Songklanakarin J. Sci. Technol.
44 (5), 1287–1291, Sep. – Oct. 2022

http://www.sjst.psu.ac.th

Original Article

Fermat numbers and Fibonacci numbers on Heron triangles

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Received: 13 April 2022; Revised: 1 September 2022; Accepted: 9 September 2022

Abstract

We mainly give necessary and sufficient conditions for being a Heron triangle in the case of certain classes of an isosceles triangles with the three sides (a, a, c), where c is an arbitrary positive integer, and a is a Fermat or Fibonacci prime.

Keywords: Heron triangle, Fermat prime, Fibonacci prime, primitive Pythagorean triple, triangle inequality

1. Introduction

A Heron triangle is a special triangle having positive integers as its three sides a, b, c and also its area. Throughout this paper, we will sometimes use (a, b, c) as a Heron triple if a triangle with its three sides a, b, c is the Heron triangle. The plethora of studies regarding several properties of such a triangle has accumulated over a long time. In 1990, Harborth and Kemnitz (Harborth & Kemnitz, 1990) studied a connection between the Fibonacci numbers and the Heron triangles. They provided the definition of a Fibonacci triangle, which is a Heron triangle whose three sides are all Fibonacci numbers, and furthermore, they intended to investigate Fibonacci triangles other than the triangle of sides 5,5,8. In 2003, Luca (Luca, 2003) studied a connection between a Fermat prime and a Heron triangle, and he proved that if three sides a, b, c of a Heron triangle are prime powers, then either $(a, b, c) = (5, 4, 3)$ or $(\mathscr{F}_m, \mathscr{F}_m, 4(\mathscr{F}_{m-1} - 1))$ for some integer $m \geq 1$, where $\mathscr{F}_m$ is a Fermat prime. In 2013, Stanica, Sarkar, Gupta, Maitra and Kar (Stanica, Sarkar, Gupta, Maitra & Kar, 2013) defined the notation $H(a, b)$ for the number of all Heron triangles for fixed natural numbers a, b. They found an upper bound for $H(p, q)$ when p and q are primes, and in this paper we will mention only the particular case $H(p, p) = 2$, where p is a prime such that $p \equiv 1 \pmod 4$.

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In this paper, we attempt to study the connections among Fermat numbers, Fibonacci numbers, and Heron triangles. Inspired by the work of Luca (Luca, 2003) mentioned above, necessary and sufficient conditions for being a Heron triangle in the case of certain classes of isosceles triangles, with the three sides a, a, c, where c is an arbitrary positive integer and a is a Fermat prime or Fibonacci prime, are eventually provided.

To complete our results, first of all, we shall present the axiom called the *triangle inequality*, which states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side. This axiom will be used many times in order to assess whether some given three positive integers can be the sides of a triangle or not.

Then, the so-called Heron's formula (Luca, 2003) will be introduced in order to find the area of any triangle, as shown below.

Theorem 1.1 (*Heron's formula*) The area of any triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ with } s = \frac{a+b+c}{2},$$

which is its semi-perimeter when a, b and c are the lengths of sides of the triangle.

In particular, if $a = b$, the triangle is isosceles and its area can be easily computed as

$$A = \frac{b}{4}\sqrt{4a^2 - b^2}\,.$$

Lastly, the main approach seen in Corollary 1.1 will be crucially needed in our work. Before proving it, let us present some properties of a Heron triangle taken from (Luca, 2003), the classical parameterization of primitive Pythagorean triples found in (Joyce, 1997) and the last two theorems taken from (Bhaskar, 2008; Burton, 2007; Stanica, Sarkar, Gupta, Maitra & Kar, 2013) as follows:

Lemma 1.1 Let a, b and c be the lengths of the sides of a Heron triangle such that $a = b$. Then $\frac{c}{2}$ is even and h_c is a positive integer when h_c is altitude from side c.

Recall that for integers a, b and c, we call (a, b, c) a Pythagorean triple if $a^2 + b^2 = c^2$, and it is said to be a primitive Pythagorean triple if it is a Pythagorean triple and gcd $(a, b, c) = 1$. In other words, (a, b, c) is a Pythagorean triple if such integers can form a right triangle where c is the length of the hypotenuse, and a, b are the lengths of the remaining two sides.

Theorem 1.2. (The classical parameterization of primitive Pythagorean triples) If (a, b, c) is a primitive Pythagorean triple, then there exist positive integers u and v such that $u > v$, $\gcd(u, v) = 1$ and $u \not\equiv v \pmod 2$ for which

$$a = u^2 - v^2,\ b = 2uv \ \text{ and } \ c = u^2 + v^2.$$

Theorem 1.3. Let p be a prime. Then $p \equiv 1 \pmod 4$ if and only if p can be written uniquely as a sum of two squares.

Theorem 1.4. Let H(a, b) be the number of a Heron triangle whose two sides a, b are fixed. If p, q are two fixed odd prime sides of a triangle, then

$$H(p, q) \text{ is } \begin{cases} = 0 \ ;\ p, q \text{ are } \equiv 3 \pmod 4 \\ = 2 \ ;\ p = q \equiv 1 \pmod 4 \\ \leq 2 \ ;\ p \neq q \text{ and exactly one of } p \text{ and } q \text{ is } \equiv 3 \pmod 4 \\ \leq 5 \ ;\ p \neq q \text{ and } p, q \text{ are } \equiv 1 \pmod 4. \end{cases}$$

The idea how to obtain the above theorem may be seen in (Ionascu, Luca & Stanica, 2007; Stanica, Sarkar, Gupta, Maitra & Kar, 2013). The particular proof of Theorem 2.4 appearing in (Ionascu, Luca & Stanica, 2007) leads us to provide exactly two isosceles triangles with the three sides p, p, c, where $p \equiv 1 \pmod 4$ and c is a positive integer as illustrated in the following corollary.

Corollary 1.1. Let p be an odd prime such that $p \equiv 1 \pmod 4$ and c be a positive integer. If (p, p, c) is a Heron triple, then there exists a unique pair of positive integers u and v for which

$$p = u^2 + v^2, \text{and } \ c = 4uv \ \text{ or } \ c = 2(u^2 - v^2),$$

where $u > v$, $\gcd(u, v) = 1$ and $u \not\equiv v \pmod 2$.

Proof. Assume that (p, p, c) is a Heron triple. Note that the corresponding triangle is isosceles and c is even by Lemma 1.1. It is not hard to see that $(\frac{c}{2}, h_c, p)$ is a Pythagorean triple. It follows by Theorem 1.2 that $p = u^2 + v^2$ for some positive integers u and v for which

$$u > v,\ \gcd(u, v) = 1,\ u \not\equiv v \pmod 2, \text{ and } \{c/2, h_c\} = \{2uv, |u^2 - v^2|\}.$$

These can happen by applying Theorem 1.3 and $p \equiv 1 \pmod 4$. Hence, we can eventually conclude that $c = 4uv$ or $c = 2(u^2 - v^2)$.

2. Fermat Numbers and Heron Triangles

Definition 2.1. Let m be a nonnegative integer. A number of the form $\mathscr{F}_m = 2^{2^m} + 1$ is called a Fermat number. We say that $\mathscr{F}_m$ is a Fermat prime if $\mathscr{F}_m$ is a prime number.

The first six Fermat numbers are

$$\mathscr{F}_0 = 3,\ \mathscr{F}_1 = 5,\ \mathscr{F}_2 = 17,\ \mathscr{F}_3 = 257,\ \mathscr{F}_4 = 65537,\ \mathscr{F}_5 = 4294967297.$$

Observe that they are prime numbers except for $\mathscr{F}_5$, which is also not a prime power since $\mathscr{F}_5 = (641)(6700417)$. Further notice that $4(\mathscr{F}_{m-1} - 1)$ can be written as an integer power of 2 for any positive integer m. The following lemma indicates that the converse of Luca's result as mentioned above fails. More precisely, there exists a Heron triple (a, b, c) for which at least one of the sides a, b and c is not a prime power, for instance $(a, b, c) = (\mathscr{F}_5, \mathscr{F}_5, 4(\mathscr{F}_5 - 1))$.

Lemma 2.1. The integral triple $(\mathscr{F}_m, \mathscr{F}_m, 4(\mathscr{F}_{m-1} - 1))$ is a Heron triple for any natural number m.

Proof. Let m be a natural number. We first show that the integers $\mathscr{F}_m, \mathscr{F}_m$ and $4(\mathscr{F}_{m-1} - 1)$ can be the sides of a triangle. It suffices to prove that the length of the longest side is less than the sum of the lengths of the remaining sides. Since $\mathscr{F}_m \neq 4(\mathscr{F}_{m-1} - 1)$, we distinguish our consideration into two cases: If $\mathscr{F}_m > 4(\mathscr{F}_{m-1} - 1)$, then we are done. If $\mathscr{F}_m < 4(\mathscr{F}_{m-1} - 1)$, then we eventually obtain that

$$2\mathscr{F}_m - 4(\mathscr{F}_{m-1} - 1) = 2\left(2^{2^{m-1}} - 1\right)^2 > 0,$$

which means that $\mathscr{F}_m + \mathscr{F}_m > 4(\mathscr{F}_{m-1} - 1)$. Secondly, let us consider

$$A_m = 2(\mathscr{F}_{m-1} - 1)\sqrt{\mathscr{F}_m^2 - 4(\mathscr{F}_{m-1} - 1)^2} = 2\mathscr{F}_{m-1}(\mathscr{F}_{m-1} - 1)(\mathscr{F}_{m-1} - 2),$$

which is always positive, and this is indeed the area of our considered triangle. Hence, the proof is complete.

Lemma 2.2. The integral triple $(\mathscr{F}_m, \mathscr{F}_m, 2(\mathscr{F}_m - 2))$ is a Heron triple for any natural number m.

Proof. Let m be a natural number. We first show that the integers $\mathscr{F}_m, \mathscr{F}_m$ and $2(\mathscr{F}_m - 2)$ can be constructed as the sides of a triangle. It suffices to prove that the length of the longest side is less than the sum of the lengths of the remaining sides. Since

$$2(\mathscr{F}_m - 2) - \mathscr{F}_m = \mathscr{F}_m - 4 = 2^{2^m} - 3 > 0,$$

it follows that $2(\mathscr{F}_m - 2)$ is the longest side. It is easy to see that $2(\mathscr{F}_m - 2) < 2\mathscr{F}_m$ as $(\mathscr{F}_m - 2) < \mathscr{F}_m$. Secondly, let us consider

$$A'_m = (\mathscr{F}_m - 2)\sqrt{\mathscr{F}_m^2 - (\mathscr{F}_m - 2)^2} = 2(\mathscr{F}_m - 2)(\mathscr{F}_{m-1} - 1),$$

which is always positive, and this is indeed the area of our considered triangle. Hence, the proof is complete.

Corollary 2.1. For a given natural number m, let A_m and A'_m be the areas of the triangles having $(\mathscr{F}_m, \mathscr{F}_m, 4(\mathscr{F}_{m-1} - 1))$ and $(\mathscr{F}_m, \mathscr{F}_m, 2(\mathscr{F}_m - 2))$ as their three sides, respectively. Then $A_m = A'_m$.

Proof. By the proofs of Lemma 2.1 and 2.2, for each natural number m, we respectively have

$$A_m = 2\mathscr{F}_{m-1}(\mathscr{F}_{m-1} - 1)(\mathscr{F}_{m-1} - 2) \text{ and } A'_m = 2(\mathscr{F}_m - 2)(\mathscr{F}_{m-1} - 1).$$

Consequently,

$$\begin{aligned} A_m &= 2\mathscr{F}_{m-1}(\mathscr{F}_{m-1} - 1)(\mathscr{F}_{m-1} - 2) \\ &= 2(2^{2^{m-1}} + 1)\, 2^{2^{m-1}}\, (2^{2^{m-1}} - 1) \\ &= 2(2^{2^m} - 1)\, 2^{2^{m-1}} \\ &= 2(\mathscr{F}_m - 2)(\mathscr{F}_{m-1} - 1) \\ &= A'_m, \end{aligned}$$

as desired.

Theorem 2.1. Let $\mathscr{F}_m$ be a Fermat prime with $m \geq 1$ and c be a positive integer. Then $(\mathscr{F}_m, \mathscr{F}_m, c)$ is a Heron triple iff $c = 4(\mathscr{F}_{m-1} - 1)$ or $c = 2(\mathscr{F}_m - 2)$.

Proof. Assume that $(\mathscr{F}_m, \mathscr{F}_m, c)$ is a Heron triple. Note that $\mathscr{F}_m$ is prime and $\mathscr{F}_m \equiv 1 \pmod 4$ for any positive integer m. It follows by Corollary 1.1 that there exists a unique pair of positive integers u and v for which

$$\mathscr{F}_m = u^2 + v^2 \text{ and } c = 4uv \text{ or } c = 2(u^2 - v^2),$$

where $u > v$, $\gcd(u, v) = 1$, and $u \not\equiv v \pmod 2$. But we have $\mathscr{F}_m = 2^{2^m} + 1$, and this implies that $c = 4(\mathscr{F}_{m-1} - 1)$ or $c = 2(\mathscr{F}_m - 2)$. The proof of sufficiency is by using Lemma 2.1 and Lemma 2.2.

3. Fibonacci Numbers and Heron Triangles

Definition 3.1. The integers appearing in the sequence F_n defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for integer $n \geq 2$ are said to be Fibonacci numbers. If F_n is a prime number, then F_n is called a Fibonacci prime.

The following lemma appearing in (Wikipedia contributors, 2022) will play an important role in this section.

Lemma 3.1. $F_{2n-1} = F_n^2 + F_{n-1}^2$ for any natural number n.

Lemma 3.2. $(F_{2n-1}, F_{2n-1}, 4F_nF_{n-1})$ is a Heron triple for any natural number $n \geq 3$.

Proof. Let n be a natural number greater than 3. Notice that $2F_{n-1} > F_n$ and $F_{n-1} < F_n$. Using Lemma 3.1, we obtain that

$$\begin{aligned} 4F_nF_{n-1} - F_{2n-1} &= 4F_nF_{n-1} - (F_n^2 + F_{n-1}^2) \\ &= (2F_nF_{n-1} - F_nF_n) + (2F_nF_{n-1} - F_{n-1}F_{n-1}) > 0, \end{aligned}$$

which implies that $4F_nF_{n-1}$ is the longest side. Let us consider

$$\begin{aligned} 2F_{2n-1} - 4F_nF_{n-1} &= 2(F_n^2 + F_{n-1}^2 - 2F_nF_{n-1}) \\ &= 2(F_n - F_{n-1})^2 > 0, \end{aligned}$$

which means that $F_{2n-1} + F_{2n-1} > 4F_nF_{n-1}$. Thus, we first conclude that the Fibonacci numbers F_{2n-1}, F_{2n-1} and $4F_nF_{n-1}$ can be the sides of a triangle. Secondly, let us consider

$$\begin{aligned} B_n &= 2F_nF_{n-1}\sqrt{F_{n-1}^2 - (2F_nF_{n-1})^2} \\ &= 2F_nF_{n-1}\sqrt{(F_{2n-1} - 2F_nF_{n-1})(F_{2n-1} + 2F_nF_{n-1})} \\ &= 2F_nF_{n-1}\sqrt{\left(F_n^2 + F_{n-1}^2 - 2F_nF_{n-1}\right)\left(F_n^2 + F_{n-1}^2 + 2F_nF_{n-1}\right)} \\ &= 2F_nF_{n-1}\sqrt{(F_n - F_{n-1})^2(F_n + F_{n-1})^2} \\ &= 2F_nF_{n-1}\left(F_n^2 - F_{n-1}^2\right), \end{aligned}$$

which is always positive, and this is indeed the area of our considered triangle. Hence, the proof is complete.

Lemma 3.3. $\left(F_{2n-1}, F_{2n-1}, 2(F_n^2 - F_{n-1}^2)\right)$ is a Heron triple for any natural number $n \geq 3$.

Proof. Let n be a natural number greater than 3. It suffices to show that the length of the longest side is less than the sum of the lengths of the remaining sides. Since $F_{2n-1} \neq 2(F_n^2 - F_{n-1}^2)$, we will separate our consideration into two cases: If $F_{2n-1} > 2(F_n^2 - F_{n-1}^2)$, then we are done. If $F_{2n-1} < 2(F_n^2 - F_{n-1}^2))$, then we obtain that

$$2F_{2n-1} - 2(F_n^2 - F_{n-1}^2) = 2(F_n^2 + F_{n-1}^2) - 2(F_n^2 - F_{n-1}^2) = 4F_{n-1}^2 > 0,$$

which leads us to finish this case. Thus, we first conclude that the Fibonacci numbers F_{2n-1}, F_{2n-1} and $2(F_n^2 - F_{n-1}^2)$ can be the sides of a triangle. Secondly, let us consider

$$\begin{aligned} B_n' &= \left(F_n^2 - F_{n-1}^2\right)\sqrt{F_{2n-1}^2 - (F_n^2 - F_{n-1}^2)^2} \\ &= \left(F_n^2 - F_{n-1}^2\right)\sqrt{\left(F_{2n-1} - \left(F_n^2 - F_{n-1}^2\right)\right)\left(F_{2n-1} + \left(F_n^2 - F_{n-1}^2\right)\right)} \\ &= \left(F_n^2 - F_{n-1}^2\right)\sqrt{\left(F_n^2 + F_{n-1}^2 - \left(F_n^2 - F_{n-1}^2\right)\right)\left(F_n^2 + F_{n-1}^2 + \left(F_n^2 - F_{n-1}^2\right)\right)} \\ &= \left(F_n^2 - F_{n-1}^2\right)\sqrt{4F_n^2F_{n-1}^2} \\ &= 2F_nF_{n-1}\left(F_n^2 - F_{n-1}^2\right), \end{aligned}$$

which is always positive, and this is indeed the area of our considered triangle. Hence, the proof is complete.

The following corollary is immediately true by taking the results from Lemma 3.2 and Lemma 3.3.

Corollary 3.1. For a given natural number $n \geq 3$, let B_n and B_n' be the areas of the triangles having $(F_{2n-1}, F_{2n-1}, 4F_nF_{n-1})$ and $\left(F_{2n-1}, F_{2n-1}, 2(F_n^2 - F_{n-1}^2)\right)$ as their three sides, respectively. Then $B_n = B_n'$.

Theorem 3.1. Let F_{2n-1} be a Fibonacci prime number with integer $n \geq 3$ and c be a positive integer. Then (F_{2n-1}, F_{2n-1}, c) is a Heron triple iff $c = 4F_nF_{n-1}$ or $c = 2(F_n^2 - F_{n-1}^2)$.

Proof. Assume that (F_{2n-1}, F_{2n-1}, c) is a Heron triple, where F_{2n-1} is a Fibonacci prime with $n \geq 3$ and c is a positive integer. Note that F_{2n-1} is always an odd prime. By Lemma 3.3, we obtain that $F_{2n+1} = F_n^2 + F_{n-1}^2 \equiv 1 (\text{mod } 4)$. It follows by referring to Corollary 1.1 that there exists a unique pair of positive integers u and v for which

$$F_{2n-1} = u^2 + v^2 \text{ and } c = 4uv \text{ or } c = 2(u^2 - v^2),$$

where $u > v$, $\gcd(u, v) = 1$, and $u \not\equiv v \ (\text{mod } 2)$. Again, we observe by Lemma 3.3 that $F_{2n-1} = F_n^2 + F_{n-1}^2$ satisfying $F_n > F_{n-1}$, $\gcd(F_n, F_{n-1}) = 1$, and $F_n \not\equiv F_{n-1} (\text{mod } 2)$. Then we eventually conclude that $c = 4F_nF_{n-1}$ or $c = 2(F_n^2 - F_{n-1}^2)$. The sufficiency is proven by referring to Lemma 3.2 and Lemma 3.3.

4. Conclusions

For a given Fermat prime $\mathscr{F}_m$ with $m \geq 1$, and a given positive integer c, we can show that $(\mathscr{F}_m, \mathscr{F}_m, c)$ is a Heron triple if and only if $c = 4(\mathscr{F}_{m-1} - 1)$ or $c = 2(\mathscr{F}_m - 2)$. This means that the converse of Luca's result holds if $\mathscr{F}_m$ is prime. Moreover, we demonstrated that (F_{2n-1}, F_{2n-1}, c) is a Heron triple iff $c = 4F_nF_{n-1}$ or $c = 2(F_n^2 - F_{n-1}^2)$ for any Fibonacci prime F_{2n-1} with $n \geq 3$ and a positive integer c. Regarding the above results, we conclude that there are many isosceles Heron triangles whose sides are Fermat primes or Fibonacci primes.

Acknowledgements

The authors wish to thank the referees for their valuable suggestions and comments to improve the article. This research was supported by the Department of Mathematics, Faculty of Science, Khon Kaen University, in the fiscal year 2022.

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