Songklanakarin J. Sci. Technol.
43 (4), 1140-1151, Jul. - Aug. 2021

Original Article

The zero-truncated discrete transmuted generalized inverse Weibull distribution and its applications

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Received: 11 April 2020; Revised: 11 July 2020; Accepted: 28 August 2020

Abstract

In this article, a new distribution for count data analysis is introduced. Firstly, the Discrete Transmuted Generalized Inverse Weibull distribution (DTGIW) is constructed. Consequently, some useful sub-models are discussed. Secondly, the Zero-Truncated Discrete Transmuted Generalized Inverse Weibull distribution (ZT-DTGIW) is introduced. We present probability mass function of the proposed distribution and some plots of those functions for illustration the behaviors of the distribution. We employed the maximum likelihood estimation (MLE) technique for model parameter estimation. For the purpose of verification of the MLE performance, the simulation study of parameter estimation using MLE is illustrated. Finally, some real data sets are applied to illustrate the goodness of fit of the proposed distribution, which is compared with the zero-truncated discrete inverse Weibull and zero-truncated Poisson distributions.

Keywords: zero-truncated distribution, discrete distribution, DTGIW, ZT-DTGIW, maximum likelihood, count data, zero-truncated data

1. Introduction

In probability theory, zero-truncated distributions are certain discrete distributions whose support is a set of positive integers. When the data to be modeled are generated excluding zero counts, zero-truncated distributions are more suitable than discrete distribution with zero counts. A typical example where zero-truncated discrete distributions are useful comes from medical science: modeling or studying duration of hospital stays in days where each patient's stay will be recorded for at least one day. In ecology, zero-truncated discrete distributions are used to model data relating to the counts, such as the number of flower heads, fly eggs, European red mites, or the number times Hares Caught of snowshoe hares captured over seven days. In sociology, these distributions are used for modeling data such as the group size of humans at park, beach or public places. Thus, zero-truncated distributions have applications in almost every branch of knowledge including biological science, medical science, psychology, demography, political science, etc (Shanker, 2017; Shanker & Shukla, 2017).

In 1960, a Zero-Truncated Poisson (ZT-P) distribution was proposed by Cohen (1960). Many researchers proposed zero-truncated distributions based on discrete distributions, i.e., the Zero-Truncated Negative Binomial (Arrabal, dos Santos Silva, & Bandeira, 2014), the Zero-Truncated Poisson-Garima (Shanker & Shukla, 2017), the Zero-Truncated Poisson-Amarendra (Shanker, 2017).

In this paper, a new zero-truncated distribution is proposed. Firstly, a new discrete distribution is introduced. It is obtained from a discretized continuous distribution based on the Transmuted Generalized Inverse Weibull (TGIW) distribution developed by Merovci, Elbatal, and Ahmed (2014), namely the Discrete Transmuted Generalized Inverse Weibull (DTGIW) distribution. Some useful sub-models of the DTGIW distribution are discussed. Secondly, the zero-truncated version of the new discrete distribution is proposed. The probability mass function (pmf) of the proposed distributions and plot some of those functions for illustration the behaviors of the distributions. Consequently, some useful

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sub-models are discussed. We employed the maximum likelihood estimation (MLE) technique for model parameter estimation. The simulation study of parameter estimation is illustrated for verification of the MLE performance before the real data analysis. Finally, some real data sets are applied to illustrate the goodness of fit of the proposed distributions, which is compared with other distributions.

2. The Discrete Transmuted Generalized Inverse Weibull Distribution

In practice, we frequently come across variables that are discrete in nature. Generally one associates the lifetime of the product with continuous non-negative lifetime distributions, however, in some situations, the lifetime can be best described through non-negative integer-valued random variables e.g. life of the equipment is measured by the number of cycles completes or the number of times it is operated prior to failure, life of a weapon is measured by the number of rounds fired until failure (Hussain & Ahmad, 2014). For example, in survival analysis, one may be interested in recording the number of days that a patient has survived since therapy or the number of days from remission to relapse. In these cases, the lifetimes are not measured on a continuous; because they are counted, they are discrete random variables. Not many of the known discrete distributions can provide accurate models for both times and counts (Alamatsaz, Dey, Dey, & Harandi, 2016). For example, the Poisson distribution is used to model counts but not times. The negative binomial distribution is not considered to be a good model for reliability, failure times, counts, etc. Discretization of a continuous lifetime model is an interesting and intuitively appealing approach to derive a lifetime model corresponding to the continuous one (Jayakumar & Sankaran, 2018). This has led to the development of new discrete distributions based on a continuous model for reliability, failure times, etc. (Alamatsaz *et al.*, 2016).

First, we provide a general definition of the proposed distribution that will subsequently reveal its probability function (Alamatsaz *et al.*, 2016; Roy, 2003, 2004) as in Theorem 1.

Definition 1. Let X be a random variable which has distributed a lifetime distribution with the cumulative density function (cdf) $G(x).$ We have the pmf of the discretized lifetime distribution as

$$f(x) = S(x) - S(x+1), x = 0,1,2,... \tag{1}$$

where $S(x)$ is a survival function of X i.e., $S(x) = 1 - G(x).$

In this section, we proposed a new discrete distribution for modelling count data based on the TGIW distribution. Let X be a TGIW random variable with the cdf as follows:

$$G_{TGIW}(x) = \exp\left[-\lambda(\beta x)^{-\alpha}\right]\left\{1+\theta-\theta\exp\left[-\lambda(\beta x)^{-\alpha}\right]\right\}, x > 0 \tag{2}$$

where the parameters $\alpha, \beta, \lambda > 0$ and $-1 \le \theta \le 1.$ The TGIW distribution is a very flexible model that approaches different distributions when its parameters are changed. It has eight sub-models as follows (Merovci *et al.*, 2014). (a) If $\lambda = 1$, they obtain the Transmuted Inverse Weibull (TIW) distribution (Khan, King & Hudson, 2013). (b) If $\theta = 0$ and $\lambda = 1,$ the TGIW distribution reduces to the Inverse Weibull (IW) distribution (Khan, Pasha, & Pasha, 2008). (c) If $\alpha = 1$ and $\lambda = 1,$ the TGIW distribution refers to the Transmuted Inverse Exponential (TIE) distribution (Oguntunde & Adejumo, 2015). (d) If $\theta = 0, \alpha = 1$ and $\lambda = 1,$ they get the Inverse Exponential (IE) distribution (Keller, Kamath, & Perera, 1982). (e) If $\alpha = 2$ and $\lambda = 1,$ they have the Transmuted Inverse Rayleigh (TIR) distribution (Ahmad, Ahmad, & Ahmed, 2014). (f) If $\theta = 0, \alpha = 2$ and $\lambda = 1,$ they get the Inverse Rayleigh (IR) distribution (Voda, 1972). (g) If $\beta = 1$ they get the Transmuted Fréchet (TF) distribution (Geetha & Poongothai, 2016). Finally, (h) if $\beta = 1$ and $\theta = 0$ they get the Fréchet (F) distribution which was developed by Maurice Fréchet in 1927 (Oguntunde, Khaleel, Ahmed, & Okagbue, 2019).

Next, in Theorem 1, we show a new distribution for modelling count data called the Discrete Transmuted Generalized Inverse Weibull (DTGIW) distribution.

Theorem 1. Let X be a random variable that has the DTGIW distribution with the parameters α, β, λ and $\theta,$ which will be denoted by X~DTGIW $(\alpha, \beta, \lambda, \theta)$. Then the pmf of X is

$$\begin{aligned} f_{DTGIW}(x) = (1+\theta)&\left\{\exp\left[-\lambda(\beta x+\beta)^{-\alpha}\right] - \exp\left[-\lambda(\beta x)^{-\alpha}\right]\right\} \\ -\theta&\left\{\exp\left[-2\lambda(\beta x+\beta)^{-\alpha}\right] - \exp\left[-2\lambda(\beta x)^{-\alpha}\right]\right\}, \end{aligned} \tag{3}$$

where $x = 0,1,2,..., \alpha, \beta, \lambda > 0$ and $-1 \le \theta \le 1.$

Proof. From Definition 1 and $G_{TGIW}(x)$ in equation (2), we replace $S(x) = 1 - G_{TGIW}(x)$ as in equation (1). Then, the pmf of X as follows:

$$
\begin{aligned}
f_{DTGIW}(x) &= \exp\left[-\lambda(\beta x+\beta)^{-\alpha}\right]\left\{1+\theta-\theta\exp\left[-\lambda(\beta x+\beta)^{-\alpha}\right]\right\} \\
&\quad -\exp\left[-\lambda(\beta x)^{-\alpha}\right]\left\{1+\theta-\theta\exp\left[-\lambda(\beta x)^{-\alpha}\right]\right\} \\
&= (1+\theta)\left\{\exp\left[-\lambda(\beta x+\beta)^{-\alpha}\right]-\exp\left[-\lambda(\beta x)^{-\alpha}\right]\right\} \\
&\quad -\theta\left\{\exp\left[-2\lambda(\beta x+\beta)^{-\alpha}\right]-\exp\left[-2\lambda(\beta x)^{-\alpha}\right]\right\}.
\end{aligned}
$$

Figure 1 illustrates the pmf behaviors of the DTGIW distribution for several values of α, β, λ and $\theta.$ The DTGIW pmf has various behaviors, such as a reverse J-shaped distribution (Figure 1: (a)-(b)) and a unimodal distribution (Figure 1: (d)-(f)), and a right-skewed shape (Figure 1: (a)-(f)). In addition, we have the sub-models of the DTGIW distribution as follows:

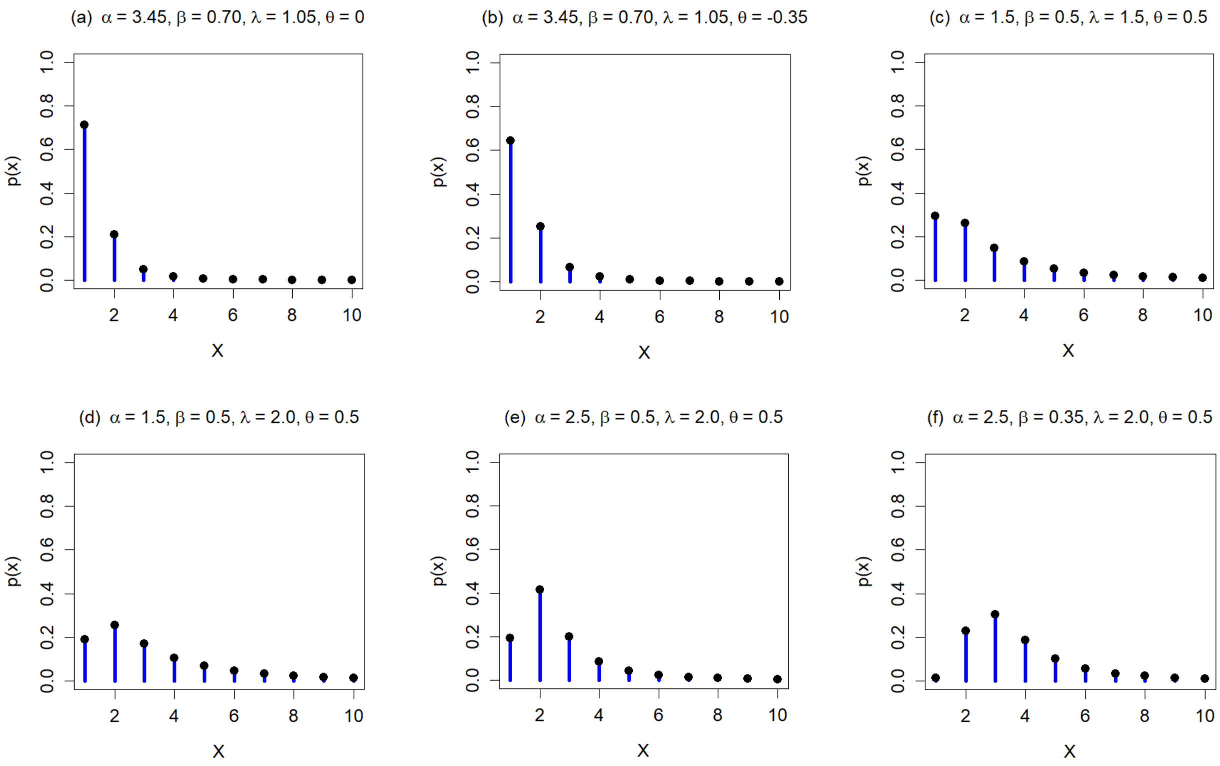

Figure 1. The pmf plot of X~DTGIW $(\alpha, \beta, \lambda, \theta)$ with the specified parameters

Corollary 1. If X~DTGIW $(\alpha, \beta, \lambda, \theta)$ and $\lambda = 1,$ then it reduces to the Discrete Transmuted Inverse Weibull (DTIW) distribution with pmf

$$
\begin{aligned}
f_{DTIW}(x) &= (1+\theta)\left\{\exp\left[-(\beta x+\beta)^{-\alpha}\right]-\exp\left[-(\beta x)^{-\alpha}\right]\right\} \\
&\quad -\theta\left\{\exp\left[-2(\beta x+\beta)^{-\alpha}\right]-\exp\left[-2(\beta x)^{-\alpha}\right]\right\},
\end{aligned}
$$

where $x = 0, 1, 2, ..., \alpha, \beta > 0$ and $-1 \le \theta \le 1.$

Proof. We get the DTIW pmf by replacing $\lambda = 1$ as in equation (3). In the same way, it is obtained by replacing the TIW survival function, i.e., $S_{TIW} = 1-(1+\theta)\exp\left[-(\beta_0/x)^{\alpha}\right] + \theta\exp\left[-2(\beta_0/x)^{\alpha}\right], \beta_0 = 1/\beta > 0$ as in equation (1), where the TIW distribution was proposed by Khan *et al.* (2013).

Corollary 2. If X~DTGIW $(\alpha, \beta, \lambda, \theta)$ when $\theta = 0$ and $\lambda = 1,$ then it reduces to the Discrete Inverse Weibull (DIW) distribution with the pmf as follows

$$
f_{DIW}(x) = \exp\left[-(\beta x+\beta)^{-\alpha}\right]-\exp\left[-(\beta x)^{-\alpha}\right], x = 0, 1, 2, ..., \alpha, \beta > 0
$$

where the DIW distribution was proposed by Jazi, Lai, and Alamatsaz (2010).

Proof. The pmf of the DIW distribution is obtained by replacing $\theta = 0$ and $\lambda = 1$ as in equation (3), or by replacing the IW survival function of $S_{IW} = 1 - \exp\left[-(\beta_0/x)^{\alpha}\right]$, $\beta_0 = 1/\beta > 0$ as in equation (1), where the IW distribution was proposed by Khan *et al.* (2008).

Corollary 3. If X~DTGIW$(\alpha, \beta, \lambda, \theta)$ for $\alpha = 1$ and $\lambda = 1$, then it reduces to the Discrete Transmuted Inverse Exponential (DTIE) distribution with pmf as

$$f_{DTIE}(x) = (1+\theta)\left\{\exp\left[-1/(\beta x + \beta)\right] - \exp\left[-1/(\beta x)\right]\right\}$$
$$-\theta\left\{\exp\left[-2/(\beta x + \beta)\right] - \exp\left[-2/(\beta x)\right]\right\},$$

where $x = 0, 1, 2, ..., \beta > 0$ and $-1 \le \theta \le 1$.

Proof. We get the pmf of the DTIE distribution by replacing the survival function of $S_{TIE}(x) = 1 - (1+\theta)\exp(-\beta_0/x) + \theta\exp(-2\beta_0/x)$ where $\beta_0 = 1/\beta > 0$ as in equation (1), or by replacing $\alpha = \lambda = 1$ as in equation (3), where the TIE distribution was proposed by Oguntunde and Adejumo (2015).

Corollary 4. If X~DTGIW$(\alpha, \beta, \lambda, \theta)$ for $\theta = 0, \alpha = 1$ and $\lambda = 1$, then it reduces to the Discrete Inverse Exponential (DIE) distribution with pmf as

$$f_{DIE}(x) = \exp\left[-1/(\beta x + \beta)\right] - \exp\left[-1/(\beta x)\right],$$

where $x = 0, 1, 2, ...$ and $\beta > 0$.

Proof. We get the DIE pmf by replacing $\alpha = \lambda = 1$ and $\theta = 0$ as in equation (3) or by replacing the survival function of the IE distribution, i.e., $S_{TIE}(x) = 1 - \exp(-\beta_0/x)$ and $\beta_0 = 1/\beta > 0$ as in equation (1), where the IE distribution was proposed by Keller *et al.* (1982).

Corollary 5. If X~DTGIW$(\alpha, \beta, \lambda, \theta)$ for $\alpha = 2$ and $\lambda = 1$, it reduces to the Discrete Transmuted Inverse Rayleigh (DTIR) distribution with pmf as

$$f_{DTIR}(x) = (1+\theta)\left\{\exp\left[-1/(\beta x + \beta)^2\right] - \exp\left[-1/(\beta x)^2\right]\right\}$$
$$-\theta\left\{\exp\left[-2/(\beta x + \beta)^2\right] - \exp\left[-2/(\beta x)^2\right]\right\},$$

where $x = 0, 1, 2, ..., \beta > 0$ and $-1 \le \theta \le 1$.

Proof. We have the DTIR pmf by replacing the survival function of the TIR distribution, i.e., $S_{TIR} = 1 - \exp\left[-(\beta_0/x)^2\right] \left\{1 + \theta - \theta\exp\left[-(\beta_0/x)^2\right]\right\}, \beta_0 = 1/\beta > 0$ as in equation (1) or by replacing $\alpha = 2$ and $\lambda = 1$ as in equation (3), where the TIR distribution was proposed by Ahmad *et al.* (2014).

Corollary 6. If X~DTGIW$(\alpha, \beta, \lambda, \theta)$ for $\theta = 0, \alpha = 2$ and $\lambda = 1$, it reduces to the Discrete Inverse Rayleigh (DIR) distribution (Hussain & Ahmad, 2014) with pmf as

$$f_{DIR}(x) = \exp\left[-1/(\beta x + \beta)^2\right] - \exp\left[-1/(\beta x)^2\right],$$

where $x = 0, 1, 2, ...$ and $\beta > 0$.

Proof. We have the pmf of the DIR distribution by replacing $\theta = 0$, $\alpha = 2$ and $\lambda = 1$ as in equation (3), or by replacing the survival function of $S_{IR} = 1 - \exp\left[-(\beta_0/x)^2\right]$, $\beta_0 = 1/\beta > 0$ as in equation (1), where the IR distribution was proposed by Voda (1972).

Corollary 7. Let X~DTGIW$(\alpha, \beta, \lambda, \theta)$ and $\beta = 1$, then it is reduces to the Discrete Transmuted Fréchet (DTF) distribution with pmf as

$$f_{DTF}(x) = (1+\theta)\left\{\exp\left[-\lambda(x+1)^{-\alpha}\right] - \exp\left[-\lambda(x)^{-\alpha}\right]\right\}$$
$$-\theta\left\{\exp\left[-2\lambda(x+1)^{-\alpha}\right] - \exp\left[-2\lambda(x)^{-\alpha}\right]\right\},$$

where $x = 0, 1, 2, ..., \alpha, \lambda > 0$ and $-1 \le \theta \le 1$.

Proof. We have the DTF pmf by replacing the survival function of the TF distribution, i.e., $S_{TF}(x)=1-\exp\left[-(\lambda_0/x)^{\alpha}\right]\left\{1+\theta-\theta\exp\left[-(\lambda_0/x)^{\alpha}\right]\right\}$, $\lambda_0=1/\lambda>0$ as in equation (1), where the TF distribution was proposed by Geetha and Poongothai (2016). In the same way, it is obtained by replacing $\beta=1$ as in equation (3).

Corollary 8. Let X~DTGIW $(\alpha,\beta,\lambda,\theta)$ for $\theta=0$ and $\beta=1$, then it reduces to the Discrete Fréchet (DF) distribution with pmf as follows

$$f_{DF}(x)=\exp\left[-\lambda(x+1)^{-\alpha}\right]-\exp\left[-\lambda(x)^{-\alpha}\right],$$

where $x=0,1,2,...,\alpha>0$ and $\lambda>0$.

Proof. We have the DF pmf by replacing $\theta=0$ and $\beta=1$ as in equation (3). In the same way, it is obtained by replacing the survival function of $S_F(x)=1-\exp\left[-(\lambda_0/x)^{\alpha}\right]$, $\lambda_0=1/\lambda>0$ as in equation (1), where the Fréchet distribution was developed by Maurice Fréchet in 1927 (Oguntunde *et al.*, 2019).

In some cases, in practice, the data to be modeled originate from a mechanism that generates data excluding zero counts; a zero-truncated distribution is a suitable model for such data. A typical example where zero-truncated discrete distributions are useful in medical science, specifically, when modeling the duration (in days, months, or years) of patients stays in hospitals. Zero-truncated distributions have applications in multiple fields, including biological science, medical science, psychology, demography, political science, engineering, etc. (Shanker, 2017).

3. The Zero-Truncated Discrete Transmuted Generalized Inverse Weibull Distribution

Zero-truncated distributions are suitable models for modeling data when the data to be modeled originate from a mechanism that generates data excluding zero counts. Suppose $f_x(x)$ is the pmf of the discrete distribution where $x=0,1,2,...$. Then the zero-truncated distribution of X can be (see Shanker and Shukla, 2017; Shanker, 2017) defined as

$$p(x)=\frac{f_x(x)}{1-f_x(0)},\ x=1,2,3,... \tag{4}$$

By replacing the pmf of the DTGIW distribution in equation (3) as in equation (4), we have

$$\begin{aligned}p_{ZT-DTGIW}(x)=&\Big\{\Big\{(1+\theta)\left\{\exp\left[-\lambda(\beta x+\beta)^{-\alpha}\right]-\exp\left[-\lambda(\beta x)^{-\alpha}\right]\right\}\\&-\theta\left\{\exp\left[-2\lambda(\beta x+\beta)^{-\alpha}\right]-\exp\left[-2\lambda(\beta x)^{-\alpha}\right]\right\}\\&\times\left\{1-\exp(-\lambda\beta^{-\alpha})\left[1+\theta-\theta\exp(-\lambda\beta^{-\alpha})\right]\right\}^{-1},\end{aligned} \tag{5}$$

where $x=1,2,3,...,\alpha,\beta,\lambda>0$ and $-1\le\theta\le1$. The random variable X has the Zero-Truncated Discrete Transmuted Generalized Inverse Weibull (ZT-DTGIW) distribution with the parameters α,β,λ and θ, which will be denoted as X~ZT-DTGIW $(\alpha,\beta,\lambda,\theta)$.

Figure 2 illustrates the pmf behaviors of the ZT-DTGIW distribution for some specified values of α,β,λ and θ. Its pmf has various behaviors, such as a reverse J-shaped distribution (Figure 2: (a)-(b)), a unimodal distribution (Figure 2: (d)-(f)), and a right-skewed shape (Figure 2: (a)-(f). Moreover, we have the special sub-models of the ZT-DTGIW distribution as in Table 1.

4. Parameter Estimation

In this section, we present the MLE to estimate the parameters of the DTGIW and ZT-DTGIW distributions.

4.1 Parameter estimation of the DTGIW distribution

Let $X_1,\ldots,X_n$ be a random sample of size n from the DTGIW distribution with the pmf as equation (3), i.e. X_i ~DTGIW $(\alpha,\beta,\lambda,\theta)$ then the log-likelihood function of X_i is given by:

$$\begin{aligned}\log L_1=\sum_{i=1}^{n}\log\Big\{&(1+\theta)\left\{\exp\left[-\lambda(\beta x_i+\beta)^{-\alpha}\right]-\exp\left[-\lambda(\beta x_i)^{-\alpha}\right]\right\}\\&-\theta\left\{\exp\left[-2\lambda(\beta x_i+\beta)^{-\alpha}\right]-\exp\left[-2\lambda(\beta x_i)^{-\alpha}\right]\right\}\Big\},\end{aligned}$$

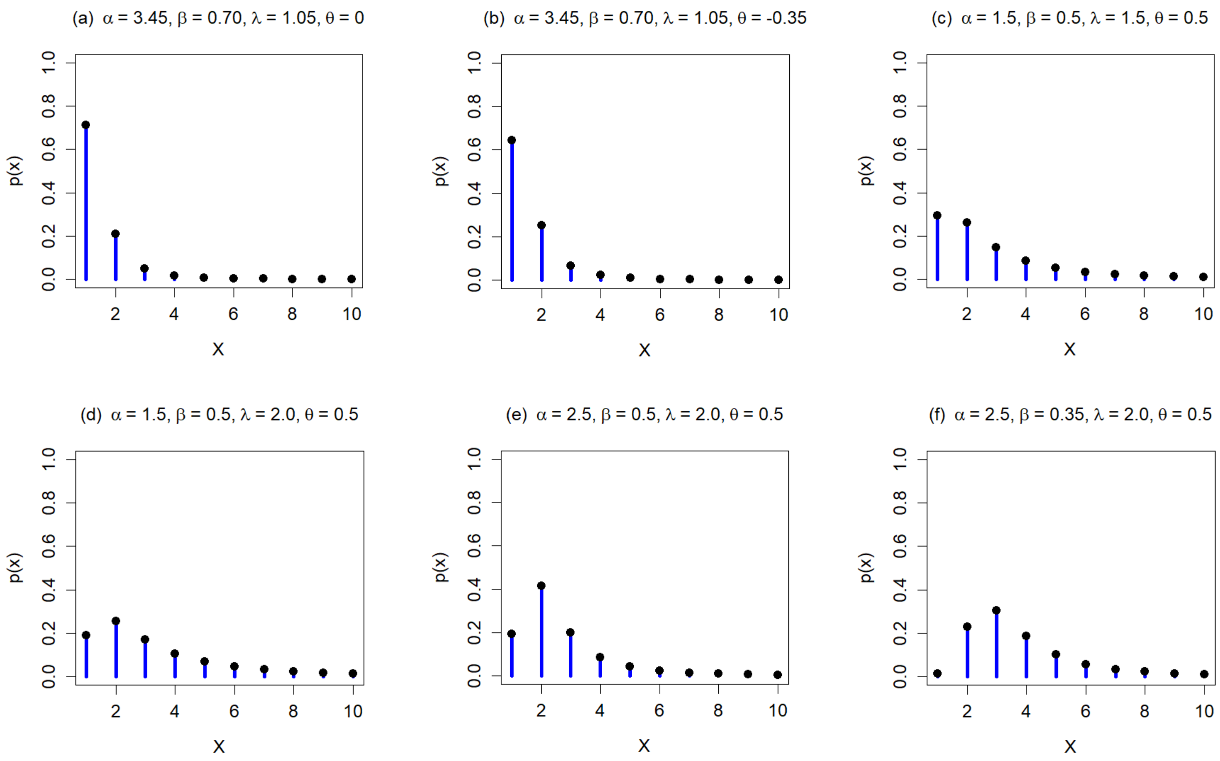

Figure 2. The pmf plots of X~ZT-DTGIW $(\alpha,\beta,\lambda,\theta)$ with the specified parameters

Table 1. Special sub-models of the ZT-DTGIW distribution

Parameters			Sub-models
β	λ	θ	
β	1	θ	Zero-Truncated Discrete Transmuted Inverse Weibull (ZT-DTIW)
β	1	0	Zero-Truncated Discrete Inverse Weibull (ZT-DIW)
β	1	θ	Zero-Truncated Discrete Transmuted Inverse Exponential (ZT-DTIE)
β	1	0	Zero-Truncated Discrete Inverse Exponential (ZT-DTIE)
β	1	θ	Zero-Truncated Discrete Transmuted Inverse Rayleigh (ZT-DTIR)
β	1	0	Zero-Truncated Discrete Inverse Rayleigh (ZT-DIR)
1	λ	θ	Zero-Truncated Discrete Transmuted Fréchet (ZT-DTF)
1	λ	0	Zero-Truncated Discrete Fréchet (ZT-DF)

To estimate the unknown parameters α,β,λ and $\theta,$ we take the partial derivatives of $\log L_1$ with respect to each parameters and equate them to zero, i.e.,

$$\frac{\partial \log L_1}{\partial\alpha}=0,\frac{\partial \log L_1}{\partial\beta}=0,\frac{\partial \log L_1}{\partial\lambda}=0,\frac{\partial \log L_1}{\partial\theta}=0\cdot$$

The maximum likelihood estimators (MLEs) of $\hat{\alpha},\hat{\beta},\hat{\lambda}$ and $\hat{\theta},$ can be obtained numerically from these non-linear equations. In this study, we solve these equations simultaneously using a numerical procedure with the Newton-Raphson method. The optim function in the optimr contribution package in R (R Core Team, 2020) is used to find the MLEs.

4.2 Parameter estimation of the ZT-DTGIW distribution

Let $X_1,\ldots,X_n$ be a random sample of size n from the ZT-DTGIW distribution with the pmf as equation (5), i.e. X_i ~ZT-DTGIW $(\alpha,\beta,\lambda,\theta)$ then the log-likelihood function of X_i is given by:

$$\log L_2 = \sum_{i=1}^{n}\log\Big\{(1+\theta)\Big\{\exp\big[-\lambda(\beta x_i+\beta)^{-\alpha}\big]-\exp\big[-\lambda(\beta x_i)^{-\alpha}\big]\Big\}$$
$$-\theta\Big\{\exp\big[-2\lambda(\beta x_i+\beta)^{-\alpha}\big]-\exp\big[-2\lambda(\beta x_i)^{-\alpha}\big]\Big\}\Big\}$$
$$-\sum_{i=1}^{n}\log\Big\{1-\exp(-\lambda\beta^{-\alpha})\big[1+\theta-\theta\exp(-\lambda\beta^{-\alpha})\big]\Big\},$$

The maximum likelihood estimates can be obtained numerically solving the equation as

$$\frac{\partial\log L_2}{\partial\alpha}=0,\ \frac{\partial\log L_2}{\partial\beta}=0,\ \frac{\partial\log L_2}{\partial\lambda}=0,\ \frac{\partial\log L_2}{\partial\theta}=0.$$

The MLEs of $\hat{\alpha},\hat{\beta},\hat{\lambda}$ and $\hat{\theta},$ can be obtained by using the optim function in the optimr package in R (R Core Team, 2020).

5. Simulation Study

The simulation study of parameter estimation is illustrated for verification of the MLE performance before application to real data is illustrated. We conducted Monte Carlo simulation studies to assess on the finite sample behavior of the maximum likelihood estimators of $\alpha,\beta,\lambda,\theta.$ All results were obtained from 1000 Monte Carlo replications ($T=1000$) and the simulations were carried out using the statistical software package R. In each replication a random sample of size n is drawn from the DTGIW $(\alpha,\beta,\lambda,\theta)$ and ZT-DTGIW $(\alpha,\beta,\lambda,\theta)$. The results of simulation study present the mean maximum likelihood estimates of the four parameters, i.e.,

$$\hat{\alpha}=\frac{1}{T}\sum_{t=1}^{T}\hat{\alpha}_t,\ \hat{\beta}=\frac{1}{T}\sum_{t=1}^{T}\hat{\beta}_t,\ \hat{\lambda}=\frac{1}{T}\sum_{t=1}^{T}\hat{\lambda}_t,\ \text{and}\ \hat{\theta}=\frac{1}{T}\sum_{t=1}^{T}\hat{\theta}_t,$$

and the Root Mean Squared Errors (RMSE) of estimators, i.e.,

$$\text{RMSE}(\hat{\alpha})=\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\alpha}_t-\alpha\right)^2},\ \text{RMSE}(\hat{\beta})=\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\beta}_t-\beta\right)^2},\ \text{RMSE}(\hat{\lambda})=\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\lambda}_t-\lambda\right)^2},\ \text{and}$$
$$\text{RMSE}(\hat{\theta})=\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\theta}_t-\theta\right)^2},$$

for sample sizes $n=30,60,100$ and 200.

5.1 Simulation study of the DTGIW distribution

Based on the cdf of $F(x)=P(X\le x)=G(x+1)$, see Alamatsaz *et al.* (2016), and Jayakumar and Babu (2019), we have the cdf of the DTGIW distribution as

$$F_{DTGIW}(x)=\exp\big[-\lambda(\beta x+\beta)^{-\alpha}\big]\Big\{1+\theta-\theta\exp\big[-\lambda(\beta x+\beta)^{-\alpha}\big]\Big\}.$$

Let $F_{DTGIW}(x)=U$ where U be a uniform random variable on [0, 1], then the quantile function of the DTGIW distribution is $Q_X(u_i)=F_{DTGIW}^{-1}(u_i)$, in which the quantile function has no closed form solution, so we have to use a numerical technique to get the quantile. R code for quantile function and the generating of a DTGIW random variable are shown as follows:

```
> qDTGIW <- function(p,alpha,beta,lambda,theta){
+  n<-length(p); x<-numeric(n);
+  for (i in 1:n){k<-0;
+  if(p[i]>=pDTGIW(k,alpha,beta,lambda,theta)){
+  while ( p[i]>=pDTGIW(k,alpha,beta,lambda,theta)) #cdf of DTGIW
+  {k<-k+1}}
+  x[i]<-k }
+ return(x)}
> rdtgiw<-function(n,alpha,beta,lambda,theta){
+  x<-numeric(); u<-runif(n);
+  x<-qDTGIW(u,alpha,beta,lambda,theta);
+ return(x)}
>
```

The true parameter values used in the data generating processes are (i) $\alpha = 2.3,\ \beta = 1.75,\ \lambda = 0.5,\ \theta = 0$ and (ii) $\alpha = 2.65,\ \beta = 0.95,\ \lambda = 0.80,\ \theta = -0.45$. In Table 2 we notice that the biases and root mean squared errors of the maximum likelihood estimators of α, β, λ and θ decay toward zero as the sample size increases, as expected.

Table 2. Statistic values of the DTGIW parameter estimation by using the MLE

n	Parameters	DTGIW (2.3,1.75,0.5,0)			DTGIW (2.65,0.95,0.80,-0.45)		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE
30	α	6.9566	4.6566	8.6110	2.7296	0.0796	0.7950
	β	1.5988	-0.1512	0.3312	0.9764	0.0264	0.0943
	λ	2.2103	1.7103	3.4201	0.9812	0.1812	0.3420
	θ	-0.1422	-0.1422	0.3393	-0.3378	0.1122	0.2681
60	α	4.0024	1.7024	5.0156	2.6632	0.0132	0.3892
	β	1.7423	-0.0077	0.2917	0.9755	0.0255	0.0694
	λ	1.0500	0.5500	1.4474	0.9282	0.1282	0.2341
	θ	-0.0798	-0.0798	0.4681	-0.3723	0.0777	0.2273
100	α	2.8565	0.5565	2.7807	2.6712	0.0212	0.3078
	β	1.7642	0.0142	0.2652	0.9742	0.0242	0.0546
	λ	0.7487	0.2487	0.8304	0.9144	0.1144	0.1816
	θ	-0.0191	-0.019 1	0.4857	-0.3933	0.0567	0.1756
200	α	2.3195	0.0195	0.7175	2.6535	0.0035	0.2169
	β	1.7603	0.0103	0.2292	0.9735	0.0235	0.0454
	λ	0.6134	0.1134	0.2894	0.8821	0.0821	0.1419
	θ	0.0622	0.0622	0.5164	-0.4177	0.0323	0.1484

5.2 Simulation study of the ZT-DTGIW distribution

A random variable X is generated from the ZT-DTGIW distribution with the true parameter values of two cases; (i) $\alpha = 3.45,\ \beta = 0.70,\ \lambda = 1.05,\ \theta = -0.35$ and (ii) $\alpha = 2.5,\ \beta = 0.5,\ \lambda = 2.0,\ \theta = 0.5$. R code for the generating of a ZT-DTGIW random variable are shown as follows:

```
> rztdtgiw<-function(n,alpha,beta,lambda,theta){
+  sampl<-c()
+  while (length(sampl)< n){
+  x<-rdtgiw(1,alpha,beta,lambda,theta)
+  if (x!=0) sampl<-c(sampl,x)}
+ return(sampl)}
>
```

In Table 3 we notice that the biases and root mean squared errors of the maximum likelihood estimators of α, β, λ and θ decay toward zero as the sample size increases, as expected.

6. Application Study

In this section, applications of the DTGIW and ZT-DTGIW distributions will be discussed with the real data sets and its goodness of fit based on the MLE. The Kolmogorov-Smirnov (K-S) and Cramer-von Mises (CVM) tests are used to compare fitting distributions, where the smaller values of test statistics give the best fit for the data. Given the cdf $F_0(x)$ of the hypothesized distribution and the empirical distribution function $F_n(x)$ of the n observed data of $X_{(j)}$ where $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n-1)} \le X_{(n)}$. The test statistics of K-S and CVM tests are, respectively

$$D = \max |F_n(x) - F_0(x)| \text{ and } W = \frac{1}{2n} + \sum_{j=1}^{n}\left[\frac{2j-1}{2n} - F_n(x_{(j)})\right].$$

In this study, the statistics of D and W are obtained by using the dgof package in R (see Arnold, Emerson, & R Core Team, 2016).

Table 3. Statistic values of the ZT-DTGIW parameter estimation by using the MLE

n	Parameters	ZT-DTGIW (3.45,0.70,1.05,-0.35)			ZT-DTGIW (2.5,0.5,2.0,0.5)		
		Estimate	Bias	RMSE	Estimate	Bias	RMSE
30	α	3.7814	0.3314	2.1692	2.7149	0.2149	0.5505
	β	0.7445	0.0445	0.1870	0.5427	0.0427	0.0804
	λ	1.1628	0.1128	0.4947	2.1649	0.1649	0.4684
	θ	-0.2482	0.1018	0.3093	0.1902	-0.3098	0.5172
60	α	3.5901	0.1401	0.9632	2.6357	0.1357	0.4423
	β	0.7244	0.0244	0.1183	0.5317	0.0317	0.0632
	λ	1.1596	0.1096	0.4537	2.1409	0.1409	0.3921
	θ	-0.2795	0.0705	0.2571	0.2844	-0.2156	0.4672
100	α	3.5109	0.0609	0.6654	2.6068	0.1068	0.3790
	β	0.7199	0.0199	0.0911	0.5267	0.0267	0.0608
	λ	1.1541	0.1041	0.4000	2.1029	0.1029	0.3618
	θ	-0.2982	0.0518	0.2199	0.3154	-0.1846	0.4648
200	α	3.4865	0.0365	0.4063	2.5534	0.0534	0.3327
	β	0.7140	0.0140	0.0678	0.5194	0.0194	0.0502
	λ	1.1323	0.0823	0.3363	2.0980	0.0980	0.3322
	θ	-0.3349	0.0151	0.1205	0.3871	-0.1129	0.4025

6.1 Application study of the DTGIW distribution

The application of the DTGIW distribution is discussed with two real data sets and compared to the Poisson distribution with the parameter μ and the DIW distribution. The first data set is the number of outbreaks of strikes in UK coal mining industries (156 observations) in four successive week periods from 1948 to 1959 (Ridout & Besbeas, 2004). The mean and variance values of these data are 0.9936 and 0.7419 respectively (under-dispersion count data). Based on the minimum D and W values from the goodness of fit with the K-S and CVM tests respectively in Table 4, we found that the DTGIW distribution gives D and W values less than the DIW and Poisson distributions. In addition, the second data set is the number of hospital stays by United States residents aged 66 and over (see Flynn, 2009), in which the mean and variance values are 0.2960 and 0.5571 respectively (over-dispersion count data). From Table 5, the results show that the DTGIW distribution gives D and W values less than the DIW and Poisson distributions. Figure 3 shows a comparison between real data sets and expected values of the fitted distributions, we found that the DTGIW distribution gives a better fit than the DIW and Poisson distributions; hence, it can be considered an important distribution for modeling data such as count data sets.

6.2 Application study of the ZT-DTGIW distribution

Two real data sets, (i) the number of counts of sites with particles from Immunogold data reported by Mathews and Appleton (1993), and (ii) the number of European red mites on apple leaves, reported by Garman (1923), (see Shanker & Shukla, 2017), are introduced in Tables 6 and 7, respectively.

According to the results of the goodness-of-fit test based on the minimum value of D and W statistics from the goodness of fit in Tables 6 and 7, we found that the ZT-DTGIW distribution gives a better fit than ZT-DIW and the ZT-P distributions (Figure 4). Therefore, the ZT-DTGIW distribution can be considered a better tool than the ZT-DIW and ZT-P distributions for modeling count data excluding zero-counts.

7. Conclusions

In this work, the DTGIW and ZT-DTGIW distributions are introduced for the analysis of count data and count data excluding zero counts, respectively. The MLE is applied to estimate the model parameters. To compare the performance, a goodness-of-fit based on the K-S and CVM tests are employed. The ZT-DTGIW distribution has eight sub-models; (i) zero-truncated discrete transmuted inverse Weibull, (ii) zero-truncated discrete inverse Weibull, (iii) zero-truncated discrete transmuted inverse exponential, (iv) zero-truncated discrete inverse exponential, (v) zero-truncated discrete transmuted inverse Rayleigh, (vi) zero-truncated discrete inverse Rayleigh, and (vii) zero-truncated discrete transmuted Fréchet, and (viii) zero-truncated discrete Fréchet distributions. From result of simulation study, notice that the biases and root mean squared errors of the maximum likelihood estimators of α, β, λ and θ decay toward zero as the sample size increases, as expected. We also note that there is small sample bias in the estimation of the parameters that index the DTGIW and ZT-DTGIW distributions. Future research should obtain bias corrections for these estimators. The result shows that; the DTGIW distribution is a better fit than the DIW and Poisson distribution for these real data sets of count data. Moreover, the ZT-DTGIW distribution seems to have the best efficiency when compared to the ZT-DIW and ZT-P distributions for fitting count data excluding zero counts.

Table 4. MLEs, test statistics, and fitted frequencies of three distributions to the strike outbreak data

Numbers of outbreaks of strikes	Observed frequency	Fitted frequencies of the distributions		
		Poisson	DIW	DTGIW
0	46	57.76	44.64	44.53
1	76	57.39	82.21	81.99
2	24	28.51	18.29	18.73
3	9	9.44	5.62	5.66
4	1	2.35	2.28	2.26
MLEs (standard error)		$\hat{\mu} = 0.9936$ (0.0798)	$\hat{\alpha} = 2.5966$ (0.2263) $\hat{\beta} = 0.9173$ (0.0348)	$\hat{\alpha} = 2.6762$ (0.2301) $\hat{\beta} = 0.9350$ (19.1958) $\hat{\lambda} = 0.7920$ (61.4544) $\hat{\theta} = -0.4298$ (0.5824)
$-\log L_1$		191.94	191.83	191.50
W		0.4155	0.0651	0.0586
D		0.0754	0.0311	0.0290

Table 5. MLEs, test statistics, and fitted frequencies of three distributions to the numbers of hospital stays

Numbers of outbreaks of strikes	Observed frequency	Fitted frequencies of the distributions		
		Poisson	DIW	DTGIW
0	3541	3277.13	3538.12	3537.75
1	599	970.03	639.38	639.22
2	176	143.56	127.18	127.78
3	48	14.17	44.73	44.86
4	20	1.05	20.63	20.64
5	12	0.06	11.15	11.13
6	5	0.00	6.69	6.67
7	1	0.00	4.32	4.30
8	4	0.00	2.95	2.93
MLEs (standard error)		$\hat{\mu} = 0.2960$ (0.0082)	$\hat{\alpha} = 2.0424$ (0.0712) $\hat{\beta} = 2.1017$ (0.0661)	$\hat{\alpha} = 2.0547$ (0.0714) $\hat{\beta} = 1.7594$ (8.7794) $\hat{\lambda} = 0.5256$ (5.3844) $\hat{\theta} = -0.3516$ (0.5040)
$-\log L_1$		3304.51	3024.92	3024.52
W		7.9694	0.0407	0.0398
D		0.0599	0.0085	0.0084

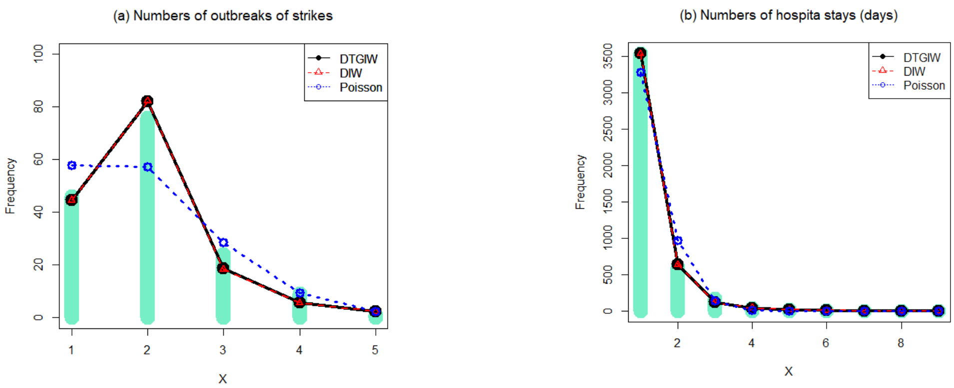

Figure 3. Plots for observed and fitted frequencies of the data; (a) the number of outbreaks of strikes; (b) the number of hospital stays

Table 6. MLEs, test statistics, and fitted frequencies of three distributions to the number of site with particles

Number of site with particles	Observed frequency	Fitted frequencies of the distributions		
		ZT-P	ZT-DIW	ZT-DTGIW
1	122	115.86	119.48	121.42
2	50	57.39	57.11	53.69
3	18	18.95	13.66	14.06
4	4	4.69	4.29	4.70
5	4	0.93	1.68	1.93
>5	0	0.18	1.78	2.20
MLEs (standard error)		$\hat{\mu} = 0.9906$ (0.0871)	$\hat{\alpha} = 3.6613$ (0.2989) $\hat{\beta} = 0.6026$ (0.0187)	$\hat{\alpha} = 3.4555$ (0.3924) $\hat{\beta} = 0.6801$ (11.941) $\hat{\lambda} = 1.0633$ (143.14) $\hat{\theta} = -0.3517$ (1.1320)
$-\log L_2$		205.95	206.67	206.04
W		0.0868	0.0232	0.0170
D		0.0310	0.0383	0.0157

Table 7. MLEs, test statistics, and fitted frequencies of three distributions to the number of European red mites on apple leaves

Number of site with particles	Observed frequency	Fitted frequencies of the distributions		
		ZT-P	ZT-DIW	ZT-DTGIW
1	38	28.67	34.23	36.67
2	17	25.68	25.00	22.14
3	10	15.34	10.03	9.48
4	9	6.87	4.48	4.51
5	3	2.46	2.27	2.41
6	2	0.74	1.28	1.41
>6	1	0.24	2.71	3.38
MLEs (standard error)		$\hat{\mu} = 1.7916$ (0.1705)	$\hat{\alpha} = 2.5536$ (0.2672) $\hat{\beta} = 0.5355$ (0.0289)	$\hat{\alpha} = 2.3309$ (0.3393) $\hat{\beta} = 1.0530$ (30.974) $\hat{\lambda} = 3.2332$ (221.68) $\hat{\theta} = -0.3530$ (0.7827)
$-\log L_2$		122.79	122.93	121.77
W		0.4150	0.1465	0.0768
D		0.1166	0.0533	0.0476

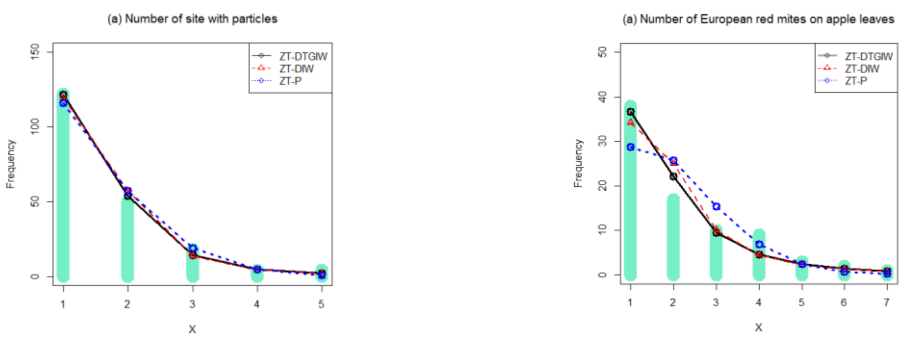

Figure 4. Plots for observed and fitted frequencies of the data; (a) the number of site with particles, (b) the number of European red mites on apple leaves

Acknowledgements

The authors would like to thank the referees for their thorough comments which helped to improve the article.

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