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Original Article

Improvements of Poisson approximation for *n*-dimensional unit cube random graph

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Abstract

This paper uses the Stein-Chen method to obtain uniform and non-uniform bounds in the Poisson approximation for the *n*-dimensional unit cube random graph. These bounds are re-established under the restriction of Poisson mean $\lambda = 1$. One bound is for the distance of two probability functions and the two other bounds are for the distance of two cumulative distribution functions. Furthermore, the last two bounds of this study are sharper than those proposed in Teerapabolarn (2014).

Keywords: Poisson approximation, *n*-dimensional unit cube random graph, Stein-Chen method

1. Introduction

In the context of graph theory, a unit cube graph or a unit hypercube graph is a cube graph or a hypercube graph whose edges have length one unit. Consider the *n*-dimensional unit hypercube graph or n -dimensional unit cube $[0,1]^n$ graph with 2^n vertices, each of degree *n*, with an edge joining pairs of vertices that differ in exactly one coordinate. However, this study focuses on n -dimensional unit cube random graph by assuming that each of the $n2^{n-1}$ edges to be randomly and independently assigned an inward direction, or outward direction, with the probability of $\frac{1}{2}$, and the interesting point of this focusing is the number of vertices at which all *n* edges point inward, which is similar to that presented in Arratia, Goldstein and Gordon (1989).

Let Ω be the set of all 2^n vertices and $|\Omega|$ size of Ω and for each $i \in \Omega$, let I_i be the indicator random variable defined by

$$I_i = \begin{cases} 1 \text{ , if vertex } i \text{ has all of its edges directed inward,} \\ 0 \text{ , otherwise.} \end{cases} \tag{1.1}$$

The probability of $I_i = 1$ is

$$p_i = P(I_i = 1) = \tfrac{1}{2^n}. \tag{1.2}$$

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Let $X=\sum_{i\in\Omega} I_i$ be the number of vertices at which all n edges point inward. However, X is also a non-negative integer-valued random variable, that is, its value is not certainty. Therefore, the aim of this topic is to estimate the value of X via probability approximation to the distribution of X. It is noted that if n is large or all p_i are small, then the distribution of X can be approximated by a Poisson distribution with mean $\lambda=E(X)=1$. In the past few years, there have been some research papers related to the topic of Poisson approximation to the distribution of X, which can be found in Arratia *et al.* (1989), Teerapabolarn and Santiwipanont (2007) and Teerapabolarn (2014, 2015). Additionally, the obtained results of all authors mentioned above were determined by the Stein-Chen method. In the first research paper of this topic proposed by Arratia *et al.* (1989), they gave a uniform bound for approximating the distribution of X by a Poisson distribution in the form

$$d_A(X,Z)\le\frac{n}{2^n} \tag{1.3}$$

for every set $A\subseteq\{0,\dots,2^{n-1}\}$, where Z is the Poisson random variable with mean $\lambda=1$ and $d_A(X,Z)=|P(X\in A)-P(Z\in A)|$ is the distance between the distributions of X and Z. The bound in (1.3) was improved to be a non-uniform bound with the same method by Teerapabolarn and Santiwipanont (2007) as follows:

$$d_A(X,Z)\le\begin{cases}\frac{n}{2^n} & \text{, if } M_A\le 1,\\ \frac{en}{2^n(M_A+1)} & \text{, if } M_A\ge 2,\end{cases} \tag{1.4}$$

where $A\subseteq\{0,\dots,2^{n-1}\}$, $M_A=\begin{cases}\max\{x\mid C_x\subseteq A\} & \text{, if } 0\in A,\\ \min\{x\mid x\in A\} & \text{, if } 0\notin A\end{cases}$ and $C_x=\{0,\dots,x\}$. Later, Teerapabolarn (2015) improved the bound in (1.4) to be a sharper bound in the form

$$d_A(X,Z)\le\frac{n}{2^n}\min\left\{1-e^{-1},\tfrac{1}{x_A}\right\}, \tag{1.5}$$

where x_A is taken to be 1 when $x_A=0$ and for $x_A>0$, $x_A=\begin{cases}\max\{x\mid C_x\subseteq A\} & \text{, if } 0\in A,\\ \min\{x\mid x\in A\}-1, & \text{if } 0\notin A.\end{cases}$

It is noted that, for $A=\{x_0\mid x_0=0,\dots,2^{n-1}\}$, the result in (1.5) becomes

$$d_{x_0}(X,Z)\le\begin{cases}\frac{n(1-e^{-1})}{2^n} & \text{, if } x_0=0,1,2,\\ \frac{n}{2^n(x_0-1)} & \text{, if } x_0=3,\dots,2^{n-1},\end{cases} \tag{1.6}$$

where $d_{x_0}(X,Z)=|P(X=x_0)-P(Z-x_0)|$ is the distance between the probability functions of X and Z. In the case of $A=C_{x_0}$, $x_0\in\{0,\dots,2^{n-1}\}$, Teerapabolarn (2014) also gave a non-uniform bound on approximation of the cumulative distribution function of X by a Poisson cumulative distribution function with mean $\lambda=1$ as follows:

$$d_{C_0}(X,Z)\le\frac{e^{-1}n}{2^n} \tag{1.7}$$

and

$$d_{C_{x_0}}(X,Z)\le\frac{n}{2^n}\min\left\{1-e^{-1},\tfrac{2(e-2)}{x_0+1},\tfrac{1}{x_0}\right\}, \tag{1.8}$$

where $x_0\in\{1,\dots,2^{n-1}\}$. From (1.6) and (1.7), a uniform bound for the Komolgorov distance $d_K(X,Z)$ is of the form

$$d_K(X,Z)\le\frac{(1-e^{-1})n}{2^n}, \tag{1.9}$$

where $d_K(X,Z)=\sup_{0\le x_0\le 2^{n-1}}|P(X\le x_0)-P(Z\le x_0)|$.

In this paper, we also use the Stein-Chen method to obtain the following results: (i) a new non-uniform bound on Poisson approximation to the probability function of *X* by improving the bound in (1.6), (ii) new non-uniform and uniform bounds on Poisson approximation to the cumulative distribution function of *X* by improving the bounds in (1.8) and (1.9), respectively, and (iii) comparing all new bounds with the corresponding bounds in (1.6), (1.8) and (1.9).

2. Methods

The Stein-Chen method is the tool for determining the main results. In this study, the main idea of the method is to determine a new bound in (1.6) and improve the bounds in Teerapabolarn (2014, 2015) by setting $\lambda=1$ in all steps of this methodology as follows. The first step, we start with Stein's equation for Poisson distribution with mean $\lambda=1$, for given *h*,

$$h(x)-P_1(h)=f(x+1)-xf(x), \tag{2.1}$$

where $P_1(h)=e^{-1}\sum_{k=0}^{\infty}h(k)\frac{1}{k!}$ and f and h are bounded real valued functions defined on $\mathbb{N}\cup\{0\}$ (Chen, 1975). With the equation (2.1), the following steps are needed to give new bounds of the main results. For $A\subseteq\mathbb{N}\cup\{0\}$, let $h_A:\mathbb{N}\cup\{0\}\to\mathbb{R}$ be defined by

$$h_A(x)=\begin{cases}1\ , \text{ if } x\in A,\\ 0\ , \text{ if } x\notin A.\end{cases} \tag{2.2}$$

From Barbour, Holst and Janson (1992), the solution f_A of (2.1) can be expressed as

$$f_A(x)=\begin{cases}(x-1)!e[P_1(h_{A\cap C_{x-1}})-P_1(h_A)P_1(h_{C_{x-1}})]\ , \text{ if } x\ge 1,\\ 0\ , \text{ if } x=0,\end{cases} \tag{2.3}$$

where $x\in\mathbb{N}$. Similarly, for $A=\{x_0\}$ and $A=C_{x_0}$ as $x_0\in\mathbb{N}\cup\{0\}$, $f_{x_0}=f_{\{x_0\}}$ and $f_{C_{x_0}}$ can be expressed as

$$f_{x_0}(x)=\begin{cases}-\frac{(x-1)!}{x_0!}P_1(h_{C_{x-1}})\ , \text{ if } x\le x_0,\\ \frac{(x-1)!}{x_0!}P_1(1-h_{C_{x-1}}), \text{ if } x>x_0,\\ 0\ , \text{ if } x=0\end{cases} \tag{2.4}$$

and

$$f_{C_{x_0}}(x)=\begin{cases}(x-1)!e[P_1(h_{C_{x-1}})P_1(1-h_{C_{x_0}})]\ , \text{ if } x\le x_0,\\ (x-1)!e[P_1(h_{C_{x_0}})P_1(1-h_{C_{x-1}})]\ , \text{ if } x>x_0,\\ 0\ , \text{ if } x=0,\end{cases} \tag{2.5}$$

respectively. Let $\Delta f_{x_0}(x)=f_{x_0}(x+1)-f_{x_0}(x)$ and $\Delta f_{C_{x_0}}(x)=f_{C_{x_0}}(x+1)-f_{C_{x_0}}(x)$. The following lemma gives a non-uniform bound for $\left|\Delta f_{x_0}\right|$, with respect to the Poisson mean $\lambda=1$, that is used to obtain the first result in the next section.

Lemma 2.1. Let $x_0\in\mathbb{N}\cup\{0\}$, then we have the following:

$$\left|\Delta f_{x_0}(x)\right|\le\delta_1(x_0) \tag{2.6}$$

for every $x\in\mathbb{N}$, where

$$\delta_1(x_0)=\begin{cases} e^{-1} & , \text{ if } x_0=0, \\ 1-e^{-1} & , \text{ if } x_0=1, \\ \frac{1+e^{-1}}{3} & , \text{ if } x_0=2, \\ \frac{1}{x_0} & , \text{ if } x_0\geq 3. \end{cases} \tag{2.7}$$

Proof. It can be obtained from Teerapabolarn and Neammanee (2005) when $x_0=0.$ For $x_0\geq 1,$ it follows from Barbour *et al.* (1992) that f_{x_0} is a negative and decreasing function in $x\in\{1,...,x_0\}$ and f_{x_0} is a positive and increasing function in $x\in\{x_0+1,...\},$ we then have

$$\begin{aligned} \left|\Delta f_{x_0}(x)\right| &\leq f_{x_0}(x_0+1)-f_{x_0}(x_0) \\ &= P_1(1-h_{C_{x_0}})+\tfrac{1}{x_0}P_1(h_{C_{x_0-1}}) \text{ (by (2.4))} \\ &= e^{-1}\sum_{k=x_0+1}^{\infty}\tfrac{1}{k!}+\tfrac{e^{-1}}{x_0}\sum_{k=0}^{x_0-1}\tfrac{1}{k!}. \end{aligned} \tag{2.8}$$

Because

$$\begin{aligned} e^{-1}\sum_{k=x_0+1}^{\infty}\tfrac{1}{k!}+\tfrac{e^{-1}}{x_0}\sum_{k=0}^{x_0-1}\tfrac{1}{k!} &\leq e^{-1}\left[\sum_{k=x_0+1}^{\infty}\tfrac{1}{k!}+\sum_{k=0}^{x_0-1}\tfrac{1}{(k+1)!}\right] \\ &= \left[\sum_{k=1}^{x_0}\tfrac{e^{-1}}{k!}+\sum_{k=x_0+1}^{\infty}\tfrac{e^{-1}}{k!}\right] \\ &= 1-e^{-1} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} e^{-1}\sum_{k=x_0+1}^{\infty}\tfrac{1}{k!}+\tfrac{e^{-1}}{x_0}\sum_{k=0}^{x_0-1}\tfrac{1}{k!} &\leq \tfrac{e^{-1}}{x_0+1}\sum_{k=x_0}^{\infty}\tfrac{1}{k!}+\tfrac{e^{-1}}{x_0}\sum_{k=0}^{x_0-1}\tfrac{1}{k!} \\ &= \tfrac{1}{x_0+1}\left(1-\sum_{k=0}^{x_0-1}\tfrac{e^{-1}}{k!}\right)+\tfrac{1}{x_0}\sum_{k=0}^{x_0-1}\tfrac{e^{-1}}{k!} \\ &= \tfrac{1}{x_0+1}\left(1+\tfrac{1}{x_0}\sum_{k=0}^{x_0-1}\tfrac{e^{-1}}{k!}\right). \end{aligned} \tag{2.10}$$

Hence, from (2.8)-(2.10), we obtain

$$\left|\Delta f_{x_0}(x)\right| \leq \min\left\{1-e^{-1},\tfrac{1}{x_0+1}\left(1+\tfrac{1}{x_0}\sum_{k=0}^{x_0-1}\tfrac{e^{-1}}{k!}\right)\right\}$$

for every $x\in\mathrm{N},$ which yields (2.6).

The bound in (2.6) is used to determine a new non-uniform bound for approximating the probability function of *X* by a Poisson probability function. In the next step, we improve the non-uniform bound of $\left|\Delta f_{C_{x_0}}\right|$ with the Poisson mean $\lambda=1$.

Lemma 2.2. Let $x,x_0\in\mathrm{N},$ then $\Delta f_{C_{x_0}}$ is an increasing function in $x\in\{x_0+1,...\}.$

Proof. We have to show that $\Delta f_{C_{x_0}}(x+1)-\Delta f_{C_{x_0}}(x)<0$ for every $x\in\{x_0+1,...\}.$ It follows from (2.5) that

$$\begin{aligned} \Delta f_{C_{x_0}}(x+1)-\Delta f_{C_{x_0}}(x) &= x!eP_1(h_{C_{x_0}})[(x+1)P_1(1-h_{C_{x+1}})-P_1(1-h_{C_x})] \\ &\quad -(x-1)!eP_1(h_{C_{x_0}})[xP_1(1-h_{C_x})-P_1(1-h_{C_{x-1}})] \end{aligned}$$

$$= (x-1)!\sum_{k=0}^{x_0}\tfrac{e^{-1}}{k!}\left[x\sum_{j=x+2}^{\infty}(x+1-j)\tfrac{1}{j!}+\sum_{j=x+1}^{\infty}(x-j)\tfrac{1}{j!}\right]$$
$$= (x-1)!\sum_{k=0}^{x_0}\tfrac{e^{-1}}{k!}\left[x\sum_{j=x+2}^{\infty}(x+1-j)\tfrac{1}{j!}+\sum_{j=x+2}^{\infty}(x+1-j)j\tfrac{1}{j!}\right]$$
$$= (x-1)!\sum_{k=0}^{x_0}\tfrac{e^{-1}}{k!}\left[\sum_{j=x+2}^{\infty}(x-j)(x+1-j)\tfrac{1}{j!}\right]$$
$$> 0.$$

Hence, $\Delta f_{C_{x_0}}$ *is an increasing function in* $x\in\{x_0+1,\ldots\}$.

Lemma 2.3. For $x_0\in\mathrm{N}\cup\{0\}$, then the following inequalities hold:

$$\left|\Delta f_{C_{x_0}}(x)\right|\le\delta_2(x_0) \tag{2.11}$$

for every $x\in\mathrm{N}$, where

$$\delta_2(x_0)=\begin{cases}e^{-1} & ,\ if\ x_0=0,\\ 1-2e^{-1} & ,\ if\ x_0=1,\\ 3(1-2.5e^{-1}) & ,\ if\ x_0=2,\\ \frac{1}{x_0+1} & ,\ if\ x_0\ge 3\end{cases} \tag{2.12}$$

and

$$\sup_{x_0\ge 0}\left|\Delta f_{C_{x_0}}(x)\right|\le e^{-1}. \tag{2.13}$$

Proof. First, we shall show that (2.11) holds for $x_0\ge 1$. Following Teerapabolarn (2008), we have that

$$\Delta f_{C_{x_0}}(x)=\begin{cases}(x-1)!eP_1(1-h_{C_{x_0}})[xP_1(h_{C_x})-P_1(h_{C_{x-1}})]>0 & ,\ \text{if}\ x\le x_0,\\ (x-1)!eP_1(h_{C_{x_0}})[xP_1(1-h_{C_x})-P_1(1-h_{C_{x-1}})]<0 & ,\ \text{if}\ x>x_0.\end{cases} \tag{2.14}$$

From (2.14), by using Teerapabolarn (2014) and Lemma 2.2, it follows that $\Delta f_{C_{x_0}}$ is a positive and increasing function in $x\in\{1,\ldots,x_0\}$ and $\Delta f_{C_{x_0}}$ is a negative and increasing function in $x\in\{x_0+1,\ldots\}$. Thus, for $x\le x_0$, we have

$$\begin{aligned}\left|\Delta f_{C_{x_0}}(x)\right| &\le (x_0-1)!eP_1(1-h_{C_{x_0}})[x_0P_1(h_{C_{x_0}})-P_1(h_{C_{x_0-1}})]\\ &=(x_0-1)!e\sum_{j=x_0+1}^{\infty}\tfrac{e^{-1}}{j!}\left[x_0\sum_{k=0}^{x_0}\tfrac{e^{-1}}{k!}-\sum_{k=0}^{x_0-1}\tfrac{e^{-1}}{k!}\right]\\ &=(x_0-1)!e\sum_{j=x_0+1}^{\infty}\tfrac{e^{-1}}{j!}\left[(x_0-1)\sum_{k=0}^{x_0}\tfrac{e^{-1}}{k!}+\tfrac{e^{-1}}{x_0!}\right]\end{aligned} \tag{2.15}$$

and for $x>x_0$, we obtain

$$\left|\Delta f_{C_{x_0}}(x)\right|\le x_0!eP_1(h_{C_{x_0}})[P_1(1-h_{C_{x_0}})-(x_0+1)P_1(1-h_{C_{x_0+1}})]$$

$$= x_0!e\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left[\sum_{k=x_0+1}^{\infty}\frac{e^{-1}}{k!}-(x_0+1)\sum_{k=x_0+2}^{\infty}\frac{e^{-1}}{k!}\right]$$

$$= x_0!\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\sum_{k=x_0+2}^{\infty}(k-x_0-1)\tfrac{1}{k!}.$$

$$= \frac{1}{(x_0+1)(x_0+2)}\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left\{1+\frac{2}{x_0+3}+\frac{3}{(x_0+3)(x_0+4)!}+\cdots\right\}$$

$$\le \frac{1}{(x_0+1)(x_0+2)}\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left\{1+\frac{1}{2}+\frac{1}{3!}+\cdots\right\}$$

$$= \frac{e-1}{(x_0+1)(x_0+2)}\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}. \tag{2.16}$$

From which, it follows that for $x_0=1$, $\left|\Delta f_{C_{x_0}}(x)\right|\le\max\left\{1-2e^{-1},\frac{1-e^{-1}}{3}\right\}=1-2e^{-1}$, for $x_0=2$, $\left|\Delta f_{C_{x_0}}(x)\right|\le\max\left\{3\left(1-2.5e^{-1}\right),\frac{2.5(1-e^{-1})}{12}\right\}=3\left(1-2.5e^{-1}\right)$ and for $x_0\ge 3$, from inequality (2.15), we have

$$\left|\Delta f_{C_{x_0}}(x)\right|\le\frac{(x_0-1)x_0!+e^{-1}}{x_0}\sum_{j=x_0+1}^{\infty}\frac{1}{j!}$$

$$=\frac{(x_0-1)x_0!+e^{-1}}{x_0}\left\{1+\frac{1}{x_0+2}+\frac{1}{(x_0+2)(x_0+3)}+\cdots\right\}$$

$$\le\frac{(x_0-1)x_0!+e^{-1}}{x_0}\left\{1+\frac{1}{x_0+2}+\frac{1}{(x_0+2)^2}+\cdots\right\}$$

$$=\frac{\left[(x_0-1)x_0!+e^{-1}\right](x_0+2)}{x_0(x_0+1)(x_0+1)!}$$

$$=\frac{1}{x_0+1}\left\{\frac{(x_0-1)(x_0+2)}{x_0(x_0+1)}+\frac{x_0+2}{ex_0(x_0+1)!}\right\}$$

$$=\frac{1}{x_0+1}\left\{1+\frac{x_0+2}{ex_0(x_0+1)!}-\frac{2}{x_0(x_0+1)}\right\}$$

$$\le\frac{1}{x_0+1}. \tag{2.17}$$

The last term of (2.17) is obtained from the fact that $\frac{\frac{x_0+2}{ex_0(x_0+1)!}}{\frac{2}{x_0(x_0+1)}}=\frac{x_0+2}{2ex_0!}<1$ for every $x_0\in\mathbb{N}$, this yields $\frac{x_0+2}{ex_0(x_0+1)!}-\frac{2}{x_0(x_0+1)}<0$. Thus, from the both inequalities in (2.16) and (2.17), we have $\left|\Delta f_{C_{x_0}}(x)\right|\le\max\left\{\frac{e-1}{(x_0+1)(x_0+2)}\sum_{j=0}^{x_0}\frac{e^{-1}}{j!},\frac{1}{x_0+1}\right\}=\frac{1}{x_0+1}$, which implies that (2.11) holds. For the inequality (2.13), by following (2.11), a uniform bound for the supremum of $\left|\Delta f_{C_{x_0}}(x)\right|$ overall steps $x_0\ge 0$ is $\max\left\{e^{-1},1-2e^{-1},3(1-2.5e^{-1}),\frac{1}{4},\frac{1}{5},\ldots\right\}=e^{-1}$, which gives the inequality (2.13).

The bounds in (2.11) and (2.13) are used to determine new non-uniform and uniform bounds for approximating the cumulative distribution function of X by a Poisson cumulative distribution with mean $\lambda=1$.

3. Main Results

The desired results of this study are uniform and non-uniform bounds for $d_{x_0}\left(X,Z\right)$, $d_{C_{x_0}}\left(X,Z\right)$ and $d_K\left(X,Z\right)$, respectively. The following theorem gives a new non-uniform bound for $d_{x_0}\left(X,Z\right)$.

Theorem 3.1. Let $x_0 \in \left\{0,...,2^{n-1}\right\}$, then we have the following:

$$d_{x_0}\left(X,Z\right) \le \frac{\delta_1(x_0)n}{2^n}, \tag{3.1}$$

where $\delta_1(x_0)$ is defined in (2.7).

Proof. Let $W_i = \sum_{j\in B_i\setminus\{i\}} I_j$, $Y_i = X - I_i - W_i = \sum_{j\notin B_i} I_j$ and $X_i = X - I_i$. Substituting x and h by X and $h_{\{x_0\}}$ respectively, and taking expectation to (2.1), we obtain

$$d_{x_0}\left(X,Z\right) = \left|E\left[f(X+1) - Xf(X)\right]\right|, \tag{3.2}$$

where $f = f_{x_0}$ is defined in (2.4). Because each I_i takes a value on 0 and 1, we have

$$\begin{aligned} E[Xf(X)] &= \sum_{i\in\Omega} E[I_i f(X_i+1)] \\ &= \sum_{i\in\Omega} E[I_i f(Y_i+1)] + \sum_{i\in\Omega} E[I_i(f(W_i+Y_i+1) - f(Y_i+1))] \end{aligned}$$

and

$$\begin{aligned} E\left[f(X+1) - Xf(X)\right] &= \sum_{i\in\Omega} E[p_i f(X+1)] - \sum_{i\in\Omega} E[I_i f(Y_i+1)] \\ &\quad -\sum_{i\in\Omega} E[I_i(f(W_i+Y_i+1) - f(Y_i+1))] \\ &= \sum_{i\in\Omega}\Big\{E\left[p_i(f(X+1) - f(Y_i+1))\right] \\ &\quad -E\left[I_i(f(W_i+Y_i+1) - f(Y_i+1))\right] + E\left[p_i f(Y_i+1) - I_i f(Y_i+1)\right]\Big\}. \end{aligned}$$

Therefore, from (3.2), we obtain

$$\begin{aligned} d_{x_0}\left(X,Z\right) \le \sum_{i\in\Omega}\Big|&E\left[p_i(f(X+1) - f(Y_i+1))\right] - E\left[I_i(f(W_i+Y_i+1) - f(Y_i+1))\right] \\ &+ E\left[p_i f(Y_i+1) - I_i f(Y_i+1)\right]\Big|. \end{aligned} \tag{3.3}$$

Following Arratia *et al.* (1989), for each $i\in\Omega$, we set $B_i = \{j\in\Omega : |i-j| = 1\} \subset \Omega$ to be the neighborhood of i such that I_i and I_j are independent for every $j \notin B_i$. It implies that I_i is independent of I_j with $|i-j| > 1$. For $|i-j| = 1$, because $P(I_i = 1, I_j = 1) = 0$, it gives $E(I_i I_j) = 0$. From these facts, we have $E\left[p_i f(Y_i+1) - I_i f(Y_i+1)\right] = 0$ and $E\left[I_i(f(W_i+Y_i+1) - f(Y_i+1))\right] = 0$, then the result in (3.3) becomes

$$\begin{aligned} d_{x_0}\left(X,Z\right) &\le \sum_{i\in\Omega} E\left|p_i(f(X+1) - f(Y_i+1))\right| \\ &\le \sup_{x\ge 1}\left|\Delta f(x)\right| \sum_{i\in\Omega} p_i E[I_i + W_i] \\ &\le \delta_1(x_0) \sum_{i\in\Omega} p_i E[I_i + W_i]. \quad \text{(by (2.6))} \end{aligned}$$

Because $\sum_{i\in\Omega} p_i E[I_i + W_i] = 2^n np_i^2 = \frac{n}{2^n}$, the result in (3.1) is obtained.

New non-uniform and uniform bounds for $d_{C_{x_0}}\left(X,Z\right)$ and $d_K\left(X,Z\right)$ are also obtained as follows:

Theorem 3.2. With the above definitions, we have the following inequalities:

$$d_{C_{x_0}}\left(X,Z\right) \le \frac{\delta_2(x_0)n}{2^n}, \tag{3.4}$$

where $x_0 \in \{1,...,2^{n-1}\}$ and $\delta_2(x_0)$ is defined in (2.12), and

$$d_K\left(X,Z\right) \le \frac{e^{-1}\, n}{2^n}. \tag{3.5}$$

Proof. Let $W_i = \sum_{j\in B_i\setminus\{i\}} I_j$, $Y_i = X - I_i - W_i = \sum_{j\notin B_i} I_j$ and $X_i = X - I_i$. Substituting X and h by X and $h_{C_{x_0}}$ respectively, and taking expectation to (2.1), we obtain

$$d_{C_{x_0}}\left(X,Z\right) = \left|E\left[f_{C_{x_0}}(X+1) - Xf_{C_{x_0}}(X)\right]\right|.$$

Using the same arguments detailed in the proof of Theorem 3.1, we have

$$\begin{aligned} d_{C_{x_0}}\left(X,Z\right) &\le \sup_{x\ge1}\left|\Delta f(x)\right|\sum_{i\in\Omega} p_i E[I_i + W_i] \\ &= \sup_{x\ge1}\left|\Delta f(x)\right|\frac{n}{2^n} \\ &\le \frac{\delta_2(x_0)n}{2^n} \quad \text{(by (2.11))} \end{aligned}$$

and

$$\begin{aligned} d_K\left(X,Z\right) &\le \sup_{0\le x_0\le 2^{n-1},x\ge1}\left|\Delta f(x)\right|\sum_{i\in\Omega} p_i E[I_i + W_i] \\ &\le \frac{e^{-1}\, n}{2^n}, \quad \text{(by (2.13))} \end{aligned}$$

which gives both results (3.4) and (3.5).

Remark. 1. The results in Theorems 3.1 and 3.2 give accurate Poisson approximations when n is large.

2. Because $e^{-1} < 1 - e^{-1}$, $\frac{1+e^{-1}}{3} < 1 - e^{-1}$ for $x_0 = 0, 2$ and $\frac{1}{x_0} < \frac{1}{x_0 - 1}$ for $x_0 = 3,...,2^{n-1}$, the bound in Theorem 3.1 is sharper than that expressed in (1.6).

3. Because $\delta_2(x_0) < \min\left\{1 - e^{-1}, \frac{2(e-2)}{x_0+1}, \frac{1}{x_0}\right\}$ and $e^{-1} < 1 - e^{-1}$, the bounds in Theorem 3.2 are sharper than those proposed in (1.8) and (1.9) that are in Teerapabolarn (2014).

4. Numerical Examples

We give two numerical examples for each bound on Poisson approximation of the results in Theorems 3.1 and 3.2. We compare the bounds in Theorems 3.1 and 3.2 with the corresponding bounds in (1.6), (1.8) and (1.9). For percentages of improvement for each comparison, we use $\frac{\text{old bound} - \text{new bound}}{\text{old bound}} \times 100\%$, denoted by improvement (%), to be the formula for measuring each improvement of the approximation.

Example 4.1. For *n = 4,* non-uniform and uniform bounds together with percentages of each improvement on Poisson approximation to the distribution of the number of vertices at which all 4 edges point inward are presented in the following table.

Table 1.

X_0	Non-uniform bounds						Uniform bounds		
	(1.6)	(3.1)	Improvement (%)	(1.8)	(3.4)	Improvement (%)	(1.9)	(3.5)	Improvement (%)
0	0.158030	0.091970	41.80	0.091970	0.091970	0			
1	0.158030	0.158030	0	0.158030	0.066060	58.20			
2	0.158030	0.113990	27.87	0.119714	0.060226	49.69			
3	0.125000	0.083333	33.33	0.083333	0.062500	25.00			
4	0.083333	0.062500	25.00	0.062500	0.050000	20.00	0.158030	0.091970	41.80
5	0.062500	0.050000	20.00	0.050000	0.041667	16.67			
6	0.050000	0.041667	16.67	0.041667	0.035714	14.29			
7	0.041667	0.035714	14.29	0.035714	0.031250	12.50			
8	0.035714	0.031250	12.50	0.031250	0.027778	11.11			

Example 4.2. For *n = 6,* non-uniform and uniform bounds together with percentages of each improvement on Poisson approximation to the distribution of the number of vertices at which all 6 edges point inward are as follows:

Table 2.

X_0	Non-uniform bounds						Uniform bounds		
	(1.6)	(3.1)	Improvement (%)	(1.8)	(3.4)	Improvement (%)	(1.9)	(3.5)	Improvement (%)
0	0.059261	0.034489	41.80	0.034489	0.034489	0			
1	0.059261	0.059261	0	0.059261	0.024773	58.20			
2	0.059261	0.042746	27.87	0.044893	0.022585	49.69			
3	0.046875	0.031250	33.33	0.031250	0.023438	25.00			
4	0.031250	0.023438	25.00	0.023438	0.018750	20.00			
5	0.023438	0.018750	20.00	0.018750	0.015625	16.67			
6	0.018750	0.015625	16.67	0.015625	0.013393	14.28			
7	0.015625	0.013393	14.28	0.013393	0.011719	12.50			
8	0.013393	0.011719	12.50	0.011719	0.010417	11.11			
9	0.011719	0.010417	11.11	0.010417	0.009375	10.00			
10	0.010417	0.009375	10.00	0.009375	0.008523	9.09			
11	0.009375	0.008523	9.09	0.008523	0.007813	8.33			
12	0.008523	0.007813	8.33	0.007813	0.007212	7.69			
13	0.007813	0.007212	7.69	0.007212	0.006696	7.15			
14	0.007212	0.006696	7.15	0.006696	0.006250	6.66			
15	0.006696	0.006250	6.66	0.006250	0.005859	6.26			
16	0.006250	0.005859	6.26	0.005859	0.005515	5.87			
17	0.005859	0.005515	5.87	0.005515	0.005208	5.57	0.059261	0.034489	41.80
18	0.013393	0.011719	12.50	0.011719	0.010417	11.11			
19	0.005208	0.004934	5.26	0.004934	0.004688	4.99			
20	0.004934	0.004688	4.99	0.004688	0.004464	4.78			
21	0.004688	0.004464	4.78	0.004464	0.004261	4.55			
22	0.004464	0.004261	4.55	0.004261	0.004076	4.34			
23	0.004261	0.004076	4.34	0.004076	0.003906	4.17			
24	0.004076	0.003906	4.17	0.003906	0.003750	3.99			
25	0.003906	0.003750	3.99	0.003750	0.003606	3.84			
26	0.003750	0.003606	3.84	0.003606	0.003472	3.72			
27	0.003606	0.003472	3.72	0.003472	0.003348	3.57			
28	0.003472	0.003348	3.57	0.003348	0.003233	3.43			
29	0.003348	0.003233	3.43	0.003233	0.003125	3.34			
30	0.003233	0.003125	3.34	0.003125	0.003024	3.23			
31	0.003125	0.003024	3.23	0.003024	0.002930	3.11			
32	0.003024	0.002930	3.11	0.002930	0.002841	3.04			

The numerical results in Examples 4.1 and 4.2 are indicated that each Poisson approximation to be quite efficient when *n* is large, which satisfies the remark mentioned above. By comparing the new and old bounds, see the three columns of improvement (%) in Tables 1 and 2, it is seen that the non-uniform bound in Theorem 3.1 is sharper that reported in (1.6) and the bounds in Theorem 3.2 are sharper than those proposed in (1.8) and (1.9) or in Teerapabolarn (2014).

5. Conclusions

The uniform and non-uniform bounds in the Poisson approximation for the *n*-dimensional unit cube random graph were re-established under the Poisson mean $\lambda = 1$ by using the Stein-Chen method. All results of this study give three improvements of Poisson approximation as follows:

(i). For probability function approximation, the new bound is sharper than the old bound with percentages of improvement 41.8%, 0%, 27.87% for $x_0 = 0,1,2$ and $\frac{100\%}{x_0}$ for $x_0 \geq 3$.

(ii). For cumulative probability approximation, the new non-uniform bound is sharper than the old non-uniform bound with percentages of improvement 58.20%, 49.69% for $x_0 = 1,2$ and $\frac{100\%}{x_0+1}$ for $x_0 \geq 3$. In addition, the new uniform bound is also sharper than the old uniform bound with percentage of improvement 41.80%.

Moreover, for cumulative probability approximation, the two bounds in this study, non-uniform and uniform bounds, are sharper than those proposed in Teerapabolarn (2014), including both theoretical and numerical results.

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