

Original Article

Some factorizations of Ramanujan’s cubic continued fraction

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Abstract

We derive new identities involving Ramanujan’s cubic continued fraction which are analogous to those of the famous Rogers-Ramanujan continued fraction. Using such new identities enables us to give the new proof of several early results of this continued fraction.

Keywords: theta function, continued fraction, infinite product

1. Introduction

Ramanujan’s general theta-function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where a, b are complex numbers with $|ab| < 1$. As customary and throughout this paper, we assume that q is a complex number with $|q| < 1$ and use the standard notation

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{and} \quad (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$

The function $f(a, b)$ can be written in terms of infinite products via Jacobi’s triple product identity (Berndt, 1991) given by

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \tag{1.1}$$

Define

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}, \tag{1.2}$$

$$\varphi(-q) := f(-q, -q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f^2(-q)}{f(-q^2)}, \tag{1.3}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f^2(-q^2)}{f(-q)}, \tag{1.4}$$

$$\chi(-q) := \frac{f(-q)}{f(-q^2)}. \tag{1.5}$$

The last equalities of Equations (1.2) to (1.4) follow from Equation (1.1).

Ramanujan’s cubic continued fraction is defined by

$$v(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \dots}}}}.$$

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In his notebooks (Berndt, 1991; Ramanujan, 1957) and in his lost notebooks (Andrews & Berndt, 2005; Ramanujan, 1988), Ramanujan found that

$$v(q) = q^{1/3} \frac{\chi(-q)}{\chi^3(-q^3)} = q^{1/3} \frac{f(-q)f^3(-q^6)}{f(-q^2)f^3(-q^3)}. \tag{1.6}$$

Moreover, Ramanujan recorded several identities involving $v(q)$ (Berndt, 1991), namely,

$$\begin{aligned} 1 + \frac{1}{v(q)} &= \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \\ 1 - 2v(q) &= \frac{\phi(-q^{1/3})}{\phi(-q^3)}, \\ 1 + \frac{1}{v^3(q)} &= \frac{\psi^4(q)}{q\psi^4(q^3)}, \\ 1 - 8v^3(q) &= \frac{\phi^4(-q)}{\phi^4(-q^3)}, \\ \frac{1}{v(q)} + 4v^2(q) &= \left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}, \\ \frac{1}{v(q)} + 4v^2(q) &= 3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}. \end{aligned} \tag{1.7}$$

This continued fraction recently has been studied by several authors. Firstly, Chan (1995) derived the formula

$$v^3(q) = v(q^3) \frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)}. \tag{1.8}$$

Next, Mahadeva Naika (2008) proved some identities

$$\begin{aligned} \frac{1}{v^2(q)} - 2v(q) &= \left(27 + \frac{f^{12}(-q^2)}{q^2 f^{12}(-q^6)}\right)^{1/3}, \\ \frac{1}{v^2(q)} - 2v(q) &= 3 + \frac{f^3(-q^{2/3})}{q^{2/3} f^3(-q^6)}. \end{aligned} \tag{1.9}$$

Later, Chan (2010) proved that

$$\begin{aligned} \frac{1}{v(q)} - 1 - 2v(q) &= \frac{f(-q^{1/3})f(-q^{2/3})}{q^{1/3}f(-q^3)f(-q^6)}, \\ \frac{1}{v^3(q)} - 7 - 8v^3(q) &= \frac{f^4(-q)f^4(-q^2)}{qf^4(-q^3)f^4(-q^6)}. \end{aligned} \tag{1.10}$$

Hirschhorn and Roselin (2009) established the 2-, 3-, 4- and 6-dissections of Ramanujan’s cubic continued fraction and its reciprocal.

In this article, we will establish several new identities for $v(q)$. In particular, Theorem 2.3 provides the results that

can be used to give another proof of early work by Ramanujan in Equation (1.7), Chan in Equation (1.8), Mahadeva Naika in Equation (1.9), and Chan in Equation (1.10).

2. Factorizations of the Ramanujan’s Cubic Continued Fraction

Firstly, we will state some identities for $f(a,b)$ used in this literature.

Lemma 2.1. (Berndt, 1991) Let $U_n = a^{n(n+1)/2} b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2} b^{n(n+1)/2}$ for each integer n. Then

$$f(U_1, V_1) = \sum_{r=0}^{k-1} U_r f\left(\frac{U_{k+r}}{U_r}, \frac{V_{k-r}}{U_r}\right), \tag{2.1}$$

for every positive integer k.

Lemma 2.2. (Berndt, 1991) We have

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f(a, a^3). \tag{2.3}$$

Using (1.1), the continued fraction $v(q)$ can be re-expressed as

$$v(q) = q^{1/3} \frac{f(q^3, q^9)}{f(q, q^2)}. \tag{2.4}$$

Throughout this section, we let $\zeta := e^{\pi i/3}$.

Theorem 2.3. We have

$$\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} = \frac{\psi(q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^2)f(-q^3)f(-q^6)}}, \tag{2.5}$$

$$\frac{1}{\sqrt{v(q)}} - \zeta \sqrt{v(q)} = \frac{\psi(-\zeta q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^2)f(-q^3)f(-q^6)}}, \tag{2.6}$$

$$\frac{1}{\sqrt{v(q)}} + \zeta^2 \sqrt{v(q)} = \frac{\psi(\zeta^2 q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^2)f(-q^3)f(-q^6)}}, \tag{2.7}$$

$$\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} = \frac{\phi(-q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q^2)}{f(-q)f(-q^3)f(-q^6)}}, \tag{2.8}$$

$$\frac{1}{\sqrt{v(q)}} + 2\zeta \sqrt{v(q)} = \frac{\phi(\zeta q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q^2)}{f(-q)f(-q^3)f(-q^6)}}, \tag{2.9}$$

$$\frac{1}{\sqrt{v(q)}} - 2\zeta^2 \sqrt{v(q)} = \frac{\phi(-\zeta^2 q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q^2)}{f(-q)f(-q^3)f(-q^6)}}. \tag{2.10}$$

Proof of Equation (2.5). By Equation (2.4), the left hand side

of Equation (2.5) becomes

$$\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} = \frac{f(q, q^2) + q^{1/3}f(q^3, q^9)}{q^{1/6}\sqrt{f(q, q^2)f(q^3, q^9)}}. \tag{2.11}$$

Using Jacobi's triple product identity Equation (1.1), we have

$$\begin{aligned} f(q, q^2)f(q^3, q^9) &= (-q; q^3)_\infty (-q^2; q^3)_\infty (q^3; q^3)_\infty (-q^3; q^{12})_\infty (-q^9; q^{12})_\infty (q^{12}; q^{12})_\infty \\ &= \frac{f(-q^2)f(-q^3)f(-q^6)}{f(-q)}. \end{aligned} \tag{2.12}$$

Putting $k = 3$, $a = 1$ and $b = q^{1/3}$ in Equation (2.1) together with Equations (2.2) and (2.3), it follows that

$$\begin{aligned} f(1, q^{1/3}) &= f(q, q^2) + f(q^2, q) + q^{1/3}f(q^3, 1), \\ 2f(q^{1/3}, q) &= 2f(q, q^2) + 2q^{1/3}f(q^3, q^9), \\ \psi(q^{1/3}) &= f(q, q^2) + q^{1/3}f(q^3, q^9). \end{aligned} \tag{2.13}$$

Substituting Equations (2.12) and (2.13) into Equation (2.11), we obtain the result.

Proof of Equation (2.6). Take $k = 3$, $a = 1$ and $b = -\zeta q^{1/3}$ in Equation (2.1).

Proof of Equation (2.7). Put $k = 3$, $a = 1$ and $b = \zeta^2 q^{1/3}$ in Equation (2.1).

Proof of Equation (2.8). By Equations (2.4) and (2.12), we have

$$\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} = \frac{(f(q, q^2) - 2q^{1/3}f(q^3, q^9))\sqrt{f(-q)}}{q^{1/6}\sqrt{f(-q^2)f(-q^3)f(-q^6)}}. \tag{2.14}$$

Take $k = 3$, $a = \zeta^2$ and $b = -\zeta q^{1/3}$ in Equation (2.1) and obtain

$$f(\zeta^2, -\zeta q^{1/3}) = f(q, q^2) + \zeta^2 f(q, q^2) - \zeta q^{1/3} f(q^3, 1).$$

Since $\frac{\zeta}{1+\zeta^2} = 1$, we arrive at

$$\frac{f(\zeta^2, -\zeta q^{1/3})}{1+\zeta^2} = f(q, q^2) - 2q^{1/3}f(q^3, q^9). \tag{2.15}$$

Using Equation (1.1), we deduce that

$$\begin{aligned} \frac{f(\zeta^2, -\zeta q^{1/3})}{1+\zeta^2} &= \frac{(-\zeta^2; q^{1/3})_\infty (\zeta q^{1/3}; q^{1/3})_\infty (q^{1/3}; q^{1/3})_\infty}{1+\zeta^2} \\ &= f(-q^{1/3}) \prod_{k=1}^{\infty} (1 + \zeta^2 q^{k/3})(1 - \zeta q^{k/3}) \\ &= f(-q^{1/3}) \prod_{k=1}^{\infty} \frac{1+q^k}{1+q^{k/3}} \\ &= f(-q^{1/3}) \prod_{k=1}^{\infty} \frac{(1-q^{2k})(1-q^{k/3})}{(1-q^k)(1-q^{2k/3})} \\ &= \frac{f^2(-q^{1/3})f(-q^2)}{f(-q^{2/3})f(-q)} \\ &= \frac{\varphi(-q^{1/3})f(-q^2)}{f(-q)}. \end{aligned} \tag{2.16}$$

By Equations (2.14), (2.15) and (2.16), we complete the proof of Equation (2.8).

Proof of Equation (2.9). Putting $k = 3$, $a = -\zeta$ and $b = q^{1/3}$ in Equation (2.1) and using the fact that

$$f(-\zeta, q^{1/3}) = \frac{(1-\zeta)\varphi(\zeta q^{1/3})f(-q^2)}{f(-q)},$$

we finish the proof of Equation (2.9).

Proof of Equation (2.10). Taking $k = 3$, $a = \zeta^2$ and $b = q^{1/3}$ in Equation (2.1) and employing

$$f(\zeta^2, q^{1/3}) = \frac{(1+\zeta^2)\varphi(-\zeta^2 q^{1/3})f(-q^2)}{f(-q)},$$

the proof of Equation (2.10) is complete.

Theorem 2.4. We have

$$\frac{1}{v(q)} - 1 - 2v(q) = \frac{f(-q^{1/3})f(-q^{2/3})}{q^{1/3}f(-q^3)f(-q^6)}, \tag{2.17}$$

$$\frac{1}{v(q)} - 1 + v(q) = \frac{\psi^4(q)\chi(-q)}{q^{1/3}\psi(q^{1/3})f^3(-q^6)}, \tag{2.18}$$

$$\frac{1}{v(q)} + 2 + 4v(q) = \frac{f(-q)\varphi^3(-q)}{q^{1/3}\varphi(-q^{1/3})f^3(-q^3)}. \tag{2.19}$$

Proof of Equation (2.17). Multiply Equations (2.5) and (2.8), we complete the proof of Equation (2.17).

Proof of Equation (2.18). We observe that

$$\frac{1}{v(q)} - 1 + v(q) = \left(\frac{1}{\sqrt{v(q)}} - \zeta\sqrt{v(q)} \right) \left(\frac{1}{\sqrt{v(q)}} + \zeta^2\sqrt{v(q)} \right).$$

Employing Equations (2.6) and (2.7) together with (1.5), we arrive at

$$\frac{1}{v(q)} - 1 + v(q) = \frac{\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3})\chi(-q)}{q^{1/3}f(-q^3)f(-q^6)}. \tag{2.20}$$

We find that

$$\begin{aligned} \psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3}) &= \frac{(\zeta^2 q^{2/3}; \zeta^2 q^{2/3})_\infty (\zeta^4 q^{2/3}; \zeta^4 q^{2/3})_\infty}{(-\zeta q^{1/3}; \zeta^2 q^{2/3})_\infty (\zeta^2 q^{1/3}; \zeta^4 q^{2/3})_\infty} \\ &= \prod_{k=0}^{\infty} \frac{(1 - \zeta^{2(k+1)} q^{2(k+1)/3})(1 - \zeta^{4(k+1)} q^{2(k+1)/3})}{(1 + \zeta^{2k+1} q^{(2k+1)/3})(1 - \zeta^{4k+2} q^{(2k+1)/3})} \\ &= \prod_{k=0}^{\infty} \frac{1 - (\zeta^{4(k+1)} + \zeta^{2(k+1)}) q^{2(k+1)/3} + q^{4(k+1)/3}}{1 - (\zeta^{4k+2} - \zeta^{2k+1}) q^{(2k+1)/3} + q^{(4k+2)/3}}. \end{aligned} \tag{2.21}$$

Since

$$\zeta^{4(k+1)} + \zeta^{2(k+1)} = \begin{cases} 2 & \text{if } 3 \mid k+1, \\ -1 & \text{otherwise,} \end{cases}$$

and

$$\zeta^{4k+2} - \zeta^{2k+1} = \begin{cases} 2 & \text{if } 3 \mid k+2, \\ -1 & \text{otherwise,} \end{cases}$$

the Equation (2.21) becomes

$$\begin{aligned} & \psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3}) \\ &= \prod_{k=0}^{\infty} \frac{(1-2q^{2k+2} + q^{4k+4})(1+q^{(6k+2)/3} + q^{(12k+8)/3})(1+q^{(6k+4)/3} + q^{(12k+4)/3})}{(1-2q^{2k+1} + q^{4k+2})(1+q^{(6k+1)/3} + q^{(12k+2)/3})(1+q^{(6k+5)/3} + q^{(12k+10)/3})} \\ &= \prod_{k=0}^{\infty} \frac{(1-q^{2k+2})^2}{(1-q^{2k+1})^2} \left(\frac{1-q^{6k+2}}{1-q^{(6k+2)/3}} \right) \left(\frac{1-q^{6k+4}}{1-q^{(6k+4)/3}} \right) \left(\frac{1-q^{(6k+1)/3}}{1-q^{6k+1}} \right) \left(\frac{1-q^{(6k+5)/3}}{1-q^{6k+5}} \right) \\ &= \frac{(q^2; q^2)_{\infty}^2 (q^2; q^6)_{\infty} (q^4; q^6)_{\infty} (q^{1/3}; q^2)_{\infty} (q^{5/3}; q^2)_{\infty}}{(q; q^2)_{\infty}^2 (q; q^6)_{\infty} (q^5; q^6)_{\infty} (q^{2/3}; q^2)_{\infty} (q^{4/3}; q^2)_{\infty}} \\ &= \frac{(q^2; q^2)_{\infty}^4 (q^3; q^6)_{\infty} (q^{1/3}; q^{2/3})_{\infty}}{(q; q^2)_{\infty}^4 (q^6; q^6)_{\infty} (q^{2/3}; q^{2/3})_{\infty}}. \end{aligned}$$

By Equation (1.4), it follows that

$$\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3}) = \frac{\psi^4(q)f(-q^3)}{\psi(q^{1/3})f^2(-q^6)}. \tag{2.22}$$

Substituting Equation (2.22) into Equation (2.20), the proof is complete.

Proof of Equation (2.19). We find that

$$\frac{1}{v(q)} + 2 + 4v(q) = \left(\frac{1}{\sqrt{v(q)}} + 2\zeta\sqrt{v(q)} \right) \left(\frac{1}{\sqrt{v(q)}} - 2\zeta^2\sqrt{v(q)} \right).$$

Utilizing Equations (2.9) and (2.10), we deduce that

$$\frac{1}{v(q)} + 2 + 4v(q) = \frac{\varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3})f(-q^2)}{q^{1/3}f(-q)f(-q^3)f(-q^6)}. \tag{2.23}$$

For the numerator of Equation (2.23), we see that

$$\begin{aligned} \varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3}) &= \frac{(-\zeta q^{1/3}; -\zeta q^{1/3})_{\infty} (\zeta^2 q^{1/3}; \zeta^2 q^{1/3})_{\infty}}{(\zeta q^{1/3}; -\zeta q^{1/3})_{\infty} (-\zeta^2 q^{1/3}; \zeta^2 q^{1/3})_{\infty}} \\ &= \prod_{k=1}^{\infty} \frac{(1 - (-\zeta q^{1/3})^k)(1 - (\zeta^2 q^{1/3})^k)}{(1 + (-\zeta q^{1/3})^k)(1 + (\zeta^2 q^{1/3})^k)} \\ &= \prod_{k=1}^{\infty} \frac{1 - (\zeta^{2k} + (-\zeta)^k)q^{k/3} + q^{2k/3}}{1 + (\zeta^{2k} + (-\zeta)^k)q^{k/3} + q^{2k/3}}. \end{aligned}$$

Since $\zeta^{2k} + (-\zeta)^k = 2$ if $3|k$ and $\zeta^{2k} + (-\zeta)^k = -1$ if $3\nmid k$, it follows that

$$\begin{aligned}
 & \varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3}) \\
 &= \prod_{k=0}^{\infty} \frac{(1-2q^{k+1} + q^{2(k+1)})(1+q^{(3k+1)/3} + q^{2(3k+1)/3})(1+q^{(3k+2)/3} + q^{2(3k+2)/3})}{(1+2q^{k+1} + q^{2(k+1)})(1-q^{(3k+1)/3} + q^{2(3k+1)/3})(1-q^{(3k+2)/3} + q^{2(3k+2)/3})} \\
 &= \prod_{k=0}^{\infty} \frac{(1-q^{k+1})^2}{(1+q^{k+1})^2} \left(\frac{1-q^{3k+1}}{1-q^{(3k+1)/3}} \right) \left(\frac{1-q^{3k+2}}{1-q^{(3k+2)/3}} \right) \left(\frac{1+q^{(3k+1)/3}}{1+q^{3k+1}} \right) \left(\frac{1+q^{(3k+2)/3}}{1+q^{3k+2}} \right) \\
 &= \left(\frac{(q; q)_{\infty}^2}{(-q; q)_{\infty}^2} \right) \left(\frac{(q; q^3)_{\infty} (q^2; q^3)_{\infty}}{(-q; q^3)_{\infty} (-q^2; q^3)_{\infty}} \right) \left(\frac{(-q^{1/3}; q)_{\infty} (-q^{2/3}; q)_{\infty}}{(q^{1/3}; q)_{\infty} (q^{2/3}; q)_{\infty}} \right) \\
 &= \left(\frac{(q; q)_{\infty}^2}{(-q; q)_{\infty}^2} \right) \left(\frac{(q; q)_{\infty} (-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty} (-q; q)_{\infty}} \right) \left(\frac{(q; q)_{\infty} (-q^{1/3}; q^{1/3})_{\infty}}{(q^{1/3}; q^{1/3})_{\infty} (-q; q)_{\infty}} \right) \\
 &= \frac{(q; q)_{\infty}^8 (q^6; q^6)_{\infty} (q^{2/3}; q^{2/3})_{\infty}}{(q^2; q^2)_{\infty}^4 (q^3; q^3)_{\infty}^2 (q^{1/3}; q^{1/3})_{\infty}^2} \\
 &= \frac{f^8(-q)f(-q^6)f(-q^{2/3})}{f^4(-q^2)f^2(-q^3)f^2(-q^{1/3})}. \tag{2.24}
 \end{aligned}$$

Substituting Equations (2.24) into (2.23), we complete the proof.

Corollary 2.5. We have

$$\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} = \frac{\psi^4(q)}{q^{1/2}\psi(q^3)} \sqrt{\frac{\chi^3(-q)}{f^3(-q^3)f^3(-q^6)}}, \tag{2.25}$$

$$\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} = \frac{f^4(-q)}{q^{1/2}\chi(-q^3)} \sqrt{\frac{\chi^5(-q)}{f^5(-q^3)f^3(-q^6)}}, \tag{2.26}$$

$$\frac{1}{v^3(q)} - 7 - 8v^3(q) = \frac{f^4(-q)f^4(-q^2)}{qf^4(-q^3)f^4(-q^6)}, \tag{2.27}$$

$$v^3(q) = v(q^3) \frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)}. \tag{2.28}$$

Proof of Equation (2.25). The identity follows from the factorization

$$\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} = \left(\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} \right) \left(\frac{1}{v(q)} - 1 + v(q) \right).$$

Proof of Equation (2.26). The identity can be obtained from

$$\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} = \left(\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} \right) \left(\frac{1}{v(q)} + 2 + 4v(q) \right).$$

Proof of Equation (2.27). Note that

$$\frac{1}{v^3(q)} - 7 - 8v^3(q) = \left(\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} \right) \left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} \right).$$

Multiply Equations (2.25) and (2.26) and then obtain

$$\frac{1}{v^3(q)} - 7 - 8v^3(q) = \frac{f^4(-q)\psi^4(q)\chi^4(-q)}{q\psi(q^3)\chi(-q^3)f^4(-q^3)f^3(-q^6)}.$$

Since $\psi^4(q)\chi^4(-q) = f^4(-q^2)$ and $\psi(q^3)\chi(-q^3) = f(-q^6)$, the result follows immediately.

Proof of Equation (2.28). Using Equations (2.18), (2.19) and (1.6), we arrive at

$$v(q^3) \frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)} = q \left(\frac{\varphi(-q)}{\psi(q)} \right) \left(\frac{\psi^4(q^3)\chi(-q^3)}{\varphi^3(-q^3)f(-q^6)} \right). \tag{2.29}$$

Utilizing Equations (1.3), (1.4) and (1.5), we get

$$\frac{\varphi(-q)}{\psi(q)} = \frac{f^3(-q)}{f^3(-q^2)}, \tag{2.30}$$

and

$$\frac{\psi^4(q^3)\chi(-q^3)}{\varphi^3(-q^3)f(-q^6)} = \frac{f^9(-q^6)}{f^9(-q^3)}. \tag{2.31}$$

Hence, substituting Equations (2.30) and (2.31) into Equation (2.29) together with Equation (1.6), we eventually conclude that

$$v(q^3) \frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)} = q \frac{f^3(-q)f^9(-q^6)}{f^3(-q^2)f^9(-q^3)} = v^3(q).$$

The following corollary mainly contains Ramanujan’s results of $v(q)$.

Corollary 2.6. We have

$$\begin{aligned} 1+v(q) &= \frac{\psi(q^{1/3})\chi(-q)}{\varphi(-q^3)}, \\ \frac{1}{v(q)}+1 &= \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \\ 1-2v(q) &= \frac{\varphi(-q^{1/3})}{\varphi(-q^3)}, \\ \frac{1}{v(q)}-2 &= \frac{\varphi(-q^{1/3})}{q^{1/3}\chi(-q)\psi(q^3)}, \\ 1+v^3(q) &= \frac{\varphi(-q)\psi^3(q)}{\varphi^3(-q^3)\psi(q^3)}, \\ \frac{1}{v^3(q)}+1 &= \frac{\psi^4(q)}{q\psi^4(q^3)}, \\ 1-8v^3(q) &= \frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \\ \frac{1}{v^3(q)}-8 &= \frac{\varphi^3(-q)\psi(q)}{q\varphi(-q^3)\psi^3(q^3)}. \end{aligned}$$

Proof. These results are immediate consequences of Theorem 2.3 and Corollary 2.5.

Corollary 2.7. We have

$$\frac{1}{v(q)}+4v^2(q) = \left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)} \right)^{1/3}, \tag{2.32}$$

$$\frac{1}{v(q)}+4v^2(q) = 3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}, \tag{2.33}$$

$$\frac{1}{v^2(q)}-2v(q) = \left(27 + \frac{f^{12}(-q^2)}{q^2f^{12}(-q^6)} \right)^{1/3}, \tag{2.34}$$

$$\frac{1}{v^2(q)}-2v(q) = 3 + \frac{f^3(-q^{2/3})}{q^{2/3}f^3(-q^6)}. \tag{2.35}$$

Proof. Employing Theorem 2.3, Corollary 2.5 and Corollary 2.6, the identities (2.32) - (2.35) follow immediately from the following factorizations

$$\begin{aligned} \left(\frac{1}{v(q)} + 4v^2(q)\right)^3 - 27 &= (1+v^3(q)) \left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)}\right)^2, \\ \frac{1}{v(q)} + 4v^2(q) - 3 &= (1+v(q)) \left(\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)}\right)^2, \\ \left(\frac{1}{v^2(q)} - 2v(q)\right)^3 - 27 &= \frac{1}{\sqrt{v^3(q)}} \left(\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)}\right)^2 \left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)}\right), \\ \frac{1}{v^2(q)} - 2v(q) - 3 &= \frac{1}{\sqrt{v^3(q)}} (1+v(q))^2 \left(\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)}\right), \end{aligned}$$

respectively.

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