^{t_1} = (1 - |z_1|)^{t_1}$ for $k < |z_1| < 1$ and $|z - z_1|^{t_1} = |z_1 - z|^{t_1} \geq (|z_1| - |z|)^{t_1} = (|z_1| - 1)^{t_1}$ for $|z_1| > 1$, we obtain that $|z - z_1|^{t_1} \geq \left|1 - |z_1|\right|^{t_1}$ for $|z_1| > k$.

For $|z| = 1$, we get that $|z - z_1| \leq |z| + |z_1| = 1 + |z_1|$ and $|\alpha - z_1| \leq |\alpha| + |z_1|$.

By combining these two results, we have that

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left|1 - |z_1|\right|^{t_1} \max_{|z|=1}|D_\alpha p_0(z)| - t_1(|\alpha| + |z_1|)(1 + |z_1|)^{t_1 - 1} \max_{|z|=1}|p_0(z)|. \tag{3.2}$$

By applying $p_0(z)$ in Theorem 8, we obtain that

$$\max_{|z|=1}|D_\alpha p_0(z)| \geq \left[\frac{(|\alpha|-1)A}{(1+|z_0|)^{t_0}} - (n - t_1) - \frac{t_0(|\alpha|+1)}{(1+|z_0|)}\right] \max_{|z|=1}|p_0(z)| + \left[\frac{(|\alpha|-1)A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}}\right] \min_{|z|=k}|p_0(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{(1+k^\mu)}$.

By substituting this result into (3.2), we have that

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1}A}{(1+|z_0|)^{t_0}} - (n - t_1)\left|1 - |z_1|\right|^{t_1} - \frac{t_0(|\alpha|+1)\left|1-|z_1|\right|^{t_1}}{(1+|z_0|)} - t_1(|\alpha| + |z_1|)(1 + |z_1|)^{t_1-1}\right] \max_{|z|=1}|p_0(z)| + \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1}A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}}\right] \min_{|z|=k}|p_0(z)|.$$

Since $|p_0(z)| = \frac{1}{|z-z_1|^{t_1}} \cdot |p(z)| \geq \frac{1}{(1+|z_1|)^{t_1}} \cdot |p(z)|$ for $|z| = 1$, $\max_{|z|=1}|p_0(z)| \geq \frac{1}{(1+|z_1|)^{t_1}} \cdot \max_{|z|=1}|p(z)|$

For $|z| = k$, we have $|p_0(z)| = \frac{1}{|z-z_1|^{t_1}} \cdot |p(z)| \geq \frac{1}{(k+|z_1|)^{t_1}} \cdot |p(z)|$ and then $\min_{|z|=k}|p_0(z)| \geq$

$\frac{1}{(k+|z_1|)^{t_1}} \cdot \min_{|z|=k}|p(z)|$

.Consequently,

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1}A}{(1+|z_0|)^{t_0}(1+|z_1|)^{t_1}} - \left(\frac{(n-t_1)\left|1-|z_1|\right|^{t_1}}{(1+|z_1|)^{t_1}} + \frac{t_0(|\alpha|+1)\left|1-|z_1|\right|^{t_1}}{(1+|z_0|)(1+|z_1|)^{t_1}}\right) - \frac{t_1(|\alpha|+|z_1|)}{(1+|z_1|)}\right] \max_{|z|=1}|p(z)| + \left[\frac{(|\alpha|-1)\left|1-|z_1|\right|^{t_1}A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right] \min_{|z|=k}|p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{(1+k^\mu)}$.

Remark 11 Consider a polynomial of degree n

$$p(z) = (z - z_m)^{t_m}(z - z_{m-1})^{t_{m-1}} \cdots (z - z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right)$$

where zeros $z_0, \ldots, z_m$ are outside $\overline{D(0,k)}$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are in $\overline{D(0,k)}$ where $k \leq 1$.

A lower bound of $\max_{|z|=1}|D_\alpha p(z)|$, where $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, can be obtained by applying Theorem 8 as in the proof of Corollary 10.

Let $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right), p_j(z) = \left(z-z_j\right)^{t_j} p_{j-1}(z),$ for $1 \leq j \leq m.$

Theorem 8 yields a lower bound of $\max_{|z|=1}|D_\alpha p_0(z)|$. Combining this lower bound together with the facts that

$$\max_{|z|=1}|p_0(z)| \geq \frac{1}{(1+|z_1|)^{t_1}} \cdot \max_{|z|=1}|p_1(z)|$$

and

$$\min_{|z|=k}|p_0(z)| \geq \frac{1}{(k+|z_1|)^{t_1}} \cdot \min_{|z|=k}|p_1(z)|,$$

we can obtain a lower bound of $\max_{|z|=1}|D_\alpha p_1(z)|$ as in Corollary 10.

Consequently, a lower bound of $\max_{|z|=1}|D_\alpha p_j(z)|$ for $2 \leq j \leq m$ can be obtained by a similar process by using a lower bound of $\max_{|z|=1}|D_\alpha p_{j-1}(z)|$ and the facts that $\max_{|z|=1}\left|p_{j-1}(z)\right| \geq \frac{1}{(1+|z_j|)^{t_j}} \cdot \max_{|z|=1}\left|p_j(z)\right|$ and $\min_{|z|=k}\left|p_{j-1}(z)\right| \geq \frac{1}{(k+|z_j|)^{t_j}} \cdot \min_{|z|=k}\left|p_j(z)\right|$ for $2 \leq j \leq m.$

4. Conclusions and Discussion

In this paper, we generalize bounds of $\max_{|z|=1}|p'(z)|$ in the inequalities (1.2) and (1.3) to bounds of $\max_{|z|=1}|D_\alpha p(z)|$ in the inequalities (3.1) and (2.3) and thereby we obtain refinements of these results. Since $\lim_{\alpha\to\infty}(D_\alpha p(z)/\alpha) = p'(z)$, our results on the polar derivative of $p(z)$ generalize results on the derivative of $p(z)$ as Theorem 3 and Theorem 8 generalize Theorem 2 and Theorem 1, respectively. Our lower bound of $\max_{|z|=1}|D_\alpha p(z)|$ is more general than lower bounds in Dewan, Singh, & Mir (2009) and Govil and McTume (2004).

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