Songklanakarin J. Sci. Technol.
46 (3), 294–301, May – Jun. 2024

http://www.sjst.psu.ac.th

Original Article]

Further results on the Diophantine equation $x^2 + 16 \cdot 7^b = y^n$ when n is even

Siti Hasana Sapar[1], Kai Siong Yow[1, 2*], and Hoa Pham[3]

[1] *Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, Serdang, Selangor, 43400 Malaysia*

[2] *School of Computer Science and Engineering, College of Engineering, Nanyang Technological University, Nanyang Ave, 639798 Singapore*

[3] *Department of Mathematics, An Giang University, Vietnam National University, Ho Chi Minh City, Vietnam*

Received: 6 November 2023; Revised: 28 March 2024; Accepted: 18 April 2024

Abstract

This work extends the results for the Diophantine equation $x^2 + 16 \cdot 7^b = y^n$ for $n = 2r$, where $x, y, b, r \in \mathbb{Z}^+$. Earlier results classified the generators of solutions, which are the pair of integers (x, y^r), into several categories and presented the general formula that determines the values of x and y^r for the respective category. The lower bound for the number of non-negative integral solutions associated with each b is also provided. We now extend the results and prove the necessary and sufficient conditions required to obtain integral solutions x and y to the equation, by considering various scenarios based on the parity of b. We also determine the values of n for which integral solutions exist.

Keywords: Diophantine equation, polynomial, generator, integral solution

1. Introduction

Diophantine equations have been studied extensively over the years, from linear (Cohn, 1993; Ko, 1965; Lebesgue, 1850) to exponential Diophantine equations (Arif & Abu Muriefah, 1997; Ismail, Atan, Sejas Viscarra, & Yow, 2023; Ismail, Atan, Yow, & Sejas Viscarra, 2021; Sapar & Yow, 2021; Yow, Sapar, & Atan, 2013). One of the most well-studied problems, namely Fermat's Last Theorem, has been investigated since 1637, and the problem was eventually solved by Wiles (1995) after more than three centuries. Diophantine equations can be applied in various areas, such as cryptography (Gao & Heindl, 2013), control systems (Kučera, 1993) and even chemistry (Crocker, 1968).

*Corresponding author
Email address: ksyow@upm.edu.my

On solving problems in Diophantine equations, the goal is to determine if there exist integral solutions that satisfy the equations. Logically, the complexity of this problem is directly proportional to the number of variables that are used in the problem. There are, however, some other factors that will affect the output of the problems, which include the parity and value of each variable. For instance, for the equation $x^2 + 2^k = y^n$, Cohn *(*1992*)* proved that there exist three families of solutions when k is odd and $n \geq 3$. Nonetheless, Arif and Abu Muriefah (1997) showed that for the case when k is even, extra conditions need to be imposed to solve the same equation. This implies that different techniques may be required to solve a Diophantine equation, particularly to solve those involving multiple variables.

Note that Diophantine equations that were studied are mostly derived from the equation $x^2 + C = y^n$ *(*Abu Muriefah & Bugeaud, 2006*)*, by manipulating the value of C using constants or prime numbers with exponents. The condition

$\gcd(x,y)=1$ is also imposed at times so that valid solutions can be found. For Diophantine equations with infinitely many solutions (with certain relaxation for some complicated cases), some generalizations of the solutions may also be given (Sapar & Yow, 2021).

In this article, we provide additional results for the Diophantine $x^2+16\cdot 7^b=y^n$ when n is even, following the generators of solutions to the equations proven in Yow, Sapar, and Low (2022). We show the necessary and sufficient conditions required for the equation to have integral solutions based on the parity of b under various scenarios. We also prove that integral solutions of the equation can only be obtained for some n.

We first give the definition and list the relevant result that will be used in our proofs. For the Diophantine equation $x^2+16\cdot 7^b=y^n$ where $n=2r$ and $b,r\in\mathbb{Z}^+$, the pair of integers (x,y^r) is a *generator* of solutions to the equation.

Theorem 1. (Yow *et al*., 2022) *Let $b,r,i\in\mathbb{Z}^+$. The generators $(x_{b,i}, y_{b,i}^r)$ of solutions to the equation $x^2+16\cdot 7^b=y^{2r}$ have the following forms:*

when $i\equiv 1 \pmod 3$,

$$x_{b,i} = 7^{\frac{1}{3}i-\frac{1}{3}}\left(2^2\cdot 7^{b-\frac{2}{3}i+\frac{2}{3}}-1\right), \tag{1}$$

$$y_{b,i}^r = 7^{\frac{1}{3}i-\frac{1}{3}}\left(2^2\cdot 7^{b-\frac{2}{3}i+\frac{2}{3}}+1\right), \tag{2}$$

when $i\equiv 2 \pmod 3$,

$$x_{b,i} = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2\cdot 7^{b-\frac{2}{3}i+\frac{4}{3}}-2\right), \tag{3}$$

$$y_{b,i}^r = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2\cdot 7^{b-\frac{2}{3}i+\frac{4}{3}}+2\right), \tag{4}$$

when $i\equiv 0 \pmod 3$,

$$x_{b,i} = 7^{\frac{1}{3}i-1}\left(7^{b-\frac{2}{3}i+2}-2^2\right), \tag{5}$$

$$y_{b,i}^r = 7^{\frac{1}{3}i-1}\left(7^{b-\frac{2}{3}i+2}+2^2\right), \tag{6}$$

where i represents the i^{th} set of non-negative integral solutions associated with each b, arranged in ascending order.

2. Solutions When $n=4$

In this section, we focus on the Diophantine equation $x^2+16\cdot 7^b=y^4$ by considering various scenarios based on the parity of b.

We first consider the scenario when b is an odd number. The following theorem gives the necessary and sufficient conditions to obtain integral solutions to the equation $x^2+16\cdot 7^b=y^4$ in such a scenario.

Theorem 2. *Let $i,t\in\mathbb{Z}^+$ and b be an odd positive integer. Then, $x_{b,i}=12\cdot 7^{2(t-1)}$ and $y_{b,i}=4\cdot 7^{t-1}$ are the integral solutions to the equation $x^2+16\cdot 7^b=y^4$ if and only if $b=4t-3$ and $i=6t-4$.*

Proof. Let $t\in\mathbb{Z}^+$. For the forward implication, suppose $x_{b,i}=12\cdot 7^{2(t-1)}$ and $y_{b,i}=4\cdot 7^{t-1}$ are the integral solutions to the equation $x^2+16\cdot 7^b=y^4$. Then, we have

$$\left(12\cdot 7^{2(t-1)}\right)^2+16\cdot 7^b = (4\cdot 7^{t-1})^4. \tag{7}$$

By simplifying Equation (7), we obtain $7^b=7^{4t-3}$, which implies that $b=4t-3$.

Note that the set $\{6t-4 \mid t\in\mathbb{Z}^+\}$ is a subset of $\{3t-1 \mid t\in\mathbb{Z}^+\}$. By Theorem 1, when $i=3t-1$, we have Equation (3) as the generator of solutions to the equation $x^2+16\cdot 7^b=y^{2r}$. By comparing the value of $x_{b,i}$ in Equation (3) to $x_{b,i}=12\cdot 7^{2(t-1)}$, we have

$$7^{\frac{1}{3}i-\frac{2}{3}}\left(2\cdot 7^{b-\frac{2}{3}i+\frac{4}{3}}-2\right) = 12\cdot 7^{2(t-1)}.$$

Since $b=4t-3$, we have

$$2\cdot 7^{4t-\frac{1}{3}i-\frac{7}{3}}-2\cdot 7^{\frac{1}{3}i-\frac{2}{3}} = 12\cdot 7^{2(t-1)}$$

$$7^{4t-\frac{1}{3}i-\frac{7}{3}}-7^{\frac{1}{3}i-\frac{2}{3}} = 6\cdot 7^{2t-2}. \tag{8}$$

Multiplying Equation (8) by $7^{-4t+\frac{1}{3}i+\frac{7}{3}}$, we obtain

$$1-7^{2\left(\frac{1}{3}i-2t\right)+\frac{5}{3}} = 6\cdot 7^{\frac{1}{3}i-2t+\frac{1}{3}}. \tag{9}$$

Let $x=7^{\frac{1}{3}i-2t}$. By rearranging Equation (9), we obtain the following quadratic equation:

$$7^{\frac{5}{3}}x^2+6\cdot 7^{\frac{1}{3}}x-1 = 0.$$

By solving the equation, we obtain,

$$x = \frac{-6\cdot 7^{\frac{1}{3}} \pm \sqrt{\left(6\cdot 7^{\frac{1}{3}}\right)^2 - 4\left(7^{\frac{5}{3}}\right)(-1)}}{2\cdot 7^{\frac{5}{3}}},$$

that is, $x = 7^{-\frac{4}{3}}$ or $x = -7^{-\frac{1}{3}}$.

By considering these two possible values of x, we have

Case 1:

$$7^{\frac{1}{3}i-2t} = 7^{-\frac{4}{3}}$$
$$\frac{1}{3}i - 2t = -\frac{4}{3}$$
$$i = 6t - 4.$$

Case 2:

$$7^{\frac{1}{3}i-2t} = -7^{-\frac{1}{3}}$$
$$7^{\frac{1}{3}i-2t+\frac{1}{3}} = -1,$$

which is a contradiction since the term on the left-hand side of the equation is positive. Hence, Case 2 needs not be considered.

Therefore, we have $b = 4t - 3$ and $i = 6t - 4$ when $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 4 \cdot 7^{t-1}$ as the integral solutions to the equation $x^2 + 16\cdot 7^b = y^4$.

Conversely, let $b = 4t - 3$ and $i = 6t - 4$. Since the set $\{6t - 4 | t \in \mathbb{Z}^+\}$ is a subset of $\{3t - 1 | t \in \mathbb{Z}^+\}$, by Theorem 1, the generators $(x_{b,i}, y_{b,i}^r)$ of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ are given by

$$x_{b,i} = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2 \cdot 7^{b-\frac{2}{3}i+\frac{4}{3}} - 2\right),$$
$$y_{b,i}^r = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2 \cdot 7^{b-\frac{2}{3}i+\frac{4}{3}} + 2\right).$$

Since $r = 2$, by substituting $b = 4t - 3$ and $i = 6t - 4$, the generators of solutions for $x_{b,i}$ and $y_{b,i}^2$ are

$$\begin{aligned} x_{b,i} &= x_{4t-3,6t-4} \\ &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2 \cdot 7^{(4t-3)-\frac{2}{3}(6t-4)+\frac{4}{3}} - 2\right) \\ &= 7^{2t-2}(2 \cdot 7 - 2) \\ &= 12 \cdot 7^{2(t-1)} \end{aligned}$$

and

$$\begin{aligned} y_{b,i}^2 &= y_{4t-3,6t-4}^2 \\ &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2 \cdot 7^{(4t-3)-\frac{2}{3}(6t-4)+\frac{4}{3}} + 2\right) \\ &= 7^{2t-2}(2 \cdot 7 + 2) \\ &= 4^2 \cdot 7^{2(t-1)}. \end{aligned}$$

Consequently, we obtain

$$y_{4t-3,6t-4} = 4 \cdot 7^{t-1}.$$

Therefore, we conclude that $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 4 \cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^4$ when $b = 4t - 3$ and $i = 6t - 4$. □

The following result shows that when b is an odd number of a fixed form, integral solutions exist for the Diophantine equation $x^2 + 16 \cdot 7^b = y^{2r}$ only when $r = 2$.

Theorem 3. *Let* $t \in \mathbb{Z}^+$, $b = 4t - 3$ *and* $i = 6t - 4$. *Then,* $x_{b,i} = 12 \cdot 7^{2(t-1)}$ *and* $y_{b,i} = 4 \cdot 7^{t-1}$ *are the integral solutions to the equation* $x^2 + 16 \cdot 7^b = y^{2r}$ *if and only if* $r = 2$.

Proof. We first prove the forward implication. Let $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 4 \cdot 7^{t-1}$ be the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. Then,

$$(12 \cdot 7^{2(t-1)})^2 + 16 \cdot 7^{4t-3} = (4 \cdot 7^{t-1})^{2r}. \quad (10)$$

By simplifying Equation (10), we have

$$(4 \cdot 7^{t-1})^4 = (4 \cdot 7^{t-1})^{2r}$$
$$2r = 4$$
$$r = 2.$$

The backward implication follows immediately from Theorem 2. This completes the proof. □

For $t \in \mathbb{Z}^+$, we show that when $b = 4t - 3$ and $i = 6t - 4$, the equation $x^2 + 16 \cdot 7^b = y^{2r}$ has no integral solutions when $r \neq 2,4$. We first state the following fact.

Fact 1. Let n be a positive integer and $p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}$ represent its prime factorisation. Then, $\gcd(x_1, x_2, \dots, x_k) = s$ if and only if n is a perfect sth power.

Theorem 4. *Let $t \in \mathbb{Z}^+$, $b = 4t-3$ and $i = 6t-4$. If $r \geq 3$ and $r \neq 4$, the equation $x^2 + 16 \cdot 7^b = y^{2r}$ has no integral solutions.*

Proof. Let t, b and i be as stated. Given $r \geq 3$ *and* $r \neq 4$, the possible values of r are listed in Table 1.

Table 1. Values of r for $b = 4t-3$ and $r \geq 3$ *and* $r \neq 4$

Case 1	$r \equiv 0 \pmod 2$,	$r = 2s, s > 1$
Case 2	$r \equiv 1 \pmod 2$,	$r = 1 + 2s, s \geq 1$

Since the set $\{6t-4 \mid t \in \mathbb{Z}^+\} \subseteq \{3t-1 \mid t \in \mathbb{Z}^+\}$, by Theorem 1, we know that the pair of Equations (3) and (4) generate the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. By substituting the values of b and i into Equation (4), we have

$$\begin{aligned} y_{b,i}^r &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2 \cdot 7^{4t-3-\frac{2}{3}(6t-4)+\frac{4}{3}} + 2\right) \\ &= 4^2 \cdot 7^{(2t-2)} \\ y_{b,i} &= \left(4 \cdot 7^{(t-1)}\right)^{\frac{2}{r}}. \\ &= \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{2}{r}}. \end{aligned} \tag{11}$$

We now consider the following two cases by substituting the two different forms of values of r into Equation (11):

Case 1: When $r = 2s$ and $s \geq 3$, we have

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{1}{s}}.$$

Since $\gcd(2, t-1) \leq 2$, $y_{b,i}$ is not a perfect sth power for $s \geq 3$ as indicated by Fact 1. Consequently, it follows that $y_{b,i}$ is not an integer, leading us to a contradiction. Thus, there exists no integral solution to the equation in this scenario.

Case 2: When $r = 1 + 2s$ and $s \geq 1$, we have

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{2}{1+2s}}.$$

Since $\gcd(2, t-1) \leq 2$, by Fact 1, $y_{b,i}$ cannot be a perfect s^{th} power for $s \geq 1$. This further implies that $y_{b,i}$ is not an integer. This is also a contradiction and hence there is no integral solution to the equation when $r = 1 + 2s$.

Combining the above two cases, we conclude that the equation $x^2 + 16 \cdot 7^b = y^{2r}$ has no integral solutions if $r \geq 3$ *and* $r \neq 4$, $b = 4t-3$ and $i = 6t-4$. □

Secondly, we give the form of integral solutions to the equation $x^2 + 16 \cdot 7^b = y^4$ when b is an even number.

Theorem 5. *Let $i, t \in \mathbb{Z}^+$ and b be an even number. Then, $x_{b,i} = 96 \cdot 7^{2(t-1)}$ and $y_{b,i} = 10 \cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^4$ if and only if $b = 4t-2$ and $i = 6t-4$.*

Proof. Let t be as stated. We first prove the forward implication. Suppose $x_{b,i} = 96 \cdot 7^{2(t-1)}$ and $y_{b,i} = 10 \cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^4$. Then, we have

$$\left(96 \cdot 7^{2(t-1)}\right)^2 + 16 \cdot 7^b = (10 \cdot 7^{t-1})^4. \tag{12}$$

By simplifying Equation (12), we obtain $7^b = 7^{4t-2}$, which implies that $b = 4t-2$.

Note that the set $\{6t-4 \mid t \in \mathbb{Z}^+\} \subseteq \{3t-1 \mid t \in \mathbb{Z}^+\}$. By Theorem 1, when $i = 3t-1$, we have Equation (3) as the generator of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. By comparing the value of $x_{b,i}$ in Equation (3) to $x_{b,i} = 96 \cdot 7^{2(t-1)}$, we have

$$7^{\frac{1}{3}i-\frac{2}{3}}\left(2 \cdot 7^{b-\frac{2}{3}i+\frac{4}{3}} - 2\right) = 96 \cdot 7^{2(t-1)}.$$

Since $b = 4t-2$, we have

$$\begin{aligned} 2 \cdot 7^{4t-\frac{1}{3}i-\frac{4}{3}} - 2 \cdot 7^{\frac{1}{3}i-\frac{2}{3}} &= 96 \cdot 7^{2(t-1)} \\ 7^{4t-\frac{1}{3}i-\frac{4}{3}} - 7^{\frac{1}{3}i-\frac{2}{3}} &= 48 \cdot 7^{2t-2}. \end{aligned} \tag{13}$$

Multiplying Equation (13) by $7^{-4t+\frac{1}{3}i+\frac{4}{3}}$, we obtain

$$1 - 7^{2(\frac{1}{3}i-2t)+\frac{2}{3}} = 48 \cdot 7^{\frac{1}{3}i-2t-\frac{2}{3}}. \tag{14}$$

Let $x = 7^{\frac{1}{3}i-2t}$ and by rearranging Equation (14), we have

$$7^{\frac{2}{3}}x^2 + 48 \cdot 7^{-\frac{2}{3}}x - 1 = 0.$$

By applying the quadratic formula, we obtain,

$$x = \frac{-48\cdot 7^{-\frac{2}{3}} \pm \sqrt{\left(48\cdot 7^{-\frac{2}{3}}\right)^2 - 4\left(7^{\frac{2}{3}}\right)(-1)}}{2\cdot 7^{\frac{2}{3}}}$$

which implies that $x = 7^{-\frac{4}{3}}$ or $x = -7^{\frac{2}{3}}$.

By testing the two possible values of x, we first obtain

$$\begin{aligned} 7^{\frac{1}{3}i-2t} &= 7^{-\frac{4}{3}} \\ \frac{1}{3}i - 2t &= -\frac{4}{3} \\ i &= 6t-4. \end{aligned}$$

In the second scenario, we have

$$\begin{aligned} 7^{\frac{1}{3}i-2t} &= -7^{\frac{2}{3}} \\ 7^{\frac{1}{3}i-2t-\frac{2}{3}} &= -1. \end{aligned}$$

This is a contradiction since the term on the left-hand side of the equation is positive. Hence, this case needs not be considered.

Hence, we have $b = 4t-2$ and $i = 6t-4$ when $x_{b,i} = 96\cdot 7^{2(t-1)}$ and $y_{b,i} = 10\cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16\cdot 7^b = y^4$.

Conversely, let $b = 4t-2$ and $i = 6t-4$. Since the set $\{6t-4 \mid t\in\mathbb{Z}^+\} \subseteq \{3t-1 \mid t\in\mathbb{Z}^+\}$, by Theorem 1, the generators of solutions $(x_{b,i}, y_{b,i}^r)$ to the equation $x^2 + 16\cdot 7^b = y^{2r}$ are given by Equations (3) and (4). Since $r = 2$, the generators of solutions for $x_{b,i}$ and $y_{b,i}^2$ in which $b = 4t-2$ and $i = 6t-4$ are

$$\begin{aligned} x_{b,i} &= x_{4t-2,6t-4} \\ &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2\cdot 7^{(4t-2)-\frac{2}{3}(6t-4)+\frac{4}{3}} - 2\right) \\ &= 7^{2t-2}(2\cdot 7^2 - 2) \\ &= 96\cdot 7^{2(t-1)} \end{aligned}$$

and

$$\begin{aligned} y_{b,i}^2 &= y_{4t-2,6t-4}^2 \\ &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2\cdot 7^{(4t-2)-\frac{2}{3}(6t-4)+\frac{4}{3}} + 2\right) \\ &= 7^{2t-2}(2\cdot 7^2 + 2) \\ &= 10^2\cdot 7^{2(t-1)}. \end{aligned}$$

As a result, we obtain

$$y_{4t-2,6t-4} = 10\cdot 7^{t-1}.$$

Therefore, we conclude that $x_{b,i} = 96\cdot 7^{2(t-1)}$ and $y_{b,i} = 10\cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16\cdot 7^b = y^4$ when $b = 4t-2$ and $i = 6t-4$. □

We next prove that when b is an even number of a fixed form, there exist integral solutions to the equation $x^2 + 16\cdot 7^b = y^{2r}$ when $r = 2$.

Theorem 6. *Let $t\in\mathbb{Z}^+$, $b = 4t-2$ and $i = 6t-4$. Then, $x_{b,i} = 96\cdot 7^{2(t-1)}$ and $y_{b,i} = 10\cdot 7^{t-1}$ are the integral solutions to the equation $x^2 + 16\cdot 7^b = y^{2r}$ if and only if $r = 2$.*

Proof. To prove the forward implication, let $x_{b,i} = 96\cdot 7^{2(t-1)}$ and $y_{b,i} = 10\cdot 7^{t-1}$ be the integral solutions to the equation $x^2 + 16\cdot 7^b = y^{2r}$. Then,

$$(96\cdot 7^{2(t-1)})^2 + 16\cdot 7^{4t-2} = (10\cdot 7^{t-1})^{2r}. \quad (15)$$

By simplifying Equation (15), we have

$$\begin{aligned} (10\cdot 7^{t-1})^4 &= (10\cdot 7^{t-1})^{2r} \\ 2r &= 4 \\ r &= 2. \end{aligned}$$

The backward implication follows according to Theorem 5. This completes the proof. □

For $t\in\mathbb{Z}^+$, we next show that when $b = 4t-2$ and $i = 6t-4$, the equation $x^2 + 16\cdot 7^b = y^{2r}$ has no integral solutions except when $r = 2$.

Theorem 7. *Let $t\in\mathbb{Z}^+$, $b = 4t-2$ and $i = 6t-4$. If $r \geq 3$, the equation $x^2 + 16\cdot 7^b = y^{2r}$ has no integral solutions.*

Proof. Let t, b and i be as stated. Given $r \geq 3$, the possible values of r are listed in Table 2.

Table 2. Values of r when $b = 4t - 2$ and $r \geq 3$

Case 1	$r \equiv 0 \pmod 2$,	$r = 2s, s > 1$
Case 2	$r \equiv 1 \pmod 2$,	$r = 1 + 2s, s \geq 1$

Since the set $\{6t - 4 \mid t \in \mathbb{Z}^+\} \subseteq \{3t - 1 \mid t \in \mathbb{Z}^+\}$, by Theorem 1, we know that the pair of Equations (3) and (4) generate the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ when $i = 3t - 1$. By substituting the values of b and i into Equation (4), we have

$$\begin{aligned} y_{b,i}^r &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2 \cdot 7^{4t-2-\frac{2}{3}(6t-4)+\frac{4}{3}} + 2\right) \\ &= 10^2 \cdot 7^{(2t-2)} \\ y_{b,i} &= \left(10 \cdot 7^{(t-1)}\right)^{\frac{2}{r}} \\ &= \left(2 \cdot 5 \cdot 7^{(t-1)}\right)^{\frac{2}{r}}. \end{aligned} \quad (16)$$

By substituting the two different forms of values of r into Equation (16), we have the following two cases:

Case 1: When $r = 2s$ and $s > 1$, we have

$$y_{b,i} = \left(2 \cdot 5 \cdot 7^{(t-1)}\right)^{\frac{1}{s}}.$$

Since $\gcd(1,1,t-1) \leq 1$, as per Fact 1, $y_{b,i}$ is not a perfect sth power for $s > 1$. Consequently, it follows that $y_{b,i}$ is not an integer, leading us to a contradiction. Thus, there is no integral solution to the equation $r = 2s$.

Case 2: When $r = 1 + 2s$ and $s \geq 1$, we obtain

$$y_{b,i} = \left(2 \cdot 5 \cdot 7^{(t-1)}\right)^{\frac{2}{1+2s}}.$$

Based on the similar argument as in Case 1, we conclude that there exists no integral solution to the equation when $r = 1 + 2s$ and $s \geq 1$.

Therefore, the desired results are obtained based on the two cases.

3. Solutions When $n = 8$

In this section, we focus on the Diophantine equation $x^2 + 16 \cdot 7^b = y^8$, and give the forms of integral solutions to the equation when b is an odd number.

Theorem 8. *Let t be a positive odd integer, $b = 4t - 3$ and $i = 6t - 4$. Then, $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 2 \cdot 7^{\frac{t-1}{2}}$ are the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ if and only if $r = 4$.*

Proof. Let t, b and i be as stated. For the forward implication, we have $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 2 \cdot 7^{\frac{t-1}{2}}$ be the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. Then,

$$\left(12 \cdot 7^{2(t-1)}\right)^2 + 16 \cdot 7^{4t-3} = \left(2 \cdot 7^{\frac{t-1}{2}}\right)^{2r}. \quad (17)$$

By simplifying Equation (17), we have

$$\begin{aligned} (4 \cdot 7^{t-1})^4 &= (4 \cdot 7^{t-1})^r \\ r &= 4. \end{aligned}$$

Conversely, suppose $r = 4$, we then have $x^2 + 16 \cdot 7^b = y^8 = (y^2)^4$. Based on Theorem 2 and since t is a positive odd integer, it follows that $y_{b,i} = 2 \cdot 7^{\frac{t-1}{2}}$ is also an integer.

Therefore, $x_{b,i} = 12 \cdot 7^{2(t-1)}$ and $y_{b,i} = 2 \cdot 7^{\frac{t-1}{2}}$ are the integral solutions to the Diophantine equation $x^2 + 16 \cdot 7^b = y^8$ when t is a positive odd integer, $b = 4t - 3$ and $i = 6t - 4$. □

In the following theorem, we demonstrate that the equation $x^2 + 16 \cdot 7^b = y^{2r}$ has integral solutions only when $r = 4$ for specific values of b and i. We disregard the scenario when $r = 2$, as there exist integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ as shown in Theorem 4.

Theorem 9. *Let t be a positive odd integer, $b = 4t - 3$ and $i = 6t - 4$. If $r \geq 3$ and $r \neq 4$, the equation $x^2 + 16 \cdot 7^b = y^{2r}$ has no integral solutions.*

Proof. Let t, b and i be as stated. Given $r \geq 3$ *and* $r \neq 4$, the possible values of r are shown in Table 3.

Table 3. Values of r when $b = 4t - 3$ and $r \geq 3$ *and* $r \neq 4$

Case 1	$r \equiv 0 \pmod 4$,	$r = 2s, s > 1$
Case 2	$r \equiv 1 \pmod 4$,	$r = 1 + 4s, s \geq 1$
Case 3	$r \equiv 2 \pmod 4$,	$r = 2 + 4s, s \geq 1$
Case 4	$r \equiv 3 \pmod 4$,	$r = 3 + 4s, s \geq 0$

Since the set $\{6t - 4 | t \in \mathbb{Z}^+\} \subseteq \{3t - 1 | t \in \mathbb{Z}^+\}$, by Theorem 1, we know that the pair of Equations (3) and (4) generate the integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ when $i = 3t - 1$. By substituting the values of b and i into Equation (4), we have

$$\begin{aligned} y_{b,i}^r &= 7^{\frac{1}{3}(6t-4)-\frac{2}{3}}\left(2 \cdot 7^{4t-3-\frac{2}{3}(6t-4)+\frac{4}{3}} + 2\right) \\ &= 4^2 \cdot 7^{(2t-2)} \\ y_{b,i} &= \left(4 \cdot 7^{(t-1)}\right)^{\frac{2}{r}} \\ &= \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{2}{r}}. \end{aligned} \tag{18}$$

By substituting the four different forms of values of r into Equation (18), we have the following four cases:

Case 1: When $r = 4s$ and $s \geq 2$, we obtain

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{1}{2s}}.$$

Since $\gcd(2, t - 1) \leq 2$, as per Fact 1, $y_{b,i}$ is not a perfect sth power for $s \geq 2$. Consequently, it follows that $y_{b,i}$ is not an integer, leading us to a contradiction. Thus, no integral solution can be obtained in this case.

Case 2: When $r = 1 + 4s$ and $s \geq 1$, we have

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{2}{1+4s}}.$$

Since $\gcd(2, t - 1) \leq 2$, by Fact 1, $y_{b,i}$ is not a perfect sth power for $s \geq 1$. As a result, it follows that $y_{b,i}$ is not an integer and this is a contradiction. Hence, no integral solution to the equation exists when $r = 1 + 4s$.

Case 3: When $r = 2 + 4s$ and $s \geq 1$, we obtain

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{1}{1+2s}}.$$

Based on a similar approach as in Case 2, we can see that there exists no integral solution to the equation when $r = 2 + 4s$ and $s \geq 1$.

Case 4: When $r = 3 + 4s$ and $s \geq 0$, we have

$$y_{b,i} = \left(2^2 \cdot 7^{(t-1)}\right)^{\frac{2}{3+4s}}.$$

We encounter another contradiction, leveraging the observation that $\gcd(2, t - 1) \leq 2$ implies that $y_{b,i}$ cannot be a perfect sth power for $s \geq 0$, as indicated by Fact 1. Thus, there is no integral solution to the equation when $r = 3 + 4s$.

Combining all the four cases as discussed above, we conclude that the Diophantine equation $x^2 + 16 \cdot 7^b = y^{2r}$ has no integral solution if $r \geq 3$ *and* $r \neq 4$ when $b = 4t - 3$, $i = 6t - 4$. □

4. Conclusions

In this article, we proved necessary and sufficient conditions that are required for the Diophantine equation $x^2 + 16 \cdot 7^b = y^{2r}$ to possess integral solutions, by using the generators of solutions (x, y^r) given in Yow *et al.* (2022). We considered various scenarios where b is either odd or even. We also determined the values of n for which integral solutions exist. We found that for $n = 4$, integral solutions exist to the equation regardless of the parity of b. For the case when $n = 8$, we concluded that integral solutions can be obtained only when b is odd, in which these solutions are more restricted, and they can be derived based on the results proven for $n = 4$.

Given the inherent difficulty in determining and formulating general integral solutions to the Diophantine equation $x^2 + 2^a \cdot 7^b = y^n$ especially when n is odd, for future investigation, we suggest decomposing the problem into smaller, more manageable subproblems. For subproblems with integral solutions of similar trends, we can consolidate and generalise these findings to establish a comprehensive solution. Alternatively, some learning approaches (Yow, Liao, Luo, & Cheng, 2023) may also be employed in addressing the problem, leveraging their demonstrated success in resolving tasks across various domains.

Acknowledgements

This research was partially supported by Universiti Putra Malaysia under Putra Grant GP − IPM/2020/9684300 and National Research Foundation Singapore (NRF-MP-2022-0001).

References

Abu Muriefah, F. S., & Bugeaud, Y. (2006). The Diophantine equation $x^2 + c = y^n$: A brief overview. *Revista Colombiana de Matemáticas, 40*(1), 31 − 37.

Arif, S. A., & Abu Muriefah, F. S. (1997). On the Diophantine equation $x^2 + 2^k = y^n$. *International Journal of Mathematics and Mathematical Sciences, 20*(2), 299 − 304.

Cohn, J. H. E. (1992). The Diophantine equation $x^2 + 2^k = y^n$. *Archives of Mathematics (Basel)*, *59,* 341 − 344.

Cohn, J. H. E. (1993). The Diophantine equation $x^2 + c = y^n$. *Acta Arithmetica*, *65*, 367 − 381.

Crocker, R. (1968). Application of Diophantine equations to problems in chemistry. *Journal of Chemical Education*, *45*(11), 731.

Gao, S., & Heindl, R. (2013). Multivariate public key cryptosystems from Diophantine equation. *Designs, Codes and Cryptography*, *67*, 1 − 18.

Ismail, S., Atan, K. A., Sejas Viscarra, S., & Yow, K. S. (2023). Gaussian integer solutions of the Diophantine equation $x^4 + y^4 = z^3$ for $x \neq y$. *Baghdad Science Journal*, *20*(5), 1751 − 1762.

Ismail, S., Atan, K. A., Yow, K. S., & Sejas Viscarra, S. (2021). On the integral solutions to the Diophantine equation $x^4 + y^4 = 2^k z^3$ where $k > 1$. *AIP Conference Proceedings*, 2365, 020029.

Ko, C. (1965). On the Diophantine equation $x^2 = y^n + 1$, $xy \neq 0$. *Scientia Sinica*, *14,* 457 − 460.

Kučera, V. (1993). Diophantine equations in control−A survey. *Automatica*, *29*(6), 1361 − 1375.

Lebesgue, V. A. (1850). Sur l'impossibilité en nombres entiers de l'équation $x^m = y^2 + 1$. *Nouvella Annals Des Mathemetics*, *78,* 26 − 35.

Sapar, S. H., & Yow, K. S. (2021). A generalisation to the Diophantine equation $x^2 + 8 \cdot 7^b = y^{2r}$. *Malaysian Journal of Science*, *40*(2), 25 − 39.

Wiles, A. J. (1995). Modular elliptic curves and Fermat's last theorem. *Annals of Mathematics*, *141*(3), 443 − 551.

Yow, K. S., Liao, N., Luo, S., & Cheng, R. (2023). Machine learning for subgraph extraction: Methods, applications and challenges. *Proceedings of the VLDB Endowment*, *26*(12), 3864 − 3867.

Yow, K. S., Sapar, S. H., & Atan, K. A. (2013). On the Diophantine equation $x^2 + 4 \cdot 7^b = y^{2r}$. *Pertanika Journal of Science and Technology*, *21*(2), 443 − 458.

Yow, K. S., Sapar, S. H. & Low, C. Y. (2022). Solutions to the Diophantine equation $x^2 + 16 \cdot 7^b = y^{2r}$. *Malaysian Journal of Fundamental and Applied Sciences*, *18*(4), 489 − 496.