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Analysis of the Ebola with a fractional-order model involving the Caputo-Fabrizio derivative

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Abstract

This paper uses a fractional-order epidemic model to describe the transmission dynamics of the Ebola virus. The proposed model uses the fractional-order derivative in Caputo-Fabrizio's sense. It calculates the time-independent solutions of the proposed model, and the next-generation matrix method is used to calculate the basic reproduction number. It provides the conditions for the existence and uniqueness of solutions to the model. Further, the conditions for generalized Ulam-Hyers-Rassias stability of the proposed model are obtained. Numerical simulations show how the proposed model's approximate solution varies for integer and fractional orders. They also show the behavior of the Ebola in terms of infections, deceased, and susceptible counts, for various contact rates. To demonstrate efficiency while using less time, CPU times are given in tabular form.

Keywords: Ebola, Caputo-Fabrizio derivative, stability, basic reproduction number, simulations

1. Introduction

The Filoviridae family member Ebola virus (EBOV) causes an inflammatory, severe, potentially fatal disease known as EVD for Ebola viral disease (while it used to be referred to as the Ebola hemorrhagic fever), affecting both humans and great apes. The first species of EBOV was discovered near the Ebola River in the Democratic Republic of Congo in central African continent, in 1976 (Bisimwa, Biamba, Aborode, Cakwira, & Akilimali, 2022; Feldmann, Sprecher, & Geisbert, 2020). With mortality rates ranging from 50% to 90% in some instances, death due to Ebola hemorrhagic fever can take place as quickly as in a few days (Hammouch, Rasul, Ouakka, & Elazzouzi, 2022). The illness requires between two and twenty-one days (typically, six to ten days) to incubate and eight hours to replicate (Feldmann *et al.*, 2020). Humans can contract EBOV through close physical contact with infected bodily fluids; blood, faeces, or vomit (World Health Organization [WHO], 2014). Over the past three decades, EBOV has been responsible for a series of epidemics (Zhang & Jain, 2020). The outbreak of 2013–16 was categorized by WHO as a Public Health Emergency of International Concern, which highlighted the difficulties associated with treating Ebola virus infections and raised concerns about society's readiness to manage future epidemics on scientific, clinical, and sociological levels.

1.1 Literature survey

Many researchers have created epidemic models to better understand the EBOV virus' disease mechanics. Some models are of integer order, but there are also fractional-order models. A fractional-order model, in contrast to the integer-order models, provides more freedom to fit the real data, which enhances the model's coherence with actual data and observations (Khajehsaeid, 2018). In Singh (2020), the author used a fractional-order GL-derivative to form an iterative numerical scheme to find numerical solutions of the epidemic model based on EBOV, and reported the CPU time usage. In Srivastava, and Saad (2020), the authors looked at three potential kernel-based numerical solutions for a fractal-fractional Ebola virus model. In Gao, Li, Li, and Zhou (2021), Shaikh, and Nisar (2019), the authors used the fractional-order CF-derivatives and a fixed-point theorem in the proposed epidemic model to show the existence and uniqueness of the governing system's solution. In Shah, Patel, and Yeolekar

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(2019), the authors proposed an integral-order model that discussed the vertical dynamics of Ebola with media impacts. In Liu, Fečkan, O'Regan, and Wang (2019), Nwajeri, Omame, and Onyenegecha (2021), the authors derived generalized UHR-stability results using fractional-order CF-derivatives in their proposed models. Fractional calculus is currently the focus of numerous studies on epidemic models. Multiple numerical and analytical methods have been developed to solve fractional calculus problems. Some other related models are discussed in Hussain, Baleanu, and Adeel (2020), Liu, Fečkan, and Wang (2020), Solís-Pérez, Gómez-Aguilar, and Atangana (2018). In Singh, Srivastava, Hammouch, and Nisar (2021), the authors analyzed the stability conditions and the numerical results of the proposed fractional-order model on COVID-19. In Singh, Baleanu, Singh, and Dutta (2021), the authors looked at a non-integer order smoking model, utilized an iterative technique to get numerical findings, and listed CPU times to illustrate the efficiency of solutions.

2. Preliminaries

Definition 1. *The fractional-order ϕ-derivative with $\phi \in (0,1]$ of function $f \in H^1[a,b]$ in Caputo's sense is defined as,*

$$ {}^{C}D_t^{\phi} f(t) = \frac{1}{\Gamma(1-\phi)} \int_a^t f'(s)(t-s)^{-\phi} ds \quad , \quad t > a \tag{1}$$

A new derivative is introduced using an exponential kernel to avoid the singularity at $t = s$ in the above expression (1).

Definition 2. (Losada & Nieto, 2015) *The new fractional-order ϕ-derivative of a function f in Caputo-Fabrizio's sense can be written as,*

$$ {}^{CF}D_t^{\phi} f(t) = \frac{(2-\phi)M(\phi)}{2(1-\phi)} \int_a^t f'(s) \exp\left[-\frac{\phi(t-s)}{(1-\phi)}\right] ds, \qquad t > a$$

where $M(\phi)$ is the normalizing constant function that depends upon ϕ.

Definition 3. *The Laplace transform of the fractional-order ϕ-derivative of a function f in Caputo-Fabrizio's sense is defined as,*

$$ \mathscr{L}\left\{ {}^{CF}D_t^{\phi} f(t), s \right\} = \frac{s\mathscr{L}\{f(t), s\} - f(0)}{\left(s + \phi(1-s)\right)}, \qquad s \geq 0.$$

Definition 4. *The fractional integral of order ϕ of a function f in Caputo-Fabrizio's sense is defined as,*

$$ {}^{CF}\mathscr{I}_t^{\phi}\left(\mathrm{f(t)}\right) = \frac{2(1-\phi)}{(2-\phi)M(\phi)} \mathrm{f(t)} + \frac{2\phi}{(2-\phi)M(\phi)} \int_a^t \mathrm{f}(\tau) d\tau, \qquad t > a.$$

3. Model Formulation

The compartments of the model are defined as follows: Susceptible that are uninfected (S), exposed to Ebola infection (E), infectious from the infection (I), Hospitalized (H), Deceased or critically sick (D), and Recovered from infection (R). The total population (N) is the sum of all the compartments. At any instance of time t,

$$N(t) = S(t) + E(t) + I(t) + H(t) + D(t) + R(t)$$

The model has the following assumptions:

• Exposed individuals $E(t)$, infectious individuals ($I(t)$), Hospitalized individuals ($H(t)$), and Deceased or critically sick individuals ($D(t)$) are carriers of the EBOV at any instance of time t.

• Whenever any susceptible person (S) comes into touch with any deceased (D), hospitalized (H), exposed (E), or infectious (I) person, it acquires EBOV at the rate $\alpha_1(E + I + H + D)S/N$.

The meaning of the parameters used in the model is given in Table 1. The governing system of the fractional-order non-linear differential equations which describe the proposed epidemic model is as follows:

$$\begin{aligned}
{}^{CF}D_t^{\phi} E(t) &= \frac{\alpha_1\left(E(t)+I(t)+D(t)+H(t)\right)S(t)}{N(t)} - (\mu + \alpha_2)E(t) \\
{}^{CF}D_t^{\phi} I(t) &= \alpha_2 E(t) - (\alpha_3 + \alpha_4 + \mu)I(t) \\
{}^{CF}D_t^{\phi} H(t) &= \alpha_3 I(t) - (\alpha_5 + \alpha_6 + \mu)H(t) \\
{}^{CF}D_t^{\phi} D(t) &= \alpha_4 I(t) + \alpha_5 H(t) - \mu D(t) \\
{}^{CF}D_t^{\phi} R(t) &= \alpha_6 H(t) - \mu R(t) \\
{}^{CF}D_t^{\phi} S(t) &= B - \mu S(t) - \frac{\alpha_1\left(E(t)+I(t)+D(t)+H(t)\right)S(t)}{N(t)}
\end{aligned} \tag{2}$$

with non-negative initial condition, $(S(0), E(0), I(0), H(0), D(0), R(0)) \in \mathbb{R}_+^6$.

Table 1. Meanings of parameters used in the proposed model.

Parameter	Meaning
B	The birth rate of the population
α_1	Contact rate of susceptible with EBOV carriers
α_2	The transmission rate of exposed getting infectious from the Ebola
α_3	The transmission rate of infectious getting hospitalized
α_4	The transmission rate of infectious getting critically ill or deceased
α_5	The transmission rate of hospitalized getting critically ill or deceased
α_6	The transmission rate of hospitalized getting recovered
μ	The death rate of the population

It can be rewritten in vector form as follows:

$$^{CF}D_t^{\phi}\vec{\psi}(t) = \vec{\mathcal{K}}(t, E(t), I(t), H(t), D(t), R(t), S(t)) \tag{3}$$

for $\phi \in (0,1]$, $t \in J = [0, b]$ with following initial condition, (4)

$$\vec{\psi}(0) = \vec{\psi}_0 = (E(0), I(0), H(0), D(0), R(0), S(0))^T \tag{5}$$

where $\vec{\psi}(t) = \big(E(t), I(t), H(t), D(t), R(t), S(t)\big)^T$
and $\vec{\mathcal{K}}(t) = (\mathcal{K}_1(t), \mathcal{K}_2(t), \mathcal{K}_3(t), \mathcal{K}_4(t), \mathcal{K}_5(t), \mathcal{K}_6(t))^T$.

4. Analysis of Model

4.1 Equilibrium points

In this section, the equilibrium points of system (2) are evaluated. These points are the steady-state solutions of the system (2). There are two equilibrium points of the proposed system to be analyzed. The Ebola-free equilibrium point, E^0 is given by:

$$E^0 = \left\{S = \frac{B}{\mu}, E = I = H = D = R = 0\right\}$$

The endemic equilibrium point E^1 is given by:

$$E^1 = \left\{S^* = \frac{A_1}{A_2}, E^* = \frac{A_3}{A_4}, I^* = \frac{A_5}{A_6}, H^* = \frac{A_7}{A_8}, D^* = \frac{A_9}{A_{10}}, R^* = \frac{A_{11}}{A_{12}}\right\}$$

where,

$A_1 = N\mu(\mu + \alpha_5 + \alpha_6)(\mu + \alpha_3 + \alpha_4)(\mu + \alpha_2)$

$A_2 = \alpha_1(\mu^3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^2 + ((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\alpha_2)$

$A_3 = -N\mu^5 - N(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^4 + (-N(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + B\alpha_1 - N(\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu^3 + (\big(B\alpha_1 - N(\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4)\big)\alpha_2 + B\alpha_1(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))\mu^2 + B((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\alpha_1\mu + \big((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5\big)B\alpha_2\alpha_1$

$A_4 = (\mu + \alpha_2)(\mu^3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^2 + ((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\alpha_2)\alpha_1$

$A_5 = (-N\mu^5 - N(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^4 + (-N(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 - N(\alpha_5 + \alpha_6)\alpha_4 - N(\alpha_5 + \alpha_6)\alpha_3 + B\alpha_1)\mu^3 + ((-N(\alpha_5 + \alpha_6)\alpha_4 - N(\alpha_5 + \alpha_6)\alpha_3 + B\alpha_1)\alpha_2 + B\alpha_1(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))\mu^2 + B((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\alpha_1\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)B\alpha_2\alpha_1)\alpha_2$

$A_6 = (\mu + \alpha_2)(\mu + \alpha_3 + \alpha_4)(\mu^3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^2 + ((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\alpha_2)\alpha_1$

$A_7 = (-N\mu^5 - N(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^4 + (-N(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 - N(\alpha_5 + \alpha_6)\alpha_4 - N\alpha_3\alpha_5 - N\alpha_3\alpha_6 + B\alpha_1)\mu^3 + ((-N((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5 + \alpha_3\alpha_6) + B\alpha_1)\alpha_2 + B\alpha_1(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))\mu^2 + B((\alpha_3 + \alpha_4\ \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\alpha_1\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_5)B\alpha_2\alpha_1)\alpha_2\alpha_3$

$A_8 = (\mu + \alpha_2)(\mu + \alpha_3 + \alpha_4)(\mu + \alpha_5 + \alpha_6)(\mu^3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^2 + ((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\alpha_2)\alpha_1$

$A_9 = (-N\mu^5 - N(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^4 + (-N(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 - N(\alpha_5 + \alpha_6)\alpha_4 - N(\alpha_5 + \alpha_6)\alpha_3 + B\alpha_1)\mu^3 + ((-N(\alpha_5 + \alpha_6)\alpha_4 - N(\alpha_5 + \alpha_6)\alpha_3 + B\alpha_1)\alpha_2 + B\alpha_1(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))\mu^2 + B((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\alpha_1\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_5)B\alpha_2\alpha_1)\alpha_2(\alpha_4\mu + (\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5$

$A_{10} = \mu A_8$

$A_{11} = (-N\mu^5 - N(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^4 + (-N(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 - N(\alpha_5 + \alpha_6)\alpha_4 - N\alpha_3\alpha_5 - N\alpha_3\alpha_6 + B\alpha_1)\mu^3 + ((-N(\alpha_5 + \alpha_6)\alpha_4 - N\alpha_3\alpha_5 - N\alpha_3\alpha_6 + B\alpha_1)\alpha_2 + B\alpha_1(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))\mu^2 + B((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\alpha_1\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\ \alpha_2\alpha_1 B)\alpha_2\alpha_6\alpha_3$

$$A_{12} = \mu(\mu+\alpha_2)(\mu+\alpha_3+\alpha_4)(\mu+\alpha_5+\alpha_6)(\mu^3+(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)\mu^2+((\alpha_3+\alpha_4+\alpha_5+\alpha_6)\alpha_2+(\alpha_5+\alpha_6)(\alpha_3+\alpha_4))\mu+((\alpha_5+\alpha_6)\alpha_4+\alpha_3\alpha_5)\alpha_2)\alpha_1$$

4.2 Basic reproduction number

The next-generation matrix method (Diekmann, Heesterbeek, & Metz, 1990; Otunuga, 2021) is used to calculate the basic reproduction number of the proposed model. The fractional-order system (2) can be rewritten as:

$$^{CF}D_t^{\phi}\vec{\psi}(t) = \vec{f}(t) - \vec{v}(t)$$

where,

$$\vec{\psi}(t) = \begin{bmatrix} E(t) \\ I(t) \\ H(t) \\ D(t) \\ R(t) \\ S(t) \end{bmatrix}, \vec{f}(t) = \begin{bmatrix} \dfrac{\alpha_1(E+I+D+H)S}{N} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{\alpha_1(E+I+D+H)S}{N} \end{bmatrix} \text{and } \vec{v}(t) = \begin{bmatrix} (\mu+\alpha_2)E \\ (\mu+\alpha_3+\alpha_4)I-\alpha_2 E \\ (\mu+\alpha_5+\alpha_6)H-\alpha_3 I \\ \mu D-\alpha_4 I-\alpha_5 H \\ \mu R-\alpha_6 H \\ \mu S - B \end{bmatrix}$$

are the column vectors with the initial condition, $\vec{\psi}(0) = \vec{\psi}_0$. Assume that F and V be the Jacobian matrices of the column vectors $\vec{f}$ and $\vec{v}$, respectively.
At the Ebola-free equilibrium point E^0,

$$F_{E^0} = F(E^0) = \begin{bmatrix} \vee & \vee & \vee & \vee & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\vee & -\vee & -\vee & -\vee & 0 & 0 \end{bmatrix} \quad (\text{where } \vee = B\alpha_1 N\mu)$$

$$V_{E^0} = V(E^0) = \begin{bmatrix} \mu+\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_2 & \mu+\alpha_3+\alpha_4 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_3 & \mu+\alpha_5+\alpha_6 & 0 & 0 & 0 \\ 0 & -\alpha_4 & -\alpha_5 & \mu & 0 & 0 \\ 0 & 0 & -\alpha_6 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

are a singular and a non-singular 6×6 matrix. The basic reproduction number is the spectral radius of the matrix FV^{-1} at Ebola-free point E^0.

$$\mathscr{R}_0 = \rho\left(F_{E^0}V_{E^0}^{-1}\right) = \left[\frac{B(\mu^3+(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)\mu^2+((\alpha_3+\alpha_4+\alpha_5+\alpha_6)\alpha_2+(\alpha_5+\alpha_6)(\alpha_3+\alpha_4))\mu+((\alpha_5+\alpha_6)\alpha_4+\alpha_3\alpha_5)\alpha_2)\alpha_1}{N(\mu+\alpha_5+\alpha_6)(\mu+\alpha_3+\alpha_4)(\mu+\alpha_2)\mu^2}\right]$$

4.3 Invariant positive region

Lemma 1. *The proposed fractional-order model for Ebola infection* (*2*) *has the feasible domain of solution*

$$\Omega = \left\{(S,E,I,H,D,R) \in \mathbb{R}_+^6 | 0 \le N \le \frac{B}{\mu}\right\}$$

that is positively invariant.
Proof. Adding all the equations in the fractional-order system (2),

$$^{CF}D_t^{\phi}N(t) = B - \mu N(t)$$

Using the Laplace transform on each side of the previous equation,

$$\mathscr{L}\left\{{}^{CF}D_t^{\phi}N(t), s\right\} = \mathscr{L}\{B-\mu N(t), s\}$$

This gives,

$$\frac{s\mathscr{L}\{N(t),s\} - N(0)}{\left(s+\phi(1-s)\right)} = \frac{B}{s} - \mu\mathscr{L}\{N(t),s\}$$

provided $s > 0$.

Solving for $\mathscr{L}\{N(t),s\}$ and taking inverse Laplace transform,

$$N(t) = N(0)\mathscr{L}^{-1}\left\{\frac{1}{s+\mu s+\mu\phi(1-s)}\right\} + B\mathscr{L}^{-1}\left\{\frac{s+\phi(1-s)}{(1+\mu-\phi\mu)s^2+\phi\mu s}\right\}$$

$$= \frac{N(0)}{1+\mu-\mu\phi}\mathscr{L}^{-1}\left\{\frac{1}{s+\left(\frac{\mu\phi}{1+\mu(1-\phi)}\right)}\right\} + \frac{B}{1+\mu-\mu\phi}\begin{bmatrix}(1-\phi)\mathscr{L}^{-1}\left\{\frac{1}{s+\frac{\mu\phi}{1+\mu(1-\phi)}}\right\} \\ +\phi\mathscr{L}^{-1}\left\{\frac{1}{s\left(s+\frac{\phi\mu}{(1+\mu-\mu\phi)}\right)}\right\}\end{bmatrix}$$

$$= \frac{N(0)}{1+\mu-\mu\phi}\left[\exp\left[-\left(\frac{\mu\phi t}{1+\mu(1-\phi)}\right)\right]\right]$$

$$+\frac{B}{1+\mu-\mu\phi}\begin{bmatrix}(1-\phi)\exp\left[-\left(\frac{\mu\phi t}{1+\mu(1-\phi)}\right)\right] \\ +\frac{1+\mu-\mu\phi}{\mu}\left[1-\exp\left[-\left(\frac{\mu\phi t}{1+\mu(1-\phi)}\right)\right]\right]\end{bmatrix}$$

$$= \frac{B}{\mu} + \left[\frac{N(0)+B(1-\phi)}{1+\mu-\mu\phi} - \frac{B}{\mu}\right]\ \exp\left[-\left(\frac{\mu\phi t}{1+\mu(1-\phi)}\right)\right]$$

and because of the asymptotic decay characteristic of the inverse exponential function as time grows, $\lim_{t\to\infty} N(t) \leq B/\mu$. Therefore, the fractional-order system has a positively oriented bounded region.

5. Existence and Uniqueness of the Solution

In this section, the existence and uniqueness of the solution of the proposed fractional-order model are shown. Applying the fractional-integral operator (${}^{CF}\mathscr{I}_t^{\phi}$) on both sides into the system of equations (2),

$$\begin{aligned}
E(t)-E(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}\left(\frac{\alpha_1(E(t)+I(t)+D(t)+H(t))S(t)}{N(t)} - (\mu+\alpha_2)E(t)\right)\\
I(t)-I(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}(\alpha_2 E(t) - (\alpha_3+\alpha_4+\mu)I(t))\\
H(t)-H(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}(\alpha_3 I(t) - (\alpha_5+\alpha_6+\mu)H(t))\\
D(t)-D(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}(\alpha_4 I(t) + \alpha_5 H(t) - \mu D(t))\\
R(t)-R(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}(\alpha_6 H(t) - \mu R(t))\\
S(t)-S(0) &= {}^{CF}\ \mathscr{I}_t^{\phi}\left(B - \mu S(t) - \frac{\alpha_1(E(t)+I(t)+D(t)+H(t))S(t)}{N(t)}\right)
\end{aligned} \tag{6}$$

Solving the right-hand side of the system (6),

$$\begin{aligned}
E(t)-E(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_1(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_1(\tau,\psi(\tau))d\tau\\
I(t)-I(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_2(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_2(\tau,\psi(\tau))d\tau\\
H(t)-H(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_3(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_3(\tau,\psi(\tau))d\tau\\
D(t)-D(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_4(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_4(\tau,\psi(\tau))d\tau\\
R(t)-R(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_5(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_5(\tau,\psi(\tau))d\tau\\
S(t)-S(0) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_6(t,\psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_6(\tau,\psi(\tau))d\tau
\end{aligned} \tag{7}$$

where kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5$, and $\mathcal{K}_6$ are defined as:

$$\begin{aligned}
\mathcal{K}_1(t,\psi(t)) &= \frac{\alpha_1(E(t)+I(t)+D(t)+H(t))S(t)}{N(t)} - (\mu+\alpha_2)E(t)\\
\mathcal{K}_2(t,\psi(t)) &= \alpha_2 E(t) - (\alpha_3+\alpha_4+\mu)I(t)\\
\mathcal{K}_3(t,\psi(t)) &= \alpha_3 I(t) - (\alpha_5+\alpha_6+\mu)H(t)\\
\mathcal{K}_4(t,\psi(t)) &= \alpha_4 I(t) + \alpha_5 H(t) - \mu D(t)\\
\mathcal{K}_5(t,\psi(t)) &= \alpha_6 H(t) - \mu R(t)\\
\mathcal{K}_6(t,\psi(t)) &= B - \mu S(t) - \frac{\alpha_1(E(t)+I(t)+D(t)+H(t))S(t)}{N(t)}
\end{aligned} \tag{8}$$

Lemma 2. *The kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5$, and $\mathcal{K}_6$ satisfy Lipschitz condition if*

$$0 \le L = sup\{L_1, L_2, L_3, L_4, L_5, L_6\} < 1$$

where $L_1 = |\alpha_1 s_1 - (\mu + \alpha_2)|, L_2 = |\mu + \alpha_3 + \alpha_4|, L_3 = |\mu + \alpha_5 + \alpha_6|, L_4 = L_5 = \mu, L_6 = |\alpha_1(1 - s_0 - r_0) - \mu|$, $s_0 = \inf_t S(t)/N(t) \le \sup_t S(t)/N(t) = s_1$ *and* $r_0 = \inf_t R(t)/N(t)$.

Proof. Let $E_1,\ E_2$ be corresponding functions for the kernel $\mathcal{K}_1$. Let I_1, I_2 be corresponding functions for the kernel $\mathcal{K}_2$. Let H_1, H_2 be corresponding functions for the kernel $\mathcal{K}_3$. Let D_1, D_2 be corresponding functions for the kernel $\mathcal{K}_4$, R_1, R_2 be corresponding functions for the kernel $\mathcal{K}_5$ and S_1, S_2 be corresponding functions for the kernel $\mathcal{K}_6$. Then,

$$\| \mathcal{K}_1(t, E_1(t)) - \mathcal{K}_1(t, E_2(t)) \| = \left\| \left(\frac{\alpha_1 S(t)}{N(t)} - (\mu + \alpha_2)\right)(E_1(t) - E_2(t)) \right\| \tag{9}$$

$$\le \underbrace{\sup_t \left|\frac{\alpha_1 S(t)}{N(t)} - (\mu + \alpha_2)\right|}_{L_1} \|E_1(t) - E_2(t)\| \tag{10}$$

Similarly,

$$\| \mathcal{K}_2(t, I_1(t)) - \mathcal{K}_2(t, I_2(t)) \| = \|(-(\mu + \alpha_3 + \alpha_4))(I_1(t) - I_2(t))\| \tag{11}$$

$$\le \underbrace{|\mu + \alpha_3 + \alpha_4|}_{L_2} \|I_1(t) - I_2(t)\| \tag{12}$$

$$\| \mathcal{K}_3(t, H_1(t)) - \mathcal{K}_3(t, H_2(t)) \| = \|(-(\mu + \alpha_5 + \alpha_6))(H_1(t) - H_2(t))\| \tag{13}$$

$$\le \underbrace{|\mu + \alpha_5 + \alpha_6|}_{L_3} \|H_1(t) - H_2(t)\| \tag{14}$$

$$\| \mathcal{K}_4(t, D_1(t)) - \mathcal{K}_3(t, D_2(t)) \| = \|(-\mu)(D_1(t) - D_2(t))\| \tag{15}$$

$$\le \underbrace{|\mu|}_{L_4} \|D_1(t) - D_2(t)\| \tag{16}$$

$$\| \mathcal{K}_5(t, R_1(t)) - \mathcal{K}_5(t, R_2(t)) \| = \|(-\mu)(R_1(t) - R_2(t))\| \tag{17}$$

$$\le \underbrace{|\mu|}_{L_5} \|R_1(t) - R_2(t)\| \tag{18}$$

$$\| \mathcal{K}_6(t, S_1(t)) - \mathcal{K}_6(t, S_2(t)) \| = \left\| \frac{\alpha_1(E(t) + I(t) + D(t) + H(t))(S_1(t) - S_2(t))}{N(t)} \right\| \tag{19}$$

$$\le \underbrace{\sup_t \left|\alpha_1\left(1 - \frac{S(t)}{N(t)} - \frac{R(t)}{N(t)}\right)\right|}_{L_6} \|S_1(t) - S_2(t)\| \tag{20}$$

For each $n \in \mathbb{N}$, we can get the following system of recursive relations using Picard's iteration,

$$\begin{aligned}
E_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_1(t, E_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_1(\tau, E_{n-1}(\tau))d\tau \\
I_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_2(t, I_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_2(\tau, I_{n-1}(\tau))d\tau \\
H_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_3(t, H_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_3(\tau, H_{n-1}(\tau))d\tau \\
D_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_4(t, D_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_4(\tau, D_{n-1}(\tau))d\tau \\
R_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_5(t, R_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_5(\tau, R_{n-1}(\tau))d\tau \\
S_n(t) &= \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}_6(t, S_{n-1}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}_6(\tau, S_{n-1}(\tau))d\tau
\end{aligned} \tag{21}$$

Now, using the Lipschitz inequalities and above recursive equations (21),

$$\| \Delta E_n(t) \| = \| E_n(t) - E_{n-1}(t) \| \tag{22}$$

$$\le \frac{2(1-\phi)L_1}{(2-\phi)M(\phi)} \| \Delta E_{n-1}(t) \| + \frac{2\phi L_1}{(2-\phi)M(\phi)}\int_0^t \| \Delta E_{n-1}(\tau) \| \, d\tau \tag{23}$$

$$\le \left(\frac{2L_1(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta E_{n-1}(t)\| \tag{24}$$

Similarly,

$$\| \Delta I_n(t) \| = \| I_n(t) - I_{n-1}(t) \| \le \left(\frac{2L_2(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta I_{n-1}(t)\| \tag{25}$$

$$\| \Delta H_n(t) \| = \| H_n(t) - H_{n-1}(t) \| \le \left(\frac{2L_3(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta H_{n-1}(t)\| \tag{26}$$

$$\| \Delta D_n(t) \| = \| D_n(t) - D_{n-1}(t) \| \le \left(\frac{2L_4(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta D_{n-1}(t)\| \tag{27}$$

$$\| \Delta R_n(t) \| = \| R_n(t) - R_{n-1}(t) \| \le \left(\frac{2L_5(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta R_{n-1}(t)\| \tag{28}$$

$$\| \Delta S_n(t) \| = \| S_n(t) - S_{n-1}(t) \| \le \left(\frac{2L_6(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)}\right) \|\Delta S_{n-1}(t)\| \tag{29}$$

Further, it can be observed by telescoping sums that,

$$E_m(t) = E_0 + \sum_{n=1}^{m} \Delta E_n(t),\ I_m(t) = I_0 + \sum_{n=1}^{m} \Delta I_n(t)$$
$$H_m(t) = H_0 + \sum_{n=1}^{m} \Delta H_n(t)\ D_m(t) = D_0 + \sum_{n=1}^{m} \Delta D_n(t)$$
$$R_m(t) = R_0 + \sum_{n=1}^{m} \Delta R_n(t), S_m(t) = S_0 + \sum_{n=1}^{m} \Delta S_n(t)$$

This proves the result.

Theorem 1. (Existence of solution) *There exists a solution of the fractional system* (2) *provided* $0 \le \theta < 1$. *where* $\theta = \left(\frac{2L(1-\phi+\phi T_{sup})}{M(\phi)(2-\phi)}\right)$ *and* $L = sup\{L_1, L_2, L_3, L_4, L_5, L_6\}$.

Proof. The functions $E(t), I(t), H(t), D(t), R(t), S(t)$ are bounded and respective kernels satisfy the Lipschitz conditions. Using the recursive formula for the inequalities (22) - (29),

$$\begin{aligned} \| \Delta E_n(t) \| &\le \left(\frac{2L_1(1-\phi+\phi T_{\text{sup}})}{M(\phi)(2-\phi)}\right)^{n-1} \|\Delta E_1(t)\| \\ &\le \theta^{n-1} \| \Delta E_1(t) \| \\ &\xrightarrow{n\to\infty} 0,\ \text{as}\ 0 \le \theta < 1. \end{aligned}$$

Similarly, it can be observed for the following sequences that

$$\| \Delta I_n(t) \|, \| \Delta H_n(t) \|, \| \Delta D_n(t) \|, \| \Delta R_n(t) \|, \| \Delta S_n(t) \| \xrightarrow{n\to\infty} 0,\ \text{as}\ 0 \le \theta < 1.$$

This proves the solutions of the fractional system exist and are of the form mentioned in (7).

Lemma 3. (Nwajeri, Panle, Omame, Obi, & Onyenegecha, 2022) *Consider the initial value problem* ${}^{CF}\psi_t^{\phi}(t) = \mathcal{K}(t, \psi(t))$, $\psi(0) = \psi_0$ *and suppose that there exists a Lipschitz constant* $L \ge 0$ *such that*

$$|\mathcal{K}(t, \psi_1(t)) - \mathcal{K}(t, \psi_2(t))| \le L|\psi_1(t) - \psi_2(t)|, \tag{30}$$

for all $t \in J = [0, b]$ *and* $\psi_1, \psi_2 \in C(J, \mathbb{R})$. *If* $L\left(\frac{2(1-\phi)+2\phi T_{\text{sup}}}{(2-\phi)M(\phi)}\right) < 1$, *Then, there exists a unique solution of the initial value problem on* $J = [0, b]$.

Proof. The uniqueness of the solution to this initial value problem is a consequence of the Banach fixed point theorem. Let $C(J, \mathbb{R})$ denote the Banach space of all the continuous functions from J to $\mathbb{R}$ with infinity norm.

$$\| f \|_\infty = \sup_t\{|f(t)| | t \in J = [0, b]\}, \quad \forall f \in C(J, \mathbb{R})$$

Consider the mapping $\omega: C^\theta(J, \mathbb{R}) \to C^\theta(J, \mathbb{R})$ defined by,

$$\omega\psi(t) = \psi_0 + \frac{2(1-\phi)}{(2-\phi)M(\phi)}(\mathcal{K}(t) - \mathcal{K}(0)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau)d\tau$$

Assume $\psi_1, \psi_2 \in C^\theta(J, \mathbb{R})$ and for each $t \in J$,

$$\begin{aligned} |\omega\psi_1(t) - \omega\psi_2(t)| \le& \frac{2(1-\phi)}{(2-\phi)M(\phi)}|\mathcal{K}(t, \psi_1(t)) - \mathcal{K}(t, \psi_2(t))| \\ &+ \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t |\mathcal{K}(\tau, \psi_1(\tau)) - \mathcal{K}(\tau, \psi_2(\tau))|d\tau \end{aligned}$$

Using inequality (30) and for each $t \in J$,

$$\begin{aligned} |\omega\psi_1(t) - \omega\psi_2(t)| \le& \frac{2L(1-\phi)}{(2-\phi)M(\phi)}|\psi_1(t) - \psi_2(t)| \\ &+ \frac{2L\phi}{(2-\phi)M(\phi)}\int_0^t |\psi_1(\tau) - \psi_2(\tau)|d\tau \\ \le& L\left(\frac{2(1-\phi)}{(2-\phi)M(\phi)} + \frac{2\phi T_{\text{sup}}}{(2-\phi)M(\phi)}\right)|\psi_1(t) - \psi_2(t)| \end{aligned}$$

Taking supremum over $t \in J$,

$$\| \omega\psi_1 - \omega\psi_2 \|_\infty \le L\left(\frac{2(1-\phi)+2\phi T_{\text{sup}}}{(2-\phi)M(\phi)}\right) \| \psi_1 - \psi_2 \|_\infty$$

Thus, ω is a contraction mapping if $0 \le L\left(\frac{2(1-\phi)+2\phi T_{\text{sup}}}{(2-\phi)M(\phi)}\right) < 1$. Consequently, by the Banach fixed point theorem, the operator ω has a fixed point say (ψ) i.e. $(\omega\psi = \psi)$ which is the required unique solution of the initial value problem on $C(J, \mathbb{R})$.

Theorem 2. (Uniqueness of solution) *The solution* (*as provided by system 7*) *of the fractional system* (2) *is unique.*

Proof. We use Lemma 3 to show the uniqueness of the solution (as mentioned in system 7) for the fractional system (2). By Lemma 2, kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5$, and $\mathcal{K}_6$ satisfies the Lipschitz conditions with constants L_1, L_2, L_3, L_4, L_5, and L_6, respectively. Let $L = \sup\{L_1, L_2, L_3, L_4, L_5, L_6\}$ and $\theta = \left(\frac{2(1-\phi)+2\phi T_{\sup}}{(2-\phi)M(\phi)}\right)$. Since a solution exists if $0 \leq \theta < 1$ (hypothesis of existence Theorem 1), the hypothesis of Lemma 3 is satisfied for each of the equations (with initial values) of the fractional-order system (2). Hence, using Lemma 3, we conclude that the fractional-order system (2) has a unique solution if $0 \leq \theta < 1$.

6. Stability

This section obtains the appropriate stability conditions for the generalized Ulam-Hyers-Rassias stability of the proposed fractional-order model.

6.1. Generalized Ulam-Hyers-Rassias stability

Definition 5. (Nwajeri *et al.*, 2021, 2022; Liu, Fečkan, O'Regan, & Wang, 2019) *The fractional-order model* ${}^{CF}D_t^{\phi}\psi(t) = \mathcal{K}(t, \psi(t))$ *is generalized Ulam-Hyers-Rassias* (*UHR*) *stable in accordance with* $\mathcal{Y}(t) \in H^1[J, \mathbb{R}^+]$ *if there exists a positive real value* ε_ϕ (*depending upon* ϕ) *such that for every solution* ψ *of the following inequality,*

$$\left| {}^{CF}D_t^{\phi}\psi(t) - \mathcal{K}\big(t, \psi(t)\big) \right| \leq \mathcal{Y}(t),$$

there exists a solution $\tilde{\psi} \in H^1(J, \mathbb{R}^+)$ *of the model with*

$$\left|\psi(t) - \tilde{\psi}(t)\right| \leq \varepsilon_\phi \mathcal{Y}(t) \qquad \text{for each } t \in J.$$

Lemma 4. *The fractional-order model* ${}^{CF}D_t^{\phi}\psi(t) = \mathcal{K}(t, \psi(t))$ (*satisfying Lipschitz condition with Lipschitz constant L depending upon kernel* $\mathcal{K}$) *is generalized UHR-stable in accordance with non-decreasing positive function* $\mathcal{Y}$ *if,*

$$0 \leq \theta = \frac{2L\big((1-\phi)+\phi T_{sup}\big)}{(2-\phi)M(\phi)} < 1 \tag{31}$$

Proof. We let $\mathcal{Y}(t)$ represent any arbitrary positive function, then there exists a positive real number η such that,

$$\left(2(1-\phi)\mathcal{Y}(t) + 2\phi \int_0^t \mathcal{Y}(\tau)d\tau\right) \leq \eta\mathcal{Y}(t). \tag{32}$$

Since the kernel of the fractional-order model satisfies Lipschitz condition with Lipschitz constant L (depending upon kernel $\mathcal{K}$) i.e.

$$\left|\mathcal{K}(t, \psi(t)) - \mathcal{K}(t, \tilde{\psi}(t))\right| \leq L\left|\psi(t) - \tilde{\psi}(t)\right| \tag{33}$$

So, using the existence and uniqueness theorem of the model, there exists a unique solution say ($\tilde{\psi}$) of the fractional-order model of the following form,

$$\tilde{\psi}(t) = \tilde{\psi}_0 + \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \tilde{\psi}(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \tilde{\psi}(\tau))d\tau \tag{34}$$

Assume that ψ is the solution of the following inequality,

$$\left| {}^{CF}D_t^{\phi}\psi(t) - \mathcal{K}(t, \psi(t)) \right| \leq \mathcal{Y}(t), \tag{35}$$

Applying fractional-order integral operator,

$$\left|\psi(t) - {}^{CF}\mathcal{J}_t^{\phi}\mathcal{K}(t, \psi(t))\right| \leq \frac{\eta\mathcal{Y}(t)}{(2-\phi)M(\phi)} \tag{36}$$

$$\begin{aligned}&\left|\psi(t) - \tilde{\psi}_0 - \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \psi(t)) - \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \psi(\tau))d\tau\right| \\ &\qquad \leq \frac{\eta\mathcal{Y}(t)}{(2-\phi)M(\phi)}\end{aligned} \tag{37}$$

Now, consider the following,

$$\begin{aligned}\left|\psi(t) - \tilde{\psi}(t)\right| &= \left|\psi(t) - \tilde{\psi}_0 - \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \tilde{\psi}(t)) - \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \tilde{\psi}(\tau))d\tau\right| \\ &= \left|\begin{array}{c}\psi(t) - \tilde{\psi}_0 - \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \tilde{\psi}(t)) - \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \tilde{\psi}(\tau))d\tau \\ + \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \psi(t)) + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \psi(\tau))d\tau \\ - \frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t, \psi(t)) - \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau, \psi(\tau))d\tau\end{array}\right|\end{aligned}$$

$$\leq \left|\psi(t)-\tilde{\psi}_0-\frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t,\psi(t))-\frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau,\psi(\tau))d\tau\right| + \frac{2(1-\phi)}{(2-\phi)M(\phi)}\left|(\mathcal{K}(t,\psi(t))-\mathcal{K}(t,\tilde{\psi}(t))\right| + \frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \left|(\mathcal{K}(\tau,\psi(\tau))-\mathcal{K}(\tau,\tilde{\psi}(\tau))\right|d\tau$$

Using Lipschitz inequality (33), we get the following:

$$\begin{aligned}\left|\psi(t)-\tilde{\psi}(t)\right| &\leq \left|\psi(t)-\tilde{\psi}_0-\frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t,\psi(t))-\frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau,\psi(\tau))d\tau\right| \\ &+ \frac{2L(1-\phi)}{(2-\phi)M(\phi)}\left|\psi(t)-\tilde{\psi}(t)\right| + \frac{2L\phi}{(2-\phi)M(\phi)}\int_0^t \left|\psi(\tau)-\tilde{\psi}(\tau)\right|d\tau \\ &= \left|\psi(t)-\tilde{\psi}_0-\frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t,\psi(t))-\frac{2\phi}{(2-\phi)M(\phi)}\int_0^t \mathcal{K}(\tau,\psi(\tau))d\tau\right| \\ &+ \frac{2L\left((1-\phi)+\phi T_{\sup}\right)}{(2-\phi)M(\phi)}\left|\psi(t)-\tilde{\psi}(t)\right|\end{aligned}$$

Using (31) and (37), we get,

$$\begin{aligned}\left|\psi(t)-\tilde{\psi}(t)\right| &\leq \frac{\eta\mathcal{Y}(t)}{(2-\phi)M(\phi)} + \theta\left|\psi(t)-\tilde{\psi}(t)\right| \\ &\leq \frac{\eta\mathcal{Y}(t)}{(1-\theta)(2-\phi)M(\phi)} = \varepsilon_\phi\mathcal{Y}(t)\end{aligned}$$

7. Numerical Scheme

In this section, a new numerical scheme is obtained to solve numerically the fractional-order system representing the proposed model. Consider the fractional-order equation ${}^{CF}D_t^{\phi}\psi(t)=\mathcal{K}(t,\psi(t))$, applying the fundamental theorem of fractional calculus an iterative scheme is obtained as follows:

$$\psi(t_{n+1})-\psi(0)=\frac{2(1-\phi)}{(2-\phi)M(\phi)}\mathcal{K}(t_n,\psi(t_n))+\frac{2\phi}{(2-\phi)M(\phi)}\int_0^{t_{n+1}}\mathcal{K}(\tau,\psi(\tau))d\tau \tag{38}$$

Replacing value of $\psi(t_n)$, we get,

$$\begin{aligned}\psi(t_{n+1}) &= \psi(t_n)+\frac{2(1-\phi)}{(2-\phi)M(\phi)}[\mathcal{K}(t_n,\psi(t_n))-\mathcal{K}(t_{n-1},\psi(t_{n-1}))] \\ &+\frac{2\phi}{(2-\phi)M(\phi)}\int_{t_n}^{t_{n+1}}\mathcal{K}(\tau,\psi(\tau))d\tau\end{aligned} \tag{39}$$

Considering uniform step-size *h* along the time axis, the integral can be approximated as in the classical two-step Adams-Bashforth scheme as follows:

$$\int_{t_n}^{t_{n+1}}\mathcal{K}(\tau,\psi(\tau))d\tau \approx \frac{3h}{2}\mathcal{K}(t_n,\psi(t_n))-\frac{h}{2}\mathcal{K}(t_{n-1},\psi(t_{n-1})) \tag{40}$$

Substituting the value of approximated integral (40) into the above equation (39),

$$\begin{aligned}\psi(t_{n+1}) &= \psi(t_n)+\frac{2(1-\phi)}{(2-\phi)M(\phi)}[\mathcal{K}(t_n,\psi(t_n))-\mathcal{K}(t_{n-1},\psi(t_{n-1}))] \\ &+\frac{2\phi}{(2-\phi)M(\phi)}\left[\frac{3h}{2}\mathcal{K}(t_n,\psi(t_n))-\frac{h}{2}\mathcal{K}(t_{n-1},\psi(t_{n-1}))\right]\end{aligned}$$

Since $M(\phi)$ is a normalizing function with $M(0)=M(1)=1$, let us assume $M(\phi)=(2-\phi^2)/(2-\phi)$ which satisfies $M(0)=M(1)=1$. Thus,

$$\psi(t_{n+1})=\psi(t_n)+\frac{2+(3h-2)\phi}{2-\phi^2}\mathcal{K}(t_n,\psi(t_n))-\frac{2+(h-2)\phi}{2-\phi^2}\mathcal{K}(t_{n-1},\psi(t_{n-1})) \tag{41}$$

Hence, the fractional-order model (3) has the following numerical scheme to obtain the numerical solutions.

$$\vec{\psi}(t_{n+1})=\vec{\psi}(t_n)+\frac{2+(3h-2)\phi}{2-\phi^2}\vec{\mathcal{K}}(t_n,\vec{\psi}(t_n))-\frac{2+(h-2)\phi}{2-\phi^2}\vec{\mathcal{K}}(t_{n-1},\vec{\psi}(t_{n-1})) \tag{42}$$

7.1. Numerical simulations

This section uses MATLAB software to perform the numerical simulations of the numerical scheme (42). The total initial population is assumed to be $N(0)=100$ and the initial values of the compartments are assumed to be $S(0)=80, E(0)=10, I(0)=5, H(0)=3, D(0)=2, R(0)=0$. The used values of the parameters are as follows: $B=10$, $\mu=0.1$, $\alpha_1=0.75$, $\alpha_2=0.85$, $\alpha_3=0.425$, $\alpha_4=0.2$, $\alpha_5=0.15$, and $\alpha_6=0.25$.

Figure 2 shows the transmission dynamics of each compartment listed in the model over time. The behavior is smooth, and it validates the theoretical results. Figure 3 shows the behavior for different values of fractional order. For relatively small values of the fractional order, the number of infectious individuals reaches the peak of approximately 19 cases but takes a relatively long time.

Figure 4 shows the behavior of the Ebola infectious cases over time for different values of the contact rate. Increasing the contact rate of susceptibles with the pathogen carriers will cause a surge in the number of Ebola infectious individuals. The contact rate is the crucial parameter in this model that directly influences the cases of Ebola. This shows that the most efficient way to control the spread of EBOV infection is to control the contact rate parameter.

Figure 5 and Figure 6 illustrate the behaviors of the Susceptibles $S(t)$ and the Deceased $D(t)$ for various values of contact rate α_1, respectively.

In Table 2, the CPU time usage is listed for different step sizes Δt and iterations n of the numerical scheme proposed. The table makes it clear that the proposed strategy increases efficiency while taking less time.

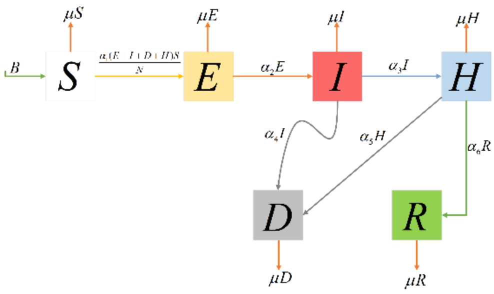

Figure 1. Flow diagram of the proposed compartmental model

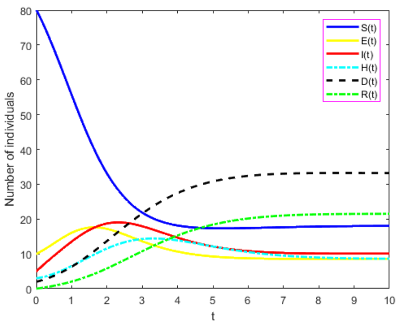

Figure 2. Smooth behavior of compartments over time using integer-order $\phi = 1$.

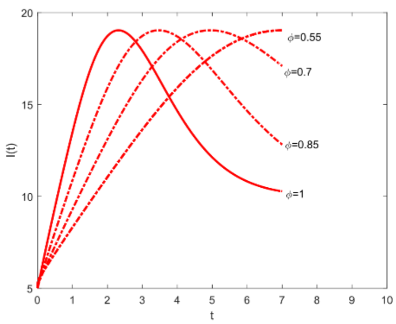

Figure 3. Effects of fractional order ϕ on infectious individuals *I(t)* over time *t*.

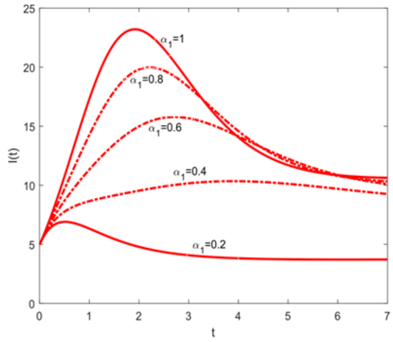

Figure 4. Effects of contact rate α_1 on infectious individuals *I(t)* over time *t* with integer-order $\phi = 1$

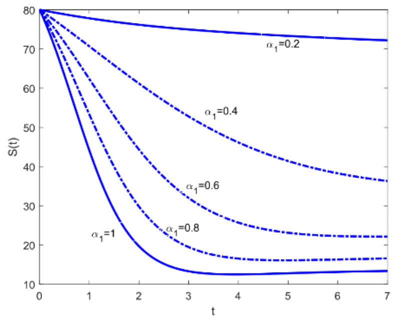

Figure 5. Effects of contact rate α_1 on susceptible individuals *S(t)* over time *t* with integer-order $\phi = 1$

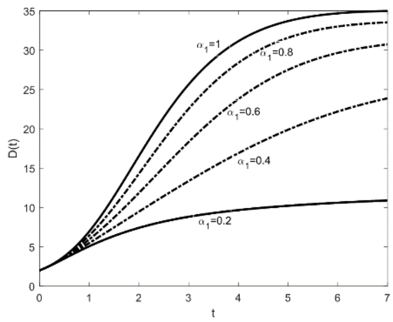

Figure 6. Effects of contact rate α_1 on deceased individuals $D(t)$ over time t with integer order $\phi = 1$

Table 2. CPU time usage for various values of Δt and n

Step size (Δt)	Number of iterations (n)	CPU time (s)
0.1	100	0.29
0.01	1000	0.34
0.001	10^4	0.41
0.0001	10^5	1.31
0.00001	10^6	2.39

8. Conclusions

In this study, an epidemic model for the Ebola disease was formulated using the Caputo-Fabrizio fractional derivative. The basic reproduction number ($\mathscr{R}_0$) is calculated using the next-generation matrix approach. We analyzed the conditions for the existence and uniqueness of the solution using a fixed point theorem approach. Additionally, the stability conditions for the generalized Ulam-Hyers-Rassias stability were found. This illustrates how the approximate solution of the proposed model differs for integer and fractional orders in numerical simulations. Additionally, the behavior of Ebola infections in deceased and vulnerable individuals at various contact rates was simulated. In the future, the authors can study this approach for other infectious diseases to get improved insights about the transmission of diseases, and the study outcomes may help the medical fraternity to work effectively.

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