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A two-sample multivariate test with one covariance matrix unknown

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Abstract

In this paper, we considered two-sample multivariate testing for testing the equality of two population mean vectors of two normal populations in this situation in which one covariance is assumed to be known and the other unknown when both the sample sizes are larger than their dimensions. We adapted a test statistic from Yao (1965) and developed its distribution. The accuracy of the proposed test is investigated by simulation study. Under simulation study, the simulated results showed that the attained significance levels of proposed tests are close to nominal significance level setting in every situation considered. All proposed tests gave excellent performance and power in every situation considered except when the sample size from population with known covariance matrix is smaller than that from population with unknown covariance matrix. The two-sided proposed test and the one-sided proposed test as $H_a : \mu_1 < \mu_2$ work very well when the dimension is less than 30. Finally, we applied the proposed tests for analyzing the real data.

Keywords: approximate degrees of freedom, covariance matrix unknown, hypothesis testing, multivariate Behrens–Fisher problem, two–sample multivariate test

1. Introduction

Usually for two multivariate means testing with the sample sizes larger the dimension, the two independent samples are assumed to be drawn from two independent normal populations both with unknown but equal covariance matrices or with unknown and unequal covariance matrices (Behrens–Fisher case). But in some scientific studies, multivariate experiments consist of standard treatment and new treatment so that the covariance of standard treatment may be treated as known from historical data while the covariance caused by the new treatment may not be the same as the standard one. The two-sample testing problem when the two variances are unknown and unequal which is known as the ‘‘Behrens–Fisher’’ problem, was studied by Welch (1938, 1947), Scheffé (1970), Kim and Cohen (1998), Schechtman and Sherman (2007). The two-sample testing problem when one variance known but the other one unknown was studied by Maity and Sherman (2006), Peng and Tong (2011). Pooling the samples may not be a good idea because the covariances may be very different from each other. Many researchers have investigated this problem and various methods of approach were suggested including Maity and Sherman (2006), who studied “The Two-sample T test with one variance unknown”, and Peng and Tong (2011), was published “A note on a two-sample T test with one variance unknown”, both of which considered the univariate case that is a special case of the Behrens-Fisher problem. In this paper, we studied under the multivariate case that is also a special cases of the Behrens-Fisher problem, and give some suggestions.

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2. Materials and Methods

Let $\mathbf{X}_{i1}, \mathbf{X}_{i2}, \ldots, \mathbf{X}_{in_i}; i = 1, 2$ be two random samples from two independent p-variate normal populations with unknown $p \times 1$ mean vector $\mu_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{ip})'$; i=1,2 and unknown $p \times p$ positive definite covariance matrix $\boldsymbol{\Sigma}_i$, $X_{ij} \sim N_p(\mu_i, \Sigma_i)$, j=1,2,..., n_i, i=1,2 in which each $n_i > p$. Without the equality assumption of the covariance matrices, referred to as the Behrens-Fisher problem, we are interested in the testing problem of $H_0:\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$ against $H_a:\boldsymbol{\mu}_1\neq\boldsymbol{\mu}_2$ in situations where one $p\times p$ positive definite covariance matrix is known but the other one is unknown. Without loss of generality, we assume that the first covariance matrix, $\boldsymbol{\Sigma}_i$, is known. The sample means and sample covariances are computed as

$$\bar{\mathbf{X}}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}\mathbf{X}_{ij} \quad \text{and} \quad \mathbf{S}_i=\frac{1}{n_i-1}\sum_{j=1}^{n_i}(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)', \quad i=1,2$$

respectively. We note that $\bar{\mathbf{X}}_1 : N_p(\boldsymbol{\mu}_1,\frac{\boldsymbol{\Sigma}_1}{n_1})$ with $\boldsymbol{\Sigma}_1$ known and $\bar{\mathbf{X}}_2 : N_p(\boldsymbol{\mu}_2,\frac{\boldsymbol{\Sigma}_2}{n_2})$ with $\boldsymbol{\Sigma}_2$ unknown.

With independence of two samples, then

$$\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2 : N_p(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2,\frac{\boldsymbol{\Sigma}_1}{n_1}+\frac{\boldsymbol{\Sigma}_2}{n_2})$$

Theorem. Given two random samples as above, the test statistic for testing hypothesis as $H_0:\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$ against $H_a:\boldsymbol{\mu}_1\neq\boldsymbol{\mu}_2$ is

$$\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right) \text{ distributes as } \frac{\upsilon p}{\upsilon\text{-}p+1}F_{p,\upsilon\text{-}p+1} \tag{1}$$

where $F_{p,\upsilon-p+1}$ is the F-distribution with degrees of freedom p and $\upsilon-p+1$ which υ is obtained from

$$\frac{1}{\upsilon}=\frac{1}{n_1-1}\left(\frac{\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}(\boldsymbol{\Sigma}_1/n_1)\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)}{\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)}\right)^2+\frac{1}{n_2-1}\left(\frac{\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}(\mathbf{S}_2/n_2)\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)}{\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2\right)}\right)^2$$

and $\mathbf{S}=\frac{\boldsymbol{\Sigma}_1}{n_1}+\frac{\mathbf{S}_2}{n_2}$ which $\mathbf{S}_2$ is the estimate of $\boldsymbol{\Sigma}_2$.

Proof. Consider the test statistic $(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2)'\mathbf{S}^{-1}(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2)$ and use the univariate Welch (1938, 1947) approximate degree of freedom (APDF) method along with parallel procedure of Yao (1965), that is, if $\mathbf{Y} : N_p(\mathbf{0},\boldsymbol{\Sigma})$ then the Wishart matrix $(n-1)\mathbf{V}$ has Wishart distribution with degree of freedom $(n-1)$ where $\mathbf{V}$ is the sample covariance matrix of $\mathbf{Y}$ and n is the sample sizes of $\mathbf{Y}$. Now consider for any arbitrary constant vector $\mathbf{b}\neq\mathbf{0}$, we have

$$\mathbf{b}'\mathbf{Y} : N\left(0,\mathbf{b}'\boldsymbol{\Sigma}\,\mathbf{b}\right) \tag{2}$$

then $(n-1)\mathbf{b}'\mathbf{V}\mathbf{b} : (\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b})\chi^2_{(n-1)}$ and

$$\frac{\dfrac{\mathbf{b}'\mathbf{Y}-0}{\sqrt{\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}}}}{\sqrt{(n-1)\dfrac{(\mathbf{b}'\mathbf{V}\mathbf{b})}{(\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b})}\Big/(n-1)}}=\frac{\mathbf{b}'\mathbf{Y}}{\sqrt{(\mathbf{b}'\mathbf{V}\mathbf{b})}} : t_{(n-1)} \tag{3}$$

where $t_{(n-1)}$ is t-distribution with degree of freedom $n-1$. Squaring both sides, we obtained

$$t_{\mathbf{b}}=\frac{(\mathbf{b}'\mathbf{Y})^2}{\mathbf{b}'\mathbf{V}\mathbf{b}} : t^2_{(n-1)}. \tag{4}$$

Bush and Olkin (1959) showed that

$$\sup_{\mathbf{b}} t_{\mathbf{b}}=t_{\mathbf{b}^*}=\frac{(\mathbf{b}^{*\prime}\mathbf{Y})^2}{\mathbf{b}^{*\prime}\mathbf{V}\mathbf{b}^*}=\mathbf{Y}'\mathbf{V}^{-1}\mathbf{Y} \tag{5}$$

where the maximizing $t_{\mathbf{b}^*}$ is $\mathbf{b}^*=\mathbf{V}^{-1}\mathbf{Y}$ and also $\mathbf{Y}'\mathbf{V}^{-1}\mathbf{Y} : T^2_{p,(n-1)}$ when $T^2_{p,(n-1)}$ has a Hotelling distribution with degree of freedom $n-1$. Then for every fixed number $\mathbf{b}\neq\mathbf{0}$, we obtain

$$P(t_{\mathbf{b}} \geq t^2_{\alpha,(n-1)}) \approx \alpha \tag{6}$$

while for the supremum all $\mathbf{b} \neq \mathbf{0}$,

$$P(t_{\mathbf{b}^*} \geq T^2_{\alpha,p,(n-1)}) \approx \alpha \cdot \tag{7}$$

In our case, for every fixed number $\mathbf{b} \neq \mathbf{0}$, $\mathbf{b}'\mathbf{S}\mathbf{b}$ is a linear combination of two chi-square variates with degree of freedom p and $n_2 - 1$ as

$$\mathbf{b}'\mathbf{S}\mathbf{b} = \mathbf{b}'\left(\frac{\boldsymbol{\Sigma}_1}{n_1}\right)\mathbf{b} + \mathbf{b}'\left(\frac{\mathbf{S}_2}{n_2}\right)\mathbf{b} \cdot \tag{8}$$

Applying the Welch (1947)-APDF method, we obtained

$$P(t_{\mathbf{b}} \geq t^2_{\alpha,df}) \approx \alpha \tag{9}$$

with degree of freedom (df) as

$$\frac{1}{df} = \frac{1}{n_1 - 1}\left(\frac{\mathbf{b}'(\boldsymbol{\Sigma}_1 / n_1)\mathbf{b}}{\mathbf{b}'\mathbf{S}\mathbf{b}}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{\mathbf{b}'(\mathbf{S}_2 / n_2)\mathbf{b}}{\mathbf{b}'\mathbf{S}\mathbf{b}}\right)^2. \tag{10}$$

Then extending this to the supremum for all $\mathbf{b} = \mathbf{S}^{-1}\mathbf{Y} \neq \mathbf{0}$, with hopefully condition exist, so that $P(t_{\mathbf{b}^*} \geq T^2_{\alpha,p,df}) \approx \alpha$ where the expression of df is

$$\frac{1}{df} = \frac{1}{n_1 - 1}\left(\frac{\mathbf{Y}'\mathbf{S}^{-1}(\boldsymbol{\Sigma}_1/n_1)\mathbf{S}^{-1}\mathbf{Y}}{\mathbf{Y}'\mathbf{S}^{-1}\mathbf{Y}}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{\mathbf{Y}'\mathbf{S}^{-1}(\mathbf{S}_2/n_2)\mathbf{S}^{-1}\mathbf{Y}}{\mathbf{Y}'\mathbf{S}^{-1}\mathbf{Y}}\right)^2. \tag{11}$$

This lead to reject $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ with critical region

$$Y'S^{-1}Y \geq T^2_{\alpha,p,df}$$

When we replace $\mathbf{Y} = \bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2$, $\upsilon = df$ and $T^2_{p,\upsilon} = \frac{\upsilon p}{\upsilon - p+1} F_{p,\upsilon-p+1}$, then $\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)' \mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)$ distributed as $T^2_{p,\upsilon} = \frac{\upsilon p}{\upsilon - p+1} F_{p,\upsilon-p+1}$ where

$$\frac{1}{\upsilon} = \frac{1}{n_1 - 1}\left(\frac{\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}(\boldsymbol{\Sigma}_1/n_1)\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)}{\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}(\mathbf{S}_2/n_2)\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)}{\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)}\right)^2. \tag{12}$$

The proof is completed.

The performance of this testing statistic will be studied by simulation technique. From the data conditions as above and with the proposed testing statistic, we can test $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against $H_a : \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$ in which H_0 will be rejected at level of significance α if

$$\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)' \mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right) \geq T^2_{\alpha,p,\upsilon} = \frac{\upsilon p}{\upsilon - p+1} F_{\alpha,p,\upsilon-p+1} \tag{13}$$

where $F_{\alpha,p,\upsilon-p+1}$ is the $(1-\alpha)^{th}$ quantile of the F-distribution with degrees of freedom p and $\upsilon - p+1$.

For the one-sided alternative test of $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against $H_a : \boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$ or $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$, we applied Follmann's technique (Follmann, 1996); with our unrestricted alternative proposed test. We propose a test statistic for testing one-sided multivariate hypothesis, $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ as

$$T = \left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)' \mathbf{S}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right) \quad \text{and} \quad \left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)' \mathbf{1}$$

where $\mathbf{1} = (1,1,...,1)'$ is $p \times 1$ vector and we will reject H_0 at significant level α if

$$T \geq T^2_{1-2\alpha,p,\upsilon} \quad \text{and} \quad (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'\mathbf{1} > 0.$$

By Theorem 2.1 of Follmann (1996), one has $(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\mathbf{1} = 0$ and the significance level is approximated by

$$P(T \geq T^2_{1-2\alpha,p,\upsilon} \cap (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'\mathbf{1} = 0) = P(T \geq T^2_{1-2\alpha,p,\upsilon})\, P((\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'\mathbf{1} = 0) = (2\alpha)\frac{1}{2} = \alpha.$$

Similarly, we can have the one-sided alternative test of $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against $H_a : \boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$, so we will reject H_0 at significant level α if

$$T \geq T^2_{1-2\alpha,p,\upsilon} \quad \text{and} \quad (\bar{\mathbf{X}}_2 - \bar{\mathbf{X}}_1)'\mathbf{1} > 0.$$

The performance of our unrestricted alternative proposed test and both one-sided alternative tests are illustrated by Monte Carlo study as Tables 1-3. The application of these proposed tests to the real data is also illustrated.

3. Simulation Study

The Monte Carlo simulation study was conducted by using R program version 3.5.1 (64 bit) for investigating the accuracy of all proposed tests on two-sample multivariate test with one covariance matrix unknown. The datasets are generated by "MASS" package version 7.3–51.1 from Venables and Ripley (2002) for p- multivariate normal distribution with the mean vector $\boldsymbol{\mu}_i$ and the positive definite covariance matrix $\boldsymbol{\Sigma}_i = W_i \boldsymbol{\Psi}_i W_i$, $W_i = \text{Diag}[w_{i1}, w_{i2}, ..., w_{ip}]$, $w_{ij} = (2 \times i) + (p - j + 1)/p$, $\boldsymbol{\Psi}_i = (\phi^{(i)}_{jk})$, $\phi^{(i)}_{jj} = 1$, $\phi^{(i)}_{jk} = (-1)^{j+k}(0.2 \times i)^{|j-k|^{0.1}}$, $i = 1,2;\ j,k = 1,2,...,p;\ j \neq k$ given by Hu, Bai, Wang, and Wang (2017). The initial value of random–number seed was set at $2^{31} - 1$. Then $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ were set to be known and unknown respectively. Each testing statistic was computed repeatedly 10,000 times in each setting. The attained significance level $(\hat{\alpha})$ and the attained power (1-β) were computed by the number of rejections under the null hypothesis divided by 10,000 and the number of rejections under the alternative hypothesis divided by 10,000, respectively. For the null hypothesis, we set the mean vectors as $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$; $\boldsymbol{\mu}_i = (\mu_{i1}, \mu_{i2}, ..., \mu_{ip})'$; i=1,2 where $\mu_{1j} = \mu_{2j}$; $\mu_{ij} \overset{iid}{\sim} U(-1,1)$, $j = 1,2,...,p$ when $U(-1,1)$ denotes uniform distribution with the support $(-1,1)$, and for the alternative two-sided test, we set $\mu_{1j} \overset{iid}{\sim} U(-1,1)$, and $\mu_{2j} \overset{iid}{\sim} U(-1.5,-1) \cup U(1,1.5)$. For the alternative one-sided test, $H_a : \boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$, the mean vectors were set to be $\mu_{1j} \overset{iid}{\sim} U(-1,1)$ and $\mu_{2j} \overset{iid}{\sim} U(-1.5,-1)$. For the alternative one-sided test, $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$, the mean vectors were set to be $\mu_{1j} \overset{iid}{\sim} U(-1,1)$ and $\mu_{2j} \overset{iid}{\sim} U(1,1.5)$. The significance level is set at $\alpha = 0.05$. We considered the performance of all proposed tests for $p = 5, 10, 30, 50$ with different condition of samples sizes as $n_1 = n_2$, $n_1 < n_2$ and $n_1 > n_2$.

Table 1. Simulation results at nominal significance levels $\alpha = 0.05$ and $n_1 = n_2$.

p	n_1, n_2	two-sided test		one sided test ($H_a : \mu_1 > \mu_2$)		one sided test ($H_a : \mu_1 < \mu_2$)	
		Type I error rate	Empirical power	Type I error rate	Empirical power	Type I error rate	Empirical power
5	20,20	.0464	.8318	.0454	.9358	.0477	.3385
	35,35	.0494	.9904	.0497	.9969	.0469	.5614
	50,50	.0452	.9996	.0475	.9999	.0488	.7329
	100,100	.0487	1.0000	.0512	1.0000	.0478	.9619
10	40,40	.0475	.9701	.0461	1.0000	.0469	.9896
	70,70	.0463	.9998	.0434	1.0000	.0501	1.0000
	100,100	.0476	1.0000	.0504	1.0000	.0511	1.0000
30	120,120	.0439	1.0000	.0459	1.0000	.0437	1.0000
	210,210	.0432	1.0000	.0450	1.0000	.0482	1.0000
	300,300	.0500	1.0000	.0488	1.0000	.0504	1.0000
50	200,200	.0410	1.0000	.0435	1.0000	.0443	1.0000
	350,350	.0458	1.0000	.0444	1.0000	.0469	1.0000
	500,500	.0482	1.0000	.0498	1.0000	.0501	1.0000
Max. $\lvert\hat{\alpha}-\alpha\rvert$		.0090	-	.0066	-	.0063	

Table 2. Simulation results at nominal significance levels $\alpha = 0.05$ and $n_1 > n_2$.

p	n_1, n_2	two-sided test		one sided test ($H_a : \mu_1 > \mu_2$)		one sided test ($H_a : \mu_1 < \mu_2$)	
		Type I error rate	Empirical power	Type I error rate	Empirical power	Type I error rate	Empirical power
5	25,20	.0528	.8532	.0551	.9459	.0486	.3656
	40,35	.0528	.9923	.0522	.9986	.0509	.5812
	60,50	.0482	.9994	.0504	1.0000	.0487	.7402
	110,100	.0506	1.0000	.0502	1.0000	.0500	.9668
10	50,40	.0541	.9784	.0528	1.0000	.0539	.9955
	80,70	.0499	.9999	.0499	1.0000	.0492	1.0000
	110,100	.0491	1.0000	.0471	1.0000	.0527	1.0000
30	140,120	.0525	1.0000	.0501	1.0000	.0528	1.0000
	230,210	.0480	1.0000	.0504	1.0000	.0481	1.0000
	350,300	.0522	1.0000	.0519	1.0000	.0516	1.0000
50	220,200	.0448	1.0000	.0456	1.0000	.0457	1.0000
	400,350	.0496	1.0000	.0485	1.0000	.0470	1.0000
	550,500	.0499	1.0000	.0504	1.0000	.0460	1.0000
Max. $\lvert\hat{\alpha}-\alpha\rvert$		.0052	-	.0051	-	.0043	

Table 3. Simulation results at nominal significance levels $\alpha = 0.05$ and $n_1 < n_2$.

p	n_1, n_2	two-sided test		one sided test ($H_a : \mu_1 > \mu_2$)		one sided test ($H_a : \mu_1 < \mu_2$)	
		Type I error rate	Empirical power	Type I error rate	Empirical power	Type I error rate	Empirical power
5	20,25	.0399	.8965	.0435	.9668	.0432	.3847
	35,40	.0462	.9943	.0477	.9991	.0438	.5965
	50,60	.0474	1.0000	.0524	1.0000	.0456	.7814
	100,110	.0503	1.0000	.0489	1.0000	.0506	.9726
10	40,50	.0433	.9898	.0429	1.0000	.0435	.9977
	70,80	.0452	1.0000	.0456	1.0000	.0475	1.0000
	100,110	.0464	1.0000	.0484	1.0000	.0496	1.0000
30	120,140	.0388	1.0000	.0436	1.0000	.0413	1.0000
	210,230	.0457	1.0000	.0441	1.0000	.0448	1.0000
	300,350	.0487	1.0000	.0467	1.0000	.0482	1.0000
50	200,220	.0388	1.0000	.0437	1.0000	.0390	1.0000
	350,400	.0407	1.0000	.0453	1.0000	.0419	1.0000
	500,550	.0444	1.0000	.0472	1.0000	.0436	1.0000
Max. $\lvert\hat{\alpha}-\alpha\rvert$		.0112	-	.0071	-	.0110	

From simulation results (Table 1-3), all the estimated attained significance level values considered are very close to the nominal significance levels setting $\alpha = 0.05$ as expected. The last row of each table provides the absolute maximum discrepancy between the nominal significance level setting and the estimated attained significance level over 13 conditions showing that the proposed test has excellent performance in every case considered except for $n_1 < n_2$, in which it gave a fairly good performance in every case considered.

For the alternative two-sided test and one-sided test, $H_a : \mu_1 > \mu_2$, overall situations considered both equal and unequal sample sizes, it was shown that the proposed test tend to give excellent power.

For the alternative one-sided test, $H_a : \mu_1 < \mu_2$, overall situations considered both equal and unequal sample sizes, it was shown that the proposed test tend to have the highest power when $p \geq 10$. For small dimension or $p < 10$ such as in this case $p = 5$, we recommend to use our proposed test when $n \geq 100$.

4. Application for Real Data Set

The data for illustrating the proposed tests are of Parkinson's disease. The dataset was created by Max Little of the University of Oxford, in collaboration with the National Centre for Voice and Speech, Denver, Colorado, who recorded the speech signals and published by Little, McSharry, Roberts, Costello, and Moroz (2007). The data from a range of biomedical voice measurements (recurrence pitch entropy density (RPDE), detrended fluctuation analysis (DFA), two nonlinear dynamical complexity measures (D2) and three nonlinear measures of fundamental frequency variation (spread1), (spread2), (PPE)) of 31 people were used to test the equality of mean vectors of two independent groups, healthy controls and patients. The sample size of the groups were 8 and 23, respectively. Without loss of generality, we assume that the covariance matrix from healthy controls group is known. After statistical computation and normality checking, we obtained the testing statistic for two-sided test equal to 43.16392 with degree of freedom 22 and p-value equal to 0.002387514. The testing result leads us to reject $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and then we can conclude that there are difference between the two mean voice signals of healthy control and patient groups. For the alternative one-sided test, $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$, we obtained the same testing statistic as two-sided test with critical value at significant level 0.05 equal to 16.71269 and $(\bar{\mathbf{X}}_2 - \bar{\mathbf{X}}_1)'\mathbf{1}$=1.892498. The testing result cannot lead us reject $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ and then we can conclude that the mean voice signals of healthy controls are less than the mean voice signals patient of the group; the mean and standard deviation are shown in Table 4. This shows that the alternative one-sided test as $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ works very well even though the dimension is small and $n_1 < n_2$.

Table 4. Mean and standard deviation voice signals of healthy controls and patients' groups

Feature	Healthy controls		Patients' groups	
	mean	SD	mean	SD
RPDE	.4426	.0784	.5148	.0964
D2	2.1545	.2231	2.4146	.0645
DFA	.6957	.0518	.7239	.0539
Spread1	-6.7593	.5637	-5.4116	.7382
Spread2	.1603	.0577	.2412	.0562
PPE	.1230	.0381	.2264	.0645

5. Conclusions

In this paper, we considered two-sample multivariate testing for testing the equality of two population mean vectors of two normal populations in situation which one covariance is assumed to be known and the other is unknown. We adapted a test statistic from Yao (1965) and developed its distribution. We also proposed one-sided alternative testing statistics of these two population means by using Follmann's technique (Follmann, 1996) with our unrestricted alternative proposed test. The accuracy of all proposed tests is investigated by simulation study. Under simulation study, the simulated results showed that all proposed tests gave attained significance levels close to nominal significance level in every situation considered, especially when the sample size from population with known covariance matrix is larger than or equal to that from the population with unknown covariance matrix. When the sample size from the population with known covariance matrix is larger than or equal to that from the population with unknown covariance matrix, the two-sided proposed test and the one-sided proposed test with $H_a : \boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$ gave excellent power in every situation considered but the one-sided proposed test with $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ gave excellent powers when the two samples sizes are larger or equal to 100. When the sample size from the population with known covariance matrix is smaller than that from the population with unknown covariance matrix, the one-sided proposed test with $H_a : \boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$ gave an excellent performance and power in every situation considered but the two-sided proposed test and the one-sided proposed test with $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ work very well when the dimension is less than 30. Over all conditions considered, we recommend to use our two-sided proposed testing statistic and two one-sided proposed testing statistics when the sample size from population with known covariance matrix is larger than or equal to those from the population with unknown covariance matrix and the sample size of both samples should be larger 50. When the sample size from the population with known covariance matrix is smaller than that from the population with unknown covariance matrix, we can use two-sided proposed test and the one-sided proposed test as $H_a : \boldsymbol{\mu}_1 < \boldsymbol{\mu}_2$ with small dimensions.

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