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Original Article

Hybrid shifted polynomial scheme for the approximate solution of a class of nonlinear partial differential equations

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Abstract

This research focuses on the development of a streamlined numerical technique founded on the hybridization of two shifted polynomial basis functions to address a specific category of nonlinear Partial Differential Equations. Within this approach, a solution based on power series is employed, utilizing Chebyshev and Legendre shifted polynomials to meet the specific conditions of the Partial Differential Equation. Plugging the candidate solution series into the provided Partial Differential Equation, and employing suitable points of collocation, a linear system of algebraic equations with unspecified hybridization coefficients was obtained and numerically solved by Gaussian elimination. Furthermore, different discretization patterns were examined to comprehend how the outcomes vary with alterations in the placement of the collocation points within the domain. Two instances were examined using the numerical method to determine the method's efficiency in terms of its reliability, effectiveness, and accuracy. The results obtained were benchmarked and validated with existing results in the literature. However, the combination of the two shifted orthogonal polynomials (Chebyshev and Legendre) greatly improved performance past that in prior literature.

Keywords: shifted Chebyshev polynomials, shifted Legendre polynomials, nonlinear partial differential equations, Klein-Gordon equations, Gaussian elimination

1. Introduction

A fundamental mathematical framework known as partial differential equations (PDEs) is used to explain a broad range of physical, engineering, and scientific phenomena. They are crucial tools for comprehending complex systems that vary in both space and time because they offer a potent method of modelling how quantities change regarding numerous independent variables.

Since the groundbreaking work of mathematicians like Leonhard Euler and Joseph-Louis Lagrange, the study of PDEs has a long and illustrious history. PDEs have shown to be essential over the years in a variety of disciplines, including fluid dynamics, heat conduction, quantum physics, and image processing. They are widely used in science and engineering because of their capacity to summarize the underlying ideas guiding the development of ongoing physical processes. Partial differential equations have played a crucial role in scientific computing. Some basic historical significance includes the modelling of natural phenomena, bridging pure and applied mathematics gap, and they have remained foundationally solid in scientific fields.

One key aspect of solving PDEs involves expressing approximate solutions in terms of orthogonal polynomials. One of the distinctive characteristics of the family of mathematical functions known as orthogonal polynomials is

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that they are orthogonal over intervals when using particular weight functions to modify the inner product. Two noteworthy sets of orthogonal polynomials commonly employed in the context of PDEs are the shifted Chebyshev and Legendre polynomials.

Shifted Chebyshev polynomials are obtained from the classical Chebyshev polynomials and are known for their orthogonality properties over an interval, such as **[**-1, 1**]**. They play a vital role in numerical analysis, approximation theory, and spectral methods for solving PDEs. These polynomials enable efficient approximations and transformations of functions, making them a valuable tool in scientific computing.

Legendre polynomials, another family of orthogonal polynomials, find applications in solving PDEs that exhibit spherical or cylindrical symmetry. Defined over the interval **[**-1, 1**]**, Legendre polynomials offer solution to a broad spectrum of problems in physics, including those involving celestial mechanics, electrostatics, and quantum mechanics. Their unique properties make them indispensable for solving PDEs in systems with radial or angular symmetry.

Abdul *et al*. **(**2023**)** proposed a hybridized approach for numerically solving PDEs, such that the temporal derivatives are approximated using both first and second order finite differences, with the idea of Lucas and Fibonacci polynomials. Forstythe and Wasow **(**2013a, 2013b**)** introduced numerical approaches using finite differences to solve partial differential equations. Akyuz-Dascioglu **(**2009**)** used the idea of polynomial approximation in terms of Chebyshev for high-order PDEs with complex conditions. Caporale and Carrato **(**2010**)** applied Chebyshev polynomials for the approximate solution of PDEs. Yuksel and Sezer **(**2013**)** introduced an approximating linear second-order PDEs approach with complex boundary conditions using Chebyshev scheme expansion. Ali and Abozar **(**2013**)** worked on utilizing the Legendre polynomials for solving PDEs. Ghimire *et al*. **(**2016**)** used Chebyshev polynomials as basis functions for the approximate solutions of elliptic PDEs. Mredula and Vakaskar **(**2017**)** applied collocation to wavelets for solving a partial differential equation. Madenci *et al*. **(**2017**)** applied peridynamic differential operator to numerically solve both linear and nonlinear PDEs. Luo *et al*. **(**2017**)** used the barycentric rational collocation approach to address a set of nonlinear parabolic PDEs. Dhiman and Tamsir **(**2018**)** explored a collocation method that utilized a modified version of cubic B-spline trigonometric functions to address Fisher**'**s reaction diffusion equation. Youssri, Ismail, and Atta **(**2023**)** addressed the time-fractional heat conduction equation in one spatial dimension, subject to nonlocal conditions in the temporal domain. Elzaki **(**2018**)** applied the Laplace variational iteration scheme for the solution of nonlinear PDEs. Karunakar and Chakraverty **(**2019**)** employed a highly effective technique relying on shifted Chebyshev polynomials to tackle PDEs. They selected a power series solution involving shifted Chebyshev polynomials to ensure compliance with the specified conditions. Jena *et al*. **(**2020**)** proposed a proficient numerical approach for addressing fractional-order delay differential equations **(**FDDEs**)**, which involves application of the shifted Legendre polynomials within collocation and Galerkin schemes. This methodology transforms FDDEs into linear and nonlinear algebraic equations. Atta and Youssri **(**2022**)** focused on an approximate spectral method for the nonlinear time-fractional partial integro-differential equation with a weakly singular kernel by setting up a new Hilbert space that satisfies the initial and boundary conditions. Samaneh *et al*. **(**2020**)** developed a method to investigate the inverse problem in the estimation of time-dependent heat and temperature source in the context of the heat equation, which includes the Dirichlet boundary conditions and the integral over-determination condition. The solution approximation for this problem employs shifted Chebyshev polynomials as the foundation of Tau method depending on the operational matrices in form of Chebyshev. Youssri and Atta **(**2024**)** constructed an explicit modal numerical solver based on the spectral Petrov-Galerkin method via a specific combination of shifted Chebyshev polynomial basis for handling the nonlinear time-fractional Burger-type partial differential equation in the Caputo sense.

2. Preliminaries of the Chebyshev and Legendre Polynomials

Here, the discussion covers the fundamentals, as well as the significant characteristics of Chebyshev and Legendre polynomials. The polynomial of nth degree with unity as the leading coefficient in the range [-1, 1] was first introduced by the Russian mathematician Pafnuty Chebyshev.

$$K_n(f) = Cos\left(n\cos^{-1} f\right), \qquad n = 0,1,2,3,4... \tag{1}$$

The Chebyshev polynomials can be determined using the recurrence formula (Theodore, 1974)

$$K_{n+1}(f) = 2 f K_n(f) - K_{n-1}(f), \quad n = 1,2,3,4... \tag{2}$$

where the first five polynomials are as follows

$$K_0(f) = 1$$
$$K_1(f) = f$$
$$K_2(f) = 2f^2 - 1$$
$$K_3(f) = 4f^3 - 3f$$
$$K_4(f) = 8f^4 - 8f^2 + 1$$

The conversion of the interval of Chebyshev polynomial [-1, 1] into another interval [0, 1] is called "Shifted Chebyshev Polynomial". The first five shifted Chebyshev polynomials are

$$K^*{}_0(f) = 1$$
$$K^*{}_1(f) = 2f - 1$$
$$K^*{}_2(f) = 8f^2 - 8f + 1$$
$$K^*{}_3(f) = 32f^3 - 48f^2 + 18f - 1$$
$$K^*{}_4(f) = 128f^4 - 256f^3 + 160f^2 - 32f + 1$$

On the other hand, Legendre polynomials are a class of complete and orthogonal polynomials in mathematics that bear Adrien-Marie Legendre's name (1782). They have a wide range of mathematical properties and are used in a diverse range of applications. Polynomials are referred to as an orthogonal system in terms of the weight function g(f) = 1 with the range [- 1, 1]. That is, each $V_m(f)$ is a polynomial of degree m satisfying

$$\int_{-1}^{1} V_m(f)V_k(f)\,df = 0 \quad \text{if} \quad m \neq k \tag{3}$$

Rodrigues' formula provides a concise representation for the Legendre polynomials as follows:

$$V_m(f) = \frac{1}{2^m\, i!}\frac{d^m}{df^m}\left(f^2 - 1\right)^m \tag{4}$$

The initial five Legendre polynomials are

$$V_0(f) = 1$$
$$V_1(f) = f$$
$$V_2(f) = \tfrac{1}{2}\left(3f^2 - 1\right)$$
$$V_3(f) = \tfrac{1}{2}\left(5f^3 - 3f\right)$$
$$V_4(f) = \tfrac{1}{8}\left(35f^4 - 30f^2 + 3\right)$$

The counterpart to Rodrigues' formula for the shifted Legendre polynomials is expressed by

$$\tilde{\tilde{V}}_m(f) = \frac{1}{m!}\frac{d^m}{df^m}\left(f^2 - f\right)^m \tag{5}$$

The initial five shifted Legendre polynomials are

$$\tilde{\tilde{V}}_0(f) = 1$$
$$\tilde{\tilde{V}}_1(f) = 2f - 1$$
$$\tilde{\tilde{V}}_2(f) = 6f^2 - 6f + 1$$
$$\tilde{\tilde{V}}_3(f) = 20f^3 - 30f^2 + 12f - 1$$
$$\tilde{\tilde{V}}_4(f) = 70f^4 - 140f^3 + 90f^2 - 20f + 1$$

3. Methodology

3.1 The hybridization collocation method

Here, we describe an approach to solve partial differential equations (PDEs); the amalgamation of Chebyshev and Legendre shifted polynomials. To exemplify the technique, we examine a partial differential equation in general characterized by the two distinct variables m and t; and with w as the dependent variable represented as follows:

$$H(w) = c(m,t) \tag{6}$$

Depending on the stipulated terms and circumstances:

$$w(m_0,t) = d(t) \quad and \quad w(m,t_0) = k(m) \tag{7}$$

where w is an unspecified function; c, d and k denote specified functions while m_0 and t_0 remain constants. To begin, we make the initial assumption of a two-variable power series solution expressed using the adjusted Chebyshev and Legendre polynomials, which must meet the criteria outlined in (8)

$$w(m,t) = \xi(m,t) + \eta(m,t)\sum_{q=0}^{N}\sum_{s=0}^{N} a_{qs}\, T_q^*(m)\tilde{\tilde{P}}_s(t) \tag{8}$$

It's significant to note $\xi\,(m, t)$ and $\eta\,(m, t)$ to be characteristics of both m and t, and their selection should be such that (8) complies with the prescribed requirements set forth in condition (7). Then, we substitute the assumed solution (8) into (6), and from this process, we derive a residual equation denoted as $Q(m,\ t) = 0$, which encompasses coefficients represented as a_{qs}. By employing appropriate points of collocation within [0, 1] by [0, 1]; we formulate a system of equations for the unspecified coefficients, denoted by a_{qs}

$$Hv = b \tag{9}$$

In this context, H stands as the well-established coefficient matrix, v represents a column vector containing Chebyshev and Legendre coefficients, and b is a predetermined vector on the right-hand side (RHS). The very important thing to note here is that both H and b consist of real values determined through the utilization of the points of collocation within the equation of residuals $L(m, t) = 0$.

Now, M_p^2 as collocation points become a requisite when for degree N polynomial with M_p = N + 1. Subsequently, the equations for determining the unspecified coefficients are solved followed by their substitution into (8) and we get an approximate solution to (6).

The discretization of the domain represents a significant consideration and should be noted. For hybridization of the method, the coefficients of the selected polynomials are required (not necessarily equal). But they are dependent on the number of associated unknowns. Meanwhile, the selected hybridization coefficients are evenly distributed over the members of the selected polynomials. As a result, different discretization patterns will be considered before the best one is finally selected to produce results that are acceptable. The hybridization of the polynomials resulted in a vibrant method that combines the features of the two polynomials; mainly the distribution of errors equally over the selected intervals.

4. Numerical Examples and their Results

Example 1: Examine the nonlinear Klein-Gordon equation featuring $\gamma = -1$ and

$$g(w) = \left(\frac{\kappa^2}{4}\right)w + w^2,$$

$$\frac{\delta^2 w}{\delta t^2} - \frac{\delta^2 w}{\delta m^2} + \frac{\kappa^2}{4}w + w^2 = m^2 \sin^2\left(\frac{\kappa}{2}t\right) \tag{10}$$

where $m \in (-1, 1),\ \ t > 0$. (12) meets the conditions $w(m,0) = 0,\ \frac{\delta w}{\delta t}(m,0) = \frac{\kappa}{2}m \ \ where\ \ m \in [-1, 1]$; as well as the Dirichlet boundary conditions

$$w(-1,t) = -\sin\left(\frac{\kappa}{2}t\right),\ \ w(1,t) = \sin\left(\frac{\kappa}{2}t\right) \ such\ that\ t \geq 0 \tag{11}$$

The theoretical solution for the differential equation is known to be $w(m,\, t) = m\sin\left(\frac{\kappa}{2}t\right)$

Assume that the bivariate series solution for the Klein-Gordon (10) is given by

$$w(m,\, t) = \frac{\kappa}{2}mt + m^3\sin\left(\frac{\kappa}{2}t\right) - \frac{\kappa}{2}m^3 t + \left(\frac{\kappa}{2}mt^2 - \frac{\kappa}{2}t^2 m^3\right)\sum_{q=0}^{3}\sum_{s=0}^{3} a_{qs}\, T_q^*(m)\, P_s^*(t) \tag{12}$$

where $\xi(m,t) = \frac{\kappa}{2}mt + m^3\sin\left(\frac{\kappa}{2}t\right) - \frac{\kappa}{2}m^3 t$ *and* $\eta(m,t) = \frac{\kappa}{2}mt^2 - \frac{\kappa}{2}t^2 m^3$. Initial conditions (11) are satisfied by the anticipated solution (12). Now, differentiate (12) partially with relation to m and t to acquire the derivatives

$$\begin{aligned}
\frac{\delta^2 w}{\delta t^2} &= \left(\kappa m - \kappa m^3\right)a_{00} + 4\left(\kappa mt - \kappa m^3 t\right)a_{01} + 12\left(\tfrac{1}{2}\kappa mt^2 - \tfrac{1}{2}\kappa m^3 t^2\right)a_{02} + \left(\kappa m - \kappa m^3\right)\\
&\left(2m-1\right)\left(2t-1\right)a_{11} + \left(\kappa m - \kappa m^3\right)\left(2m-1\right)\left(6t^2 - 6t + 1\right)a_{12} + 2\left(\kappa mt - \kappa m^3 t\right)\left(2m-1\right)\left(12t-6\right)a_{12}\\
&+ \left(\kappa m - \kappa m^3\right)\left(2m-1\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{13} + 2\left(\kappa mt - \kappa m^3 t\right)\left(2m-1\right)\left(60t^2 - 60t + 12\right)a_{13}\\
&+ \left(\tfrac{1}{2}\kappa mt^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(2m-1\right)\left(120t - 60\right)a_{13} + \left(\kappa m - \kappa m^3\right)\left(8m^2 - 8m + 1\right)\left(2t-1\right)a_{21} + \left(\kappa m - \kappa m^3\right)\\
&\left(8m^2 - 8m + 1\right)\left(6t^2 - 6t + 1\right)a_{22} + 2\left(\kappa mt - \kappa m^3 t\right)\left(8m^2 - 8m + 1\right)\left(12t - 6\right)a_{22} + \left(\kappa m - \kappa m^3\right)\\
&\left(8m^2 - 8m + 1\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{23} + 2\left(\kappa mt - \kappa m^3 t\right)\left(8m^2 - 8m + 1\right)\left(60t^2 - 60t + 12\right)a_{23}\\
&+ \left(\tfrac{1}{2}\kappa mt^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(8m^2 - 8m + 1\right)\left(120t - 60\right)a_{23} + \left(\kappa m - \kappa m^3\right)\left(32m^3 - 48m^2 + 18m - 1\right)\\
&\left(2t-1\right)a_{31} + \left(\kappa m - \kappa m^3\right)\left(32m^3 - 48m^2 + 18m - 1\right)\left(6t^2 - 6t + 1\right)a_{32} + 2\left(\kappa mt - \kappa m^3 t\right)\\
&\left(32m^3 - 48m^2 + 18m - 1\right)\left(12t - 6\right)a_{32} + \left(\kappa m - \kappa m^3\right)\left(32m^3 - 48m^2 + 18m - 1\right)\\
&\left(20t^3 - 30t^2 + 12t - 1\right)a_{33} + 2\left(\kappa mt - \kappa m^3 t\right)\left(32m^3 - 48m^2 + 18m - 1\right)\left(60t^2 - 60t + 12\right)a_{33}\\
&+ \left(\tfrac{1}{2}\kappa mt^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(32m^3 - 48m^2 + 18m - 1\right)\left(120t - 60\right)a_{33} + 4\left(\kappa mt - \kappa m^3 t\right)\left(2m-1\right)a_{11}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\frac{\delta^2 w}{\delta m^2} &= -3\kappa mt^2\left(2m-1\right)\left(2t-1\right)a_{11} - 3\kappa mt^2\left(2m-1\right)\left(6t^2 - 6t + 1\right)a_{12} - 3\kappa mt^2\left(2m-1\right)(20t^3 - 30t^2\\
&+ 12t - 1)a_{13} - 3\kappa mt^2\left(8m^2 - 8m + 1\right)\left(2t-1\right)a_{21} - 3\kappa mt^2\left(8m^2 - 8m + 1\right)\left(6t^2 - 6t + 1\right)a_{22} - 3\kappa mt^2\\
&\left(8m^2 - 8m + 1\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{23} - 3\kappa mt^2\left(32m^3 - 48m^2 + 18m - 1\right)\left(2t-1\right)a_{31} - 3\kappa mt^2\\
&\left(32m^3 - 48m^2 + 18m - 1\right)\left(6t^2 - 6t + 1\right)a_{32} - 3\kappa mt^2\left(32m^3 - 48m^2 + 18m - 1\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{33}\\
&+ 2\left(\tfrac{1}{2}\kappa t^2 - \tfrac{3}{2}\kappa m^2 t^2\right)\left(16m - 8\right)a_{20} + 4\left(\tfrac{1}{2}\kappa t^2 - \tfrac{3}{2}\kappa m^2 t^2\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{13} + 2\left(\tfrac{1}{2}\kappa t^2 - \tfrac{3}{2}\kappa m^2 t^2\right)\\
&\left(96m^2 - 96m + 18\right)a_{30} + \left(\tfrac{1}{2}\kappa t^2 - \tfrac{3}{2}\kappa m^2 t^2\right)\left(192m - 96\right)a_{30} + 4\left(\tfrac{1}{2}\kappa t^2 - \tfrac{3}{2}\kappa m^2 t^2\right)\left(6t^2 - 6t + 1\right)a_{12} +\\
&4\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\left(2t-1\right)a_{11} + 16\left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)\left(2t-1\right)a_{21} + 16\left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)\left(6t^2 - 6t + 1\right)\\
&a_{22} + 4\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)a_{10} + 16\left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)a_{20} + 6m\sin\left(\frac{1}{2}\kappa t\right) - 3\kappa mt^2 a_{00} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\\
&(16m - 8)\left(2t-1\right)a_{21} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)(16m - 8)\left(6t^2 - 6t + 1\right)a_{22} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)(16m - 8)\\
&\left(20t^3 - 30t^2 + 12t - 1\right)a_{23} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\left(96m^2 - 96m + 18\right)\left(2t-1\right)a_{31} + \left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\\
&(192m - 96)\left(2t-1\right)a_{31} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\left(96m^2 - 96m + 18\right)\left(6t^2 - 6t + 1\right)a_{32} + \left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)\\
&(192m - 96)\left(6t^2 - 6t + 1\right)a_{32} + 2\left(\frac{1}{2}\kappa t^2 - \frac{3}{2}\kappa m^2 t^2\right)\left(96m^2 - 96m + 18\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{33} +\\
&\left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)(192m - 96)\left(20t^3 - 30t^2 + 12t - 1\right)a_{33} - 3\kappa mt^2\left(8m^2 - 8m + 1\right)a_{20} - 3\kappa mt^2\\
&\left(6t^2 - 6t + 1\right)a_{02} - 3\kappa mt^2\left(20t^3 - 30t^2 + 12t - 1\right)a_{03} - 3\kappa mt^2\left(32m^3 - 48m^2 + 18m - 1\right)a_{30}\\
&- 3\kappa mt^2\left(2t-1\right)a_{01} - 3\kappa mt^2\left(2m-1\right)a_{10} + 16\left(\frac{1}{2}\kappa mt^2 - \frac{1}{2}\kappa m^3 t^2\right)\left(20t^3 - 30t^2 + 12t - 1\right)a_{23} - 3\kappa mt
\end{aligned} \tag{14}$$

Now, substitute (12), (13) and (14) into (10) to have the residual equation $L(m, t) = 0$. Then, collocation is at sixteen points, since our N = 3, within the domain [-1, 1] x [-1, 1] i.e.

$\left(-1, \frac{3}{4}\right), \left(-1, \frac{1}{2}\right), \left(-1, \frac{1}{4}\right), \left(-\frac{3}{4}, \frac{3}{4}\right), \left(-\frac{3}{4}, \frac{1}{4}\right), \left(-\frac{1}{4}, \frac{3}{4}\right), \left(-\frac{1}{4}, \frac{1}{4}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{4}\right), \left(1, \frac{1}{2}\right), \left(1, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right).$

which in turns and gives a system of sixteen linear algebraic equations. Having solved the system by Gaussian elimination (or by computational software), we have the numerical solution

$a_{00} = -0.6623183606,\ a_{01} = 0.5497199121,\ a_{02} = -0.3153758451,\ a_{03} = 0.09683077730$

$a_{10} = 0.1774167310,\ a_{11} = -0.2188744071,\ a_{12} = 0.1150522479,\ a_{13} = -0.08151337732$

$a_{20} = 0.01504675306,\ a_{21} = -0.02879743084,\ a_{22} = 0.01277593301,\ a_{23} = 0.02003622298$

$a_{30} = -0.003101315309,\ a_{31} = 0.01340175865,\ a_{32} = -0.001691759673,\ a_{33} = 0.01645813390$

Substituting all a'_{qs} in (12), we obtain an approximate solution for the differential equations as

$$
\begin{aligned}
u(m, t) &= \tfrac{1}{2}\kappa m t + m^3 \sin\left(\tfrac{1}{2}\kappa t\right) - \tfrac{1}{2}\kappa m^3 t - 0.3311591802\ \kappa m t^2 + 0.3311591802\ \kappa m^3 t^2 \\
&+ 0.5497199121\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)(2t-1) - 0.3153758451\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(6t^2 - 6t + 1\right) \\
&+ 0.09683077730\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(20t^3 - 30t^2 + 12t - 1\right) + 0.1774167310\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right) \\
&(2m-1) - 0.2188744071\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)(2m-1)(2t-1) + 0.1150522479\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right) \\
&(2m-1)\left(6t^2 - 6t + 1\right) - 0.08151337732\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)(2m-1)\left(20t^3 - 30t^2 + 12t - 1\right) \\
&+ 0.01504675306\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(8m^2 - 8m + 1\right) - 0.02879743084\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right) \\
&\left(8m^2 - 8m + 1\right)(2t-1) + 0.01277593301\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(8m^2 - 8m + 1\right)\left(6t^2 - 6t + 1\right) + \\
&0.02006322298\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(8m^2 - 8m + 1\right)\left(20t^3 - 30t^2 + 12t - 1\right) - 0.003101315309 \\
&\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(32m^3 - 48m^2 + 18m - 1\right) + 0.01340175865\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right) \\
&\left(32m^3 - 48m^2 + 18m - 1\right)(2t-1) - 0.001691759673\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(32m^3 - 48m^2 + 18m - 1\right) \\
&\left(6t^2 - 6t + 1\right) + 0.01645813390\left(\tfrac{1}{2}\kappa m t^2 - \tfrac{1}{2}\kappa m^3 t^2\right)\left(32m^3 - 48m^2 + 18m - 1\right)\left(20t^3 - 30t^2 + 12t - 1\right)
\end{aligned}
$$

But the exact solution of (10) is $u(m, t) = m \sin\left(\frac{\kappa}{2} t\right)$. In Table 1, the numerical error norms at different times are shown.

Table 1. Comparison of error norms when m = 1 at different times t

m	*t*	Rashidinia *et al*. (2010) method $O(k^2 + k^2h^2 + h^2)$	Rashidinia *et al*. (2010) method $O(k^2 + k^2h^2 + h^4)$	Hongchun *et al*. (2018)	Proposed/ Present method at N = 2	Proposed/ Present method at N = 3
1.0	0.1	2.71 E-05	2.71 E-05	6.5955E-12	4.3176E-18	1.2514E-21
1.0	0.3	8.97 E-06	8.97 E-06	6.1985E-13	3.1052E-19	4.0841E-22
1.0	0.5	1.49 E-05	1.49 E-05	1.4550E-13	4.8002E-19	5.1673E-20
1.0	0.7	1.05 E-05	1.05 E-05	1.8202E-12	6.1025E-17	6.0144E-21
1.0	1.0	3.36 E-05	3.36 E-05	2.3494E-11	7.0203E-17	7.0183E-19

4.1 Discussion of results

It is evident from Table 1 above that the approximate results gotten with the current scheme performs well and is in a satisfactory agreement with the theoretical solution of Example 1. Collocation points are picked within the boundary and have proven good based on the results obtained. This method also requires choosing different discretization patterns to obtain a better approximate solution for the PDEs. We could see from the solution that pattern at different schemes (N = 2 and N = 3) were observed, both schemes performed better compared to that of the available ones. Figures 1 to 3 show the exact and approximate results, and the error obtained for Example 1 at the various times t.

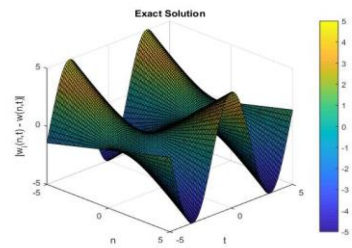

Figure 1. The theoretical solution of the nonlinear Klein-Gordon PDE at various times t, Example 1

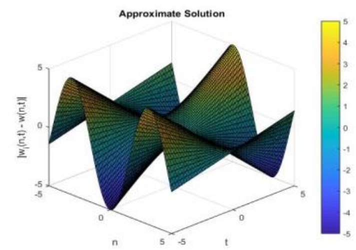

Figure 2. The approximate solutions of the nonlinear Klein-Gordon PDE at different times t, Example 1

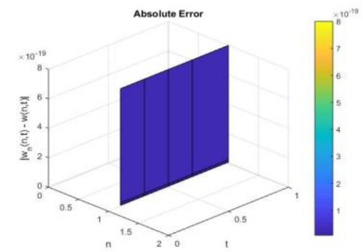

Figure 3. Errors in the approximate solution of Example 1

Example 2: Let us consider another example of nonlinear Klein-Gordon PDE with $\gamma = -1$ and $g(w) = w^2$

$$\frac{\delta^2 w}{\delta t^2} - \frac{\delta^2 w}{\delta m^2} + w^2 = -m\cos t + m^2 \cos^2 t \tag{15}$$

where (15) satisfies the initial conditions and $m \in (-1, 1)$, $t > 0$

$$w(m, 0) = m, \ \frac{\delta w}{\delta t}(m, 0) = 0 \ \ where \ \ m \in [-1, 1]$$

as well as the Dirichlet boundary conditions

$$w(-1, t) = -\cos t, \ w(1, t) = \cos t, \ t \geq 0 \tag{16}$$

The theoretical solution for the DE is $w(m, t) = m \cos t$. Let the assumption for the bivariate series solution of (15) be written as:

$$w(m, t) = m\cos t + (m^2 t^2 - t^2) \sum_{q=0}^{2} \sum_{s=0}^{2} a_{qs} T_q^*(m) P_s^*(t) \tag{17}$$

where $\xi(m,t) = m\cos t$ *and* $\eta(m,t) = (m^2 t^2 - t^2)$ in (17) satisfy conditions (16). Now, differentiate (17) w.r.t. m and t partially to get the derivatives. Following the same procedure as in Example 1, the unknown coefficients obtained numerically are
a_{00} = - 1.811101092, a_{01} = 1.356771719, a_{02} = - 0.2324188716, a_{10} = 0.2113387523,
a_{11} = - 0.04687048676, a_{12} = 0.05621444390, a_{20} = 0.04162676963, a_{21} = - 0.004205565904
a_{22} = - 0.008362225938. Substituting the coefficients into (17), we have the solution (15)

$$\begin{aligned} w(m, t) = &-1.380586404 m^2 t^4 + 4.246748738 m^2 t^3 + 0.3290864892 m^3 t^2 + 0.2997608768 m^4 t^2 \\ &+ m\cos t - 1.196153064 m^3 t^3 + 1.196153064 m t^3 + 1.075960172 m^3 t^4 - 1.075960172 m t^4 \\ &+ 0.3340977906 m^4 t^3 - 0.4013868450 m^4 t^4 - 3.977006133 m^2 t^2 + 3.677245256 t^2 - \\ &0.3290864892 m t^2 + 1.781973249 t^4 - 4.580846528 t^3 \end{aligned}$$

But the theoretical solution of (15) is $w(m, t) = m \cos t$. Table 2 below enumerates the errors at various times t.

Table 2. Comparison of error norms when m = 1 at various times t

m	*t*	Rashidinia *et al*. (2010) method $O(k^2 + k^2h^2 + h^2)$	Rashidinia *et al*. (2010) method $O(k^2 + k^2h^2 + h^4)$	Hongchun *et al*. (2018)	Proposed/ Present method at N = 2	Proposed/ Present method at N = 3
1.0	0.1	7.01 E-09	4.91 E-09	1.5457E-12	1.2146E-12	1.0124E-18
1.0	0.3	6.59 E-09	4.69 E-09	2.3010E-13	9.4012E-12	4.0634E-19
1.0	0.5	1.29 E-09	9.46 E-09	7.6876E-13	7.5018E-11	6.1315E-19
1.0	0.7	7.47 E-09	5.11 E-09	2.1556E-12	2.4587E-10	9.1145E-19
1.0	1.0	5.84 E-09	3.98 E-09	5.8307E-10	8.0017E-10	6.2357E-18

From the table above, it is clearly observed that the approximate results obtained using the present scheme perform favourably well and are in a satisfactory agreement with the theoretical solution of Example 2. Collocation points are picked within the boundary and have proven good based on the results obtained. This method also requires choosing different discretization patterns to obtain better approximate solutions of the PDE. We could see from the solution that pattern at different schemes (N = 2 and N = 3) were observed, both schemes performed well but at N = 3 the results perform better here than the other ones. Figures 4 to 6 below show the exact and approximate results, and the error between them in Example 2 at various times t.

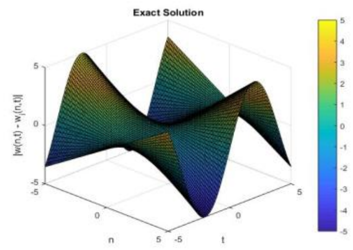

Figure 4. The theoretical solution of the nonlinear Klein-Gordon PDE at various times t, see Example 2

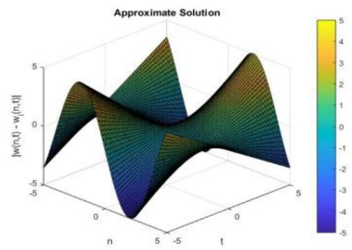

Figure 5. The approximate solution of the nonlinear Klein-Gordon PDE at various times t, see Example 2

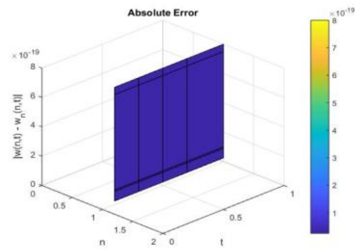

Figure 3. Errors mesh for Example 2

5. Conclusion and Remarks

A numerical approach that combines collocation with hybridizing the Legendre and Chebyshev shifted polynomials was successfully applied to solve a set of nonlinear Klein-Gordon PDEs. With this approach, a set of solutions in terms of the hybridized shifted polynomials was assumed such that satisfied the required boundary conditions of the PDE. Comparing the approach with the previous methods described in the literature, the adjustment to the assumption has been successful in yielding an approximate solution with fewer terms. The interval of examples considered was [0, 1], and as such requires shifting the polynomials. The numerical computation adopts the use of collocation methods, while the convergence was established using known exact solutions. Two examples were considered, and their results were good in comparison with the available literature. The outcomes demonstrated the effectiveness of the suggested approach and its ability to achieve convergence with fewer terms. However, it is recommended that higher than two-dimensional nonlinear PDEs with complex boundary conditions or irregular domains should be considered.

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