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*Short Communication*

# A closed-form formula for the conditional expectation of the extended CIR process

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## Abstract

This paper is an extension to a recent paper by Rujivan (2016), in which we derive a closed-form formula for the conditional expectation of the valuation process, defined by

$$V_{t,T} := e^{-\int_t^T r(s)ds} f\left(v_T\right) + \int_t^T h\left(v_s\right) e^{-\int_t^s r(u)du} ds$$

for $0 \le t \le T$, where $v_t$ is assumed to follow the extended Cox-Ingersoll-Ross process, for $f\left(v\right) = v^{\gamma_1}$ and $h\left(v\right) = v^{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \mathbf{R}$, and any integrable function $r$. Our newly-derived formula can be used to price a contingent claim $\left(f, r, h\right)$ in which $f\left(v_t\right)$, $r\left(t\right)$, and $h\left(v_t\right)$ for $t \in \left[0, T\right]$ represent, respectively, a terminal payoff, an interest rate process, and a payoff rate process.

**Keywords:** extended CIR process, conditional expectation, closed-form formula

## 1. Introduction

The Cox-Ingersoll-Ross (CIR) process has form of

$$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma\sqrt{v_t}\,dW_t \tag{1.1}$$

where $v_t$ is an instantaneous variance, $\theta$, $\kappa$, and $\sigma$ are parameters, and $W_t$ is a standard Brownian motion under a probability space $\left(\Omega, F, P\right)$ with a filtration $\left(F_t\right)_{t\ge 0}$. A general class of the CIR process is that of the class of the extended Cox-Ingersoll-Ross (ECIR) process,

$$dv_t = \kappa\left(t\right)\left(\theta\left(t\right) - v_t\right)dt + \sigma\left(t\right)\sqrt{v_t}\,dW_t \tag{1.2}$$

where all of the parameters are set to be smooth and bounded time-dependent parameter functions, i.e., $\theta\left(t\right)$, $\kappa\left(t\right)$, and $\sigma\left(t\right)$. Although the CIR process is the most common model used to describe the dynamics of the instantaneous variance or

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interest rates in the Heston model of stochastic volatility or in stochastic interest rate models (Lech & Oosterlee, 2011), there is much empirical evidence supporting the theory that the data generating process governing the dynamics of many economics variables might vary over time, because of economic climate changes or time effects. In that case, the ECIR process is more suitable for describing the data than the corresponding CIR process, because the ECIR process uses time-dependent parameter functions to present possible time varying expected trends and volatilities of the market and the economy. Very recently, many researchers in commodity markets such as Schneider and Tavin (2015), and Arismendi, Back, Prokopczuk, Paschke, and Rudolf (2016), described seasonal stochastic volatility by using the ECIR process in which $\theta(t)$ represents the long-term mean variance level of commodity prices, which is assumed to be a function of time.

In the context of option pricing when the underlying process is assumed to follow the ECIR process (1.2), we define the valuation process of a contingent claim $(f,r,h)$ by

$$V_{t,T} := e^{-\int_t^T r(s)ds} f(v_T) + \int_t^T h(v_s) e^{-\int_t^s r(u)du} ds \tag{1.3}$$

for real-valued functions $f, r$ and $h$. In this context, the processes $f(v_t)$, $r(t)$, and $h(v_t)$ for $t \in [0,T]$ represent, respectively, a terminal payoff, an interest rate process, and a payoff rate process. According to the theorem for option pricing proposed by Karatzas and Shreve (1991) (see page 378), the fair price of the contingent claim $(f,r,h)$ at a cur-rent time $t$ is the conditional expectation of the evaluation process (1.3) with respect to the risk-neutral probability measure $P$ and current $\sigma-$ field $F_t$, such as

$$E^P\left[V_{t,T} \mid F_t\right] = E^P\left[V_{t,T} \mid v_t = v\right] \tag{1.4}$$

for $t \in [0,T]$ and $v > 0$, where we denote by $E^P[X \mid F_t]$, the conditional expectation of a random variable $X$ with respect to the probability measure $P$ and $\sigma-$ field $F_t$.

Next, we define

$$X_{t,T} := e^{-\int_t^T r(s)ds} \tag{1.5}$$

$$Y_{t,T} := \int_t^T h(v_s) e^{-\int_t^s r(u)du} ds \tag{1.6}$$

for $t \in [0,T]$. Hence, the valuation process (1.3) can be expressed as

$$V_{t,T} = X_{t,T} f(v_T) + Y_{t,T} \tag{1.7}$$

and the conditional expectation (1.4) can be explicitly written in terms of a triple integral as

$$E^P\left[V_{t,T} \mid v_t = v\right] = \int_{D_Y}\int_{D_X}\int_{D_V} \left(x f(v) + y\right) p_{vxy}(v,x,y,t+\tau \mid v,t)\, dvdxdy \tag{1.8}$$

for $\tau = T - t \geq 0$ where $p_{vxy}(v,x,y,t+\tau \mid v,t)$ denotes the joint-transition density of the processes $v_t, X_{t,T}$, and $Y_{t,T}$ defined on the domains $D_V \subseteq \mathbf{R}^+, D_X \subseteq \mathbf{R}^+$, and $D_Y \subseteq \mathbf{R}^+$, respectively.

In terms of computation, various analytical or numerical methods can be employed to obtain exact or numeri-cal solutions for the triple integral on the RHS of (1.8) providing that the joint-transition density $p_{vxy}$ is available in closed-form. However, to derive $p_{vxy}$ in closed-form, we need to solve the forward Kolmogorov equation, associated with the processes $v_t, X_{t,T}$, and $Y_{t,T}$ (Karatzas & Shreve, 19 91) and this is a difficult and complicated task in general for arbitrary real-valued functions $f$, $h$, and $r$.

In some special cases, the conditional expectation (1.4) has a closed-form formula. For example, Dufresne (2001) proposed a closed-form formula for the case $f(v) = v^\gamma$ for any $\gamma > \frac{-2\kappa\theta}{\sigma^2}$ and $h = r = 0$ in which $v_t$ is assumed to fol-low the CIR process (1.1). Recently, Rujivan (2016) extended Dufresne's (2001) work to the ECIR processes (1.2) for any $\gamma \in \mathbf{R}$.

In this study, we adopt the analytical approach presented by Rujivan (2016) to derive a closed-form formula for the conditional expectation (1.4) for $f(v) = v^{\gamma_1}$ and $h(v) = v^{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \mathbf{R}$, and any integrable function $r$. Very interestingly, the derivation of our approach has completely avoided the utilization of the joint-transition density $p_{vxy}$.

There are two major contributions of this paper. First, our closed-form formula produces the exact value of the conditional expectation (1.4) without employing numerical integration or Monte-Carlo (MC) simulations. Clearly, this can substantially reduce the computational burden as shown in Rujivan (2016), which is a major drawback of numerical integration and MC method. Second, our closed-form formula has a simple form, which can be easily used by practitioners. With these contributions, our closed-form formula should be valuable in both theoretical and practical senses.

The following two assumptions proposed by Maghsoodi (1996) are needed, in order to ensure that the stochastic differential equation (SDE) (1.2) has a pathwise unique strong solution, in which $v_t$ avoids zero a.s. $P$ for all $t\in(0,T]$.

**Assumption 1** The parameter functions $\theta(t)$, $\kappa(t)$, and $\sigma(t)$ are strictly positive and continuous on $[0,T]$ such that the dimension of the ECIR process (1.2), defined by $\delta(t):=\dfrac{4\theta(t)\kappa(t)}{\sigma^2(t)}$, is bounded.

**Assumption 2** The inequality $\delta(t)\ge 2$ holds for all $t\in[0,T]$.

## 2. Main Results

Suppose $v_t$ follows the ECIR process (1.2) and Assumptions 1-2 hold. We denote

$$U_E(v,\tau):=E^P\left[V_{t,T}\,|\,v_t=v\right] \tag{2.1}$$

for $v>0$ and $\tau=T-t\ge 0$. On the other hand, if $v_t$ follows the CIR process (1.1), we write $U_C(v,\tau)$ instead of $U_E(v,\tau)$.

**Theorem 2.1.** Suppose that $f$ and $h$ can be written as $f(v)=v^{\gamma_1}$ and $h(v)=v^{\gamma_2}$ for any $\gamma_1,\gamma_2\in\mathbf{R}$, and $r$ is integrable on $[0,T]$. Then, the conditional expectation (1.4) can be expressed as

$$U_E(v,\tau)=\left(e^{-\int_t^T r(s)ds}\right)U_E^{(\gamma_1)}(v,\tau)+\sum_{k=0}^{\infty}\left(\int_t^T A_{\gamma_2-k}(s-t)e^{-\int_t^s r(u)du}ds\right)v^{\gamma_2-k} \tag{2.2}$$

for $v>0$ and $\tau=T-t\ge 0$ where the functions $U_E^{(\gamma)}(v,\tau)$ and $A_{\gamma-k}(s-t),k=0,1,...,$ for any $\gamma\in\mathbf{R}$ are given by

$$U_E^{(\gamma)}(v,\tau)=A_\gamma(\tau)v^\gamma+\sum_{k=0}^{\infty}A_{\gamma-k}(\tau)v^{\gamma-k} \tag{2.3}$$

$$A_\gamma(\tau)=e^{-\gamma\int_0^\tau \kappa(T-s)ds} \tag{2.4}$$

$$A_{\gamma-k}(\tau)v^{\gamma-k}=e^{-(\gamma-k)\int_0^\tau\kappa(T-s)ds}\int_0^\tau e^{(\gamma-k)\int_0^\eta\kappa(T-\varsigma)d\varsigma}P_{\gamma-k+1}(T-\eta)A_{\gamma-k+1}(\eta)d\eta \tag{2.5}$$

$$P_{\gamma-k+1}(\tau)=(\gamma-k+1)\left\{\frac{1}{2}(\gamma-k)\sigma^2(\tau)+\kappa(\tau)\theta(\tau)\right\} \tag{2.6}$$

for $k=1,2,...$. In particular, if $\gamma_1=m_1$ and $\gamma_2=m_2$ are non-negative integers, then

$$U_E(v,\tau)=\left(e^{-\int_t^T r(s)ds}\right)U_E^{(m_1)}(v,\tau)+\sum_{j=0}^{m_2}\left(\int_t^T A_j(s-t)e^{-\int_t^s r(u)du}ds\right)v^j \tag{2.7}$$

for $v>0$ and $\tau=T-t\ge 0$, where the functions $U_E^{(n)}(v,\tau)$ and $A_j(s-t),\ j=0,1,...,n,$ for any non-negative integer n are given by

$$U_E^{(n)}(v,\tau)=A_n(\tau)v^n+\sum_{j=0}^{n-1}A_j(\tau)v^j \tag{2.8}$$

$$A_n(\tau)=e^{-n\int_0^\tau\kappa(T-s)ds} \tag{2.9}$$

$$A_j(\tau)=e^{-j\int_0^\tau\kappa(T-s)ds}\int_0^\tau e^{j\int_0^\eta\kappa(T-\varsigma)d\varsigma}P_{j+1}(T-\eta)A_{j+1}(\eta)d\eta \tag{2.10}$$

where $P_{j+1}(\tau)=(j+1)\left\{\frac{1}{2}j\sigma^2(\tau)+\kappa(\tau)\theta(\tau)\right\}$ *for* $j=n-1,...,0.$

**Proof.** From (1.3)–(1.4), we have

$$U_E(v,\tau)=E^P\left[e^{-\int_t^T r(s)ds}f(v_T)+\int_t^T h(v_s)e^{-\int_t^s r(u)du}ds\,|\,v_t=v\right]$$

$$=\left(e^{-\int_t^T r(s)ds}\right)E^P\left[f\left(v_T\right)\mid v_t=v\right]+\int_t^T E^P\left[h\left(v_s\right)\mid v_t=v\right]e^{-\int_t^s r(u)du}ds$$

$$=\left(e^{-\int_t^T r(s)ds}\right)E^P\left[v_T^{\gamma_1}\mid v_t=v\right]+\int_t^T E^P\left[v_s^{\gamma_2}\mid v_t=v\right]e^{-\int_t^s r(u)du}ds. \tag{2.11}$$

Using the closed-form formula (2.2) written in Theorem 2.1 by Rujivan (2016) to compute the $\gamma_i^{\text{th}}$ conditional moments on the RHS of (2.11) for $i=1,2,$ we thus obtain

$$E^P\left[v_s^{\gamma_i}\mid v_t=v\right]=A_{\gamma_i}\left(\tau\right)v^{\gamma_i}+\sum_{k=1}^{\infty}A_{\gamma_i-k}\left(\tau\right)v^{\gamma_i-k} \tag{2.12}$$

For $s\in[t,T]$. Inserting (2.12) into the RHS of (2.11) yields (2.12).

On the other hand, when $\gamma_1=m_1$ and $\gamma_2=m_2$ are non-negative integers, we adopt the closed-form formula (2.13) written in Theorem 2.2. by Rujivan (2016) to obtain

$$E^P\left[v_s^{m_i}\mid v_t=v\right]=A_{m_i}\left(\tau\right)v^{m_i}+\sum_{j=0}^{m_i-1}A_j\left(\tau\right)v^j \tag{2.13}$$

for $i=1,2,$ and $s\in[t,T]$. Inserting (2.13) into the RHS of (2.11) yields (2.7).

The following corollary can readily be deduced from Theorem 2.1.

**Corollary 2.1.** Suppose $f$ and $h$ can be written as $f(v)=\sum_{k=0}^{n_f}a_kv^k$ and $h(v)=\sum_{k=0}^{n_h}b_kv^k$ for $v>0$ and for some sequences of real numbers $\left(a_0,...,a_{n_f}\right)$ and $\left(b_0,...,b_{n_h}\right)$ in which $a_{n_f}$ and $b_{n_h}$ are not zero and $r$ is integrable on $[0,T]$. Then, the conditional expectation (1.4) can be expressed as

$$U_E\left(v,\tau\right)=\left(e^{-\int_t^T r(s)ds}\right)\sum_{m_1=0}^{n_f}a_{m_1}U_E^{(m_1)}\left(v,\tau\right)+\sum_{m_2=0}^{n_h}b_{m_2}\sum_{j=0}^{m_2}\left(\int_t^T A_j\left(s-t\right)e^{-\int_t^s r(u)du}ds\right)v^j \tag{2.14}$$

for $v>0$ and $\tau=T-t\geq 0,$ where the functions $U_E^{(n)}\left(v,\tau\right)$ and $A_j\left(s-t\right),\ j=0,1,...,n,$ for any non-negative integer $n$ are given in (2.8)-(2.10), respectively.

**Proof.** From (0.3)-(0.4), we have

$$U_E\left(v,\tau\right)=\left(e^{-\int_t^T r(s)ds}\right)\sum_{m_1=0}^{n_f}a_{m_1}E^P\left[v_T^{m_1}\mid v_t=v\right]+\int_t^T\sum_{m_2=0}^{n_h}b_{m_2}E^P\left[v_s^{m_2}\mid v_t=v\right]e^{-\int_t^s r(u)du}ds. \tag{2.15}$$

Applying (2.13) to the conditional expectations on the RHS of (2.15) yields (2.14)

The integral terms on the RHS of (2.2) and (2.7) can be worked out when $v_t$ follows the CIR process (1.1), as shown in the following Theorem.

**Theorem 2.2.** According to Theorem 2.1., if $v_t$ follows the CIR process (1.1) and $r = r_0$ is a constant then

$$U_C(v,\tau) = \sum_{k=0}^{\infty}\left\{c_k^{(\gamma_1)}\frac{e^{-(r_0+\gamma_1\kappa)\tau}}{k!}\left(\frac{e^{\kappa\tau}-1}{\kappa}\right)^k\right\}v^{\gamma_1-k} + \left\{c_k^{(\gamma_2)}\frac{1}{\kappa^k}\sum_{i=0}^{k}\frac{(-1)^{k-i+1}}{(k-i)!i!}\left(\frac{e^{-(r_0+(\gamma_2-i)\kappa)\tau}-1}{r_0+(\gamma_2-i)\kappa}\right)\right\}v^{\gamma_2-k} \tag{2.14}$$

for $v > 0$ and $\tau = T - t \geq 0$, where we define

$$c_0^{(\gamma)} = 1 \text{ and } c_k^{(\gamma)} = \prod_{l=1}^{k}(\gamma - l + 1)\left(\frac{1}{2}(\gamma - l)\sigma^2 + \kappa\theta\right) \text{ for } k = 1,2,\ldots \text{ and } \gamma \in \mathbf{R}$$

In particular, if $\gamma_1 = m_1$ and $\gamma_2 = m_2$ are non-negative integers then

$$U_C(v,\tau) = \sum_{j=0}^{\max(m_1,m_2)}\left\{d_j^{(m_1)}\frac{e^{-(r_0+m_1\kappa)\tau}}{(m_1-j)!}\left(\frac{e^{\kappa\tau}-1}{\kappa}\right)^{m_1-j} + d_j^{(m_2)}\frac{1}{\kappa^{m_2-j}}\sum_{i=0}^{m_2-j}\frac{(-1)^{m_2-j-i+1}}{(m_2-j-i)!i!}\left(\frac{e^{-(r_0+(m_2-i)\kappa)\tau}-1}{r_0+(m_2-i)\kappa}\right)\right\}v^j \tag{2.17}$$

for $v > 0$ and $\tau = T - t \geq 0$, where for any non-negative integer $N$, we define

$$d_j^{(N)} = 0 \text{ for } j > N, d_N^{(N)} = 1 \text{ and } d_j^{(N)} = \prod_{l=1}^{N-j}(N - l + 1)\left(\frac{1}{2}(N - l)\sigma^2 + \kappa\theta\right) \text{ for } j < N.$$

**Proof.** When $v_t$ follows the CIR process (1.1), the function on the LHS of (2.2) can be written as

$$U_C(v,\tau) = e^{-r_0\tau}U_C^{(\gamma_1)}(v,\tau) + \sum_{k=0}^{\infty}\left(\int_t^T A_{\gamma_2-k}(s-t)e^{-r_0(s-t)}ds\right)v^{\gamma_2-k} \tag{2.18}$$

where $U_C^{(\gamma_1)}(v,\tau)$ can be obtained using the closed-form formula (2.18) written in Theorem 2.3. by Rujivan (2016) with $\gamma = \gamma_1$ as

$$U_C^{(\gamma_1)}(v,\tau) = \sum_{k=0}^{\infty}\left\{c_k^{(\gamma_1)}\frac{e^{-\gamma_1\kappa\tau}}{k!}\left(\frac{e^{\kappa\tau}-1}{\kappa}\right)\right\}v^{\gamma_1-k}. \tag{2.19}$$

Therefore, we now obtain the first term on the RHS of (2.18). Next, we apply the binomial expansion to the term $(e^{\kappa u}-1)^k$ in order to compute the integral terms on the RHS of (2.18) as follows. For any $k = 0,1,\ldots,$

$$\begin{aligned}\int_t^T A_{\gamma_2-k}(s-t)e^{-r_0(s-t)}ds &= \int_0^\tau A_{\gamma_2-k}(u)e^{-r_0u}du \\ &= c_k^{(\gamma_2)}\frac{1}{k!\kappa^k}\int_0^\tau\left(e^{\kappa u}-1\right)^k e^{-(r_0+\gamma_2\kappa)u}du \\ &= c_k^{(\gamma_2)}\frac{1}{k!\kappa^k}\sum_{i=0}^{k}\frac{k!(-1)^{k-i}}{(k-i)!i!}\int_0^\tau e^{-(r_0+(\gamma_2-i)\kappa)u}du \\ &= c_k^{(\gamma_2)}\frac{1}{\kappa^k}\sum_{i=0}^{k}\frac{(-1)^{k-i+1}}{(k-i)!i!}\left(\frac{e^{-(r_0+(\gamma_2-i)\kappa)\tau}-1}{r_0+(\gamma_2-i)\kappa}\right).\end{aligned} \tag{2.20}$$

By analogy with the proof for obtaining (2.20) and (2.16), but using the closed-form formula (2.25) written Theorem 2.4. by Rujivan (2016), the closed-form formula (2.17) can be derived in a similar fashion.

## 3. Conclusions

This paper has proposed closed-form formulas for the conditional expectation of the valuation process, defined by $V_{t,T} := e^{-\int_t^T r(v_s)ds} f(v_T) + \int_t^T h(v_s) e^{-\int_t^s r(v_u)du} ds$ for $0 \le t \le T$, where $v_t$ is assumed to follow the CIR process (1.1) and extended CIR process (1.2), for $f(v) = v^{\gamma_1}$ and $h(v) = v^{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \mathbf{R}$, and any integrable function $r$. Moreover, we have provided a closed-form formula for the conditional expectation of $V_{t,T}$ when $f$ and $h$ are polynomial functions. Clearly, our results will be very useful to obtain a closed-form approximation for the conditional expectation of $V_{t,T}$ when $f$ and $h$ can be approximated by series of polynomial func-tions, which will be left to future research with results shown in a forthcoming paper.

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