Original Article

Analytical evaluation of beam deformation problem using approximate methods

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Abstract

The beam deformation equation has very wide applications in structural engineering. As a differential equation, it has its own problem concerning existence, uniqueness and methods of solutions. Often, original forms of governing differential equations used in engineering problems are simplified, and this process produces noise in the obtained answers.

This paper deals with the solution of second order of differential equation governing beam deformation using four analytical approximate methods, namely the Perturbation, Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variational Iteration Method (VIM). The comparisons of the results reveal that these methods are very effective, convenient and quite accurate for systems of non-linear differential equation.

Key words: Beam, differential equation, Perturbation, analytical methods, non-linear

1. Introduction

Linear and nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real life problems are still very difficult to solve. Therefore, approximate analytical solutions were introduced.

Often, assumptions are made and empirical models are implemented in order to overcome difficulties in solving the equation due to high interdependence of some of the parameters involved. Analytical solutions often fit under classical perturbation methods (Aude *et al.*, 1998; Fillo and Geer, 1996a, 1996b). However, as with other analytical techniques, certain limitations restrict the wide application of perturbation methods, most important of which is the dependence of these methods on the existence of a small parameter in the equation. Disappointingly, the majority of nonlinear problems have no small parameter at all. Even in cases where a small parameter does exist, the determination of such a parameter does not seem to follow any strict rule, and is rather problem-specific. Furthermore, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption.

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Recently, several new techniques have been presented to overcome the mentioned difficulties. Some of these techniques include Variational Iteration Method (VIM) (He, 1997,1999a, 2006a, 2007, 2008a ; He and Wu, 2007; Ganji *et al.*, 2007; Ganji and Sadighi, 2007; Barari *et al.*, 2008a, 2008b, 2008c; Fouladi *et al.*, 2010), decomposition method (Adomian, 1983, 1986, 1994), Homotopy Perturbation Method (HPM) (He, 1999b, 2003,2006b,2008b; Ghotbi *et al.*, 2008; Ganji and Sadighi, 2006; Choobbasti, 2008a, 2008b; Mirmoradi *et al.*, 2009; Hosein Nia *et al.*, 2008) Homotopy Analysis Method (Ghasempour *et al.*, 2009; Sohouli *et al.*, 2010, Omidvar *et al.*, 2010a; Sajid and hayat, 2009; Fooladi *et al.*, 2009; Kimiaeifar *et al.*, 2009a, 2009b,2010; Moghimi *et al.*, 2010), Energy Balance Method (Jamshidi and Ganji, 2010; Momeni *et al.*,2010) etc.

The non-linear differential equation of beam deformation under static load (Figure 1) is shown in the following Eq. (1).

$$\frac{(\frac{d^2y}{dx^2})}{(1+(\frac{dy}{dx})^2)^{\frac{3}{2}}} = \frac{M}{EI}, \tag{1}$$

Note that E is the elastic modulus and that I is the second moment of area. I must be calculated with respect to axis perpendicular to the applied loading. x is the length from left bearing. Mostly, it is assumed that the part $\frac{dy}{dx}$ is equal to zero and the problem is converted to an easier form in order to analyze it. In current research the authors apply new analytical methods to achieve an exact solution of the beam deformation problem.

The boundary conditions usually model supports, they can also model point loads, moments, or other effects, as represented below:

$$y(x = 0) = 0$$
$$\frac{dy}{dx}(x = \frac{l}{2}) = 0, \tag{2}$$

where, l represents length of the beam. Herein, the authors present approximate methods namely HPM, VIM, HAM and perturbation methods to solve the above equation in order to obtain the deformation of beam under static loading.

2. Methods

2.1 Basic Idea of Perturbation Method

Perturbation method is based on assuming a small parameter. The approximate solution obtained by the perturbation methods, in most cases, are valid only for small value of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we can't rely fully on approximations, because there is no criterion on which the small parameter should exist. Thus, it is essential to check the validity of approximations numerically and/or experimentally (Aude *et al.*, 1998; Fillo and Geer, 1996a, 1996b).

For very small ε, let us assume a regular perturbation expansion and calculate the first three terms; thus we assume

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon\theta_2 \tag{3}$$

With substituting Eq. (3) in the Eq. (1) and after expansion and rearranging based on coefficient of ε-term, we have:
And finally with three-term expansion:

$$\varepsilon^0: \text{Differential equation in } \theta_0(\tau) = f(u), \tag{4}$$

$$\varepsilon^1: \text{Differential equation in } \theta_1(\tau) \text{ and } \theta_0(\tau) = 0, \tag{5}$$

$$\varepsilon^2: \text{Differential equation in } \theta_2(\tau), \theta_1(\tau) \text{ and } \theta_0(\tau) = 0 \tag{6}$$

And finally with three-term expansion:

$$\theta(\tau) = \varepsilon^0\theta_0(\tau) + \varepsilon^1\theta_1(\tau) + \varepsilon^2\theta_2(\tau) \tag{7}$$

2.2 Basic Idea of Homotopy Perturbation Method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{8}$$

with the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{9}$$

where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally speaking, the operator A can be divided in to a linear part L and a nonlinear part $N(u)$. Eq. (8) can therefore, be written as:

$$L(u) + N(u) - f(r) = 0, \tag{10}$$

By the homotopy technique, it is reduced to:

$$v(r, p) : \Omega \times [0,1] \rightarrow R$$

which satisfies

$$H(v, p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0,$$
$$p \in [0,1], r \in \Omega, \tag{11}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (8), which satisfies the boundary conditions. Obviously, from Eqs. (11) and (12) then it is reduced to:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{13}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{14}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called Homotopy. According to the HPM, the embedding parameter p can be first used as a "small parameter", and assume that the solutions of Eqs. (11) and (12) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ..., \tag{15}$$

Setting $p = 1$ yields in the approximate solution of Eq. (11) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{16}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (16) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He made the following suggestions (He, 1999b):

• The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$.

• The norm of $L^{-1} \dfrac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

2.3 Basic Idea of Variational Iteration Method

To clarify the basic ideas of VIM the following differential equation is considered:

$$Lu + Nu = g(t), \tag{17}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau) \right) d\tau \tag{18}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theorem. The subscript n indicates the n*th* approximation and u_n is considered as a restricted variation.

2.4 Application of Perturbation Method

Consider Eq. (1) for beam with conditions as follows:

$$E.I = 1000 \ N.m^2, L \ (\text{length of the beam}) = 1 \ m, F = 10 \ N \tag{19}$$

To solve Eq. (1) by means of Perturbation, we consider the following process. First we change the Eq. (1) to the following form:

$$\left(\frac{d^2 y}{dx^2}\right)^2 - \left(\frac{M}{EI}\right)^2 . \left(1 + \varepsilon \left(\frac{dy}{dx}\right)^2\right)^3 = 0$$

$$M = \frac{F}{2} x, \tag{20}$$

Where, x is length from the left support. It is assumed that $\theta(x)$ is:

$$\theta(x) = \theta_0(x) + \varepsilon \theta_1(x) + \varepsilon^2 \theta_2(x) \tag{21}$$

Substituting Eq.(21) in to Eq.(20) and rearranging the resultant equation based on powers of ε-terms, one has:

$$\varepsilon^0 : \left(\frac{d}{dx}\theta_0(x)\right)^2 - \frac{F^2 x^2}{4EI} = 0, \tag{22}$$

$$\varepsilon^1 : 2\left(\frac{d^2}{dx^2}\theta_0(x)\right)\left(\frac{d^2}{dx^2}\theta_1(x)\right) - \frac{3F^2 x^2 \left(\frac{d}{dx}\theta_0(x)\right)^2}{4(EI)^2} = 0, \tag{23}$$

$$\varepsilon^2 : -\frac{3F^2 x^2 \left(\frac{d}{dx}\theta_0(x)\right)^4}{4(EI)^2} - \frac{3F^2 x^2 \left(\frac{d}{dx}\theta_0(x)\right)\left(\frac{d}{dx}\theta_1(x)\right)}{2(EI)^2}$$
$$+ 2\left(\frac{d^2}{dx^2}\theta_0(x)\right)\left(\frac{d^2}{dx^2}\theta_2(x)\right) + \left(\frac{d^2}{dx^2}\theta_1(x)\right)^2 = 0, \tag{24}$$

$\theta(x)$ might be written as follows by solving the Eqs. (22), (23) and (24):

$$\theta_0(x) = -\frac{Fx^3}{12EI} + \frac{Fl^2 x}{16EI}, \tag{25}$$

$$\theta_1(x) = \frac{3F^3 \left(\frac{8}{21}x^7 - \frac{2}{5}l^2 x^5 + \frac{1}{6}l^4 x^3\right)}{1024(EI)^3} - \frac{F^3 l^6 x}{8192(EI)^3} \tag{26}$$

$$\theta_2(x) : \frac{15F^5 \left(\frac{128}{55}x^{11} - \frac{32}{9}l^2 x^9 + \frac{16}{7}l^4 x^7 - \frac{4}{5}l^6 x^5 + \frac{1}{6}l^8 x^3\right)}{1048576(EI)^5}$$
$$- \frac{3F^5 l^{10} x}{8388608(EI)^5}, \tag{27}$$

In the same manner, the rest of components were obtained using the Maple package. According to the Perturbation, we can conclude that:

$$y(x) = \lim \theta(x) = \theta_0(x) + \theta_1(x) + \theta_2(x)... ,$$
$$\varepsilon \to 1 \tag{28}$$

Therefore, substituting the values of $\theta_0(x)$, $\theta_1(x)$ and $\theta_2(x)$ from Eqs. (25), (26) and (27) in to Eq. (28) yields:

$$y(x) = -\frac{Fx^3}{12EI} + \frac{Fl^2x}{16EI} + \frac{3F^3(\frac{8}{21}x^7 - \frac{2}{5}l^2x^5 + \frac{1}{6}l^4x^3)}{1024(EI)^3}$$

$$-\frac{F^3l^6x}{8192(EI)^3} + \frac{15F^5(\frac{128}{55}x^{11} - \frac{32}{9}l^2x^9 + \frac{16}{7}l^4x^7 - \frac{4}{5}l^6x^5 + \frac{1}{6}l^8x^3)}{1048576(EI)^5}$$

$$-\frac{3F^5l^{10}x}{8388608(EI)^5}, \tag{29}$$

2.5 Application of Homotopy Perturbation Method

To solve Eq. (1) by means of HPM, the following process is considered. First Eq. (1) is changed to the following form:

$$(\frac{d^2}{dx^2}v(x)) - (\frac{M}{EI}).(1 + \frac{3}{2}(\frac{d}{dx}v(x))^2) =$$

$$M = \frac{F}{2}x$$

$$\lim(1+\alpha)^\beta = 1 + \beta\alpha$$

$$\alpha \to 0 \tag{30}$$

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + ..., \tag{31}$$

Here it is assumed that $v_2 = 0$. To solve Eq. (30) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation. A Homotopy can be constructed as follows:

$$H(v,p) = (1-p)(\frac{\partial^2}{\partial x^2}v(x)) + p((\frac{d^2v(x)}{dx^2})$$

$$-(\frac{Fx}{2EI}).(1 + \frac{3}{2}(\frac{dv(x)}{dx})^2)) = 0, \tag{32}$$

Substituting $v = v_0 + pv_1 + ...$ in to Eq. (32) and rearranging the resultant equation based on the powers of p-terms, one has:

$$p^0 : \frac{\partial^2 v_0(x)}{\partial x^2} = 0, \tag{33}$$

$$p^1 : (\frac{\partial^2}{\partial x^2}v_1(x)) - \frac{Fx}{2EI} - \frac{3Fx(\frac{d}{dx}v_0(x))^2}{4EI} = 0, \tag{34}$$

$$p^3 : (\frac{d^2}{dx^2}v_1(x)) - \frac{3Fx(\frac{d}{dx}v_1(x))^2}{4EI} = 0, \tag{35}$$

Where, $V(x)$ may be written as follows by solving the Eqs. (33), (34) and (35):

$$v_0(x) = 0, \tag{36}$$

$$v_1(x) = \frac{Fx^3}{12EI} - \frac{Fl^2x}{16EI}, \tag{37}$$

$$v_3(x) = \frac{3F^3(\frac{8}{21}x^7 - \frac{2}{5}l^2x^5 + \frac{1}{6}l^4x^3)}{1024(EI)^3} - \frac{F^3l^6x}{8192(EI)^3}, \tag{38}$$

In the same manner, the rest of components were obtained using the Maple package. According to the HPM, we can conclude that:

$$y(x) = \lim v(x,t) = v_0(x) + v_1(x) + ... ,$$
$$p \to 1 \tag{39}$$

Therefore, substituting the values of $v_0(x)$, $v_1(x)$ and $v_2(x)$ from Eqs. (36), (37) and (38) in to Eq. (39) yields:

$$y(x) = \frac{Fx^3}{12EI} - \frac{Fl^2x}{16EI} + \frac{3F^3(\frac{8}{21}x^7 - \frac{2}{5}l^2x^5 + \frac{1}{6}l^4x^3)}{1024(EI)^3} - \frac{F^3l^6x}{8192(EI)^3}, \tag{40}$$

2.6 Application of Variational Iteration Method

In this Section, variational iteration method is developed for solving beam deformation equation. Consider beam deformation equation (Eq. 1). To solve Eq. (1) via VIM at first, Eq. (1) is developed to Eq. (41).

$$\frac{d^2v(x)}{dx^2} - (\frac{M}{EI})(1 + \frac{3}{2}(\frac{dv(x)}{dx})^2) = 0$$

$$M = \frac{Fx}{2}$$

$$\lim(1+\alpha)^\beta = 1 + \beta\alpha$$

$$\alpha \to 0 \tag{41}$$

where x is displacement form left support and $y(x)$ is the deformation of the beam In order to find the Lagrangian multiplier, which can be identified by substituting Eq. (41) into Eq. (18), upon making it stationary leads to the following:

$$1 - \lambda'|_{x=\tau} = 0$$
$$\lambda|_{x=\tau} = 0$$
$$\lambda''|_{x=\tau} = 0 \tag{42}$$

Solving the system of Eq. (42), yields:

$$\lambda(\tau) = x - \tau, \tag{43}$$

and the variational iteration formula is obtained in the form:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left\{ (x-\tau)\left(\frac{d^2}{d\tau^2}y_n(\tau) - \left(1 + \frac{3}{2}\left[\frac{d}{d\tau}y_n(\tau)\right]^2\right)\right) * \frac{F\tau}{2EI} \right\} d\tau, \tag{44}$$

Now, for initial approximation ($y_0(x)$) the following procedure is considered.

- The value of $\dfrac{d}{dx}y(x)$ is small therefore we assumed $\dfrac{d}{dx}y(x) = 0$, then the Eq. 44 changes to Eq. 45.

$$\left(\frac{d^2}{dx^2}y(x)\right) = \left(\frac{M}{EI}\right)$$
$$M = \frac{F}{2}x, \tag{45}$$

- The Eq. 45 can be solved easily considering bellow boundary condition

$$y(x=0) = 0$$
$$\frac{dy}{dx}\left(x=\frac{l}{2}\right) = 0, \tag{46}$$

- The answer of Eq. 45 considering Eq.(46) is shown in Eq.(47) which is the initial approximation.

$$y(x) = y_0(x) = \frac{Fx^3}{12EI} - \frac{FL^2x}{16EI} \tag{47}$$

Using the above variational formula (44), we have:

$$y_1(x) = y_0(x) + \int_0^x \left\{ (x-\tau)\left(\frac{d^2}{d\tau^2}y_0(\tau) - \left(1 + \frac{3}{2}\left[\frac{d}{d\tau}y_0(\tau)\right]^2\right)\right) * \frac{F\tau}{2EI} \right\} d\tau, \tag{48}$$

Substituting Eq. (47) in to Eq. (48) and after simplification, we have:

$$y_1(x) =$$
$$-\frac{Fx(-17920x^2(EI)^2 + 13440L^2(EI)^2 + 240F^2x^6 - 252F^2L^2x^4 + 105F^2L^4x^2)}{215040(EI)^3} \tag{49}$$

as it is apparent through comparisons, just one iteration leads us to high accuracy of solutions.

2.7 Application of homotopy Analysis Method

In this Section, Homotopy Analysis Method (HAM) is used to obtain the exact solution of nonlinear governing equation for deformation of a beam. Nonlinear operator is defined as follows:

$$N[y(x;q)] = \frac{d^2 y(x;q)}{dx^2} - \left(\frac{M}{EI}\right)\left(1 + \frac{3}{2}\left(\frac{dy}{dx}\right)^2\right), \tag{50}$$

Where $q \in [0, 1]$ is the embedding parameter, As the embedding parameter increases from *0* to *1*, $U(x; q)$, varies from the initial guess, $U_0(x)$, to the exact solution, $U(x)$.

$$y(x;0) = U_0(x), \qquad y(x;1) = U(x), \tag{51}$$

Expanding $y(x;q)$, in Taylor series with respect to q results in:

$$y(x;q) = U_0(x) + \sum_{m=1}^{\infty} U_m(x)q^m, \tag{52}$$

where,

$$U_m(x) = \frac{1}{m!}\frac{\partial^m y(x;q)}{\partial q^m}\bigg|_{q=0}, \tag{53}$$

Homotopy analysis method can be expressed by many different base functions (Kimiaeifar *et al.*, 2009a, 2009b), according to the governing equations; it is straightforward to use a base function in the form of:

$$U(x) = \sum_{m=1}^{\infty} b_m x^m, \tag{54}$$

b_m, *is* the coefficients to be determined. The linear operators L is chosen as:

$$L[y(x;q)] = \frac{\partial^2 y(x;q)}{\partial x^2}, \tag{55}$$

Eq. (55) results in:

$$L[c_1 + c_2 x] = 0, \tag{56}$$

Where c_1 and c_2 are the integral constants. According to the rule of solution expression and the initial conditions, the initial approximations, U_0 as well as the integral constants, c_1 and c_2 are expressed as (Moghimi *et al.*, 2010):

$$U_0(\eta) = c_1 + c_2 x \qquad c_1 = c_2 = 0, \tag{57}$$

The zeroth order deformation equation and their boundary condition for *y(x)* are:

$$(1-q)L[y(x;q) - U_0(x)] = q\hbar H(x) N[y(x;q)], \tag{58}$$

$$y(0;q) = 0, \qquad \frac{dy\left(\frac{L}{2};q\right)}{dx} = 0, \tag{59}$$

According to the rule of solution expression and from Eq.(52), the auxiliary function $H(x)$ can be chosen as follows, (Kimiaeifar, 2010):

$$H(x) = 1 \tag{60}$$

Differentiating Eq.(58), *m* times, with respect to the embedding parameter *q* and then setting *q=0* in the final expression and dividing it by *m!*, it is reduced to:

$$U_m(x) = \chi_m U_{m-1}(x) + \hbar\left(\int_0^x \int_0^x H(x)R_m(U_{m-1})dxdx + c_1 + c_2 x\right), \tag{61}$$

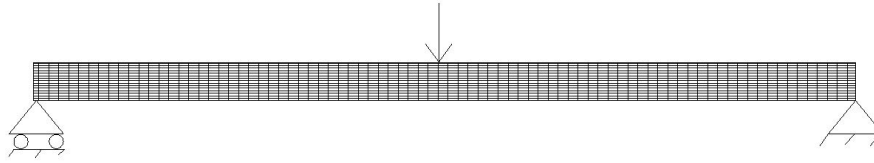$$U_m(0) = 0, \qquad U_m'\left(\frac{L}{2}\right) = 0, \tag{62}$$
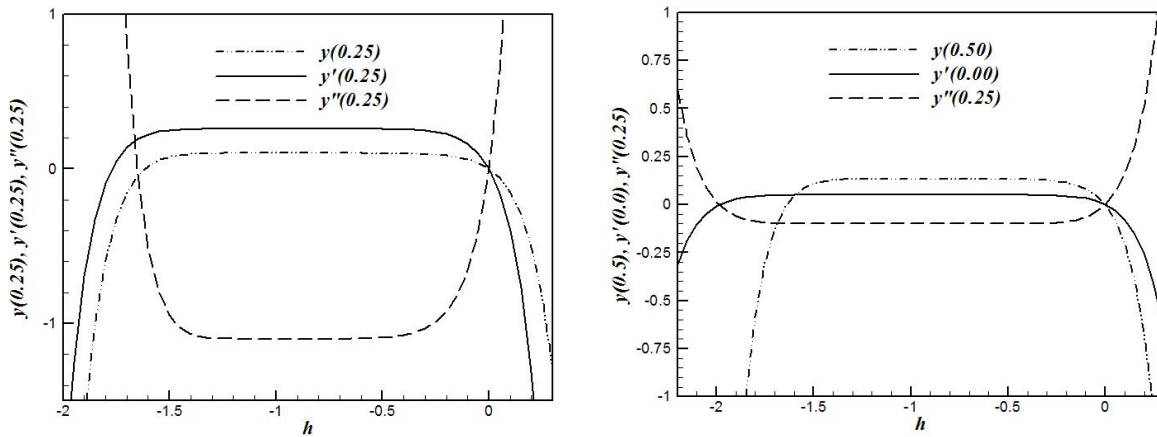
Figure 1. A simply supported beam.

Figure 2. h curve to show the convergence region

Eq. (61) is the *m*th order deformation equation for *y(x)*, where

$$R_m(U_{m-1}) = \frac{d^2 U_{m-1}(x)}{dx^2} - \frac{M}{EI}\left(\chi_m + \frac{3}{2}\sum_{z=0}^{m-1}\frac{dU_z(x)}{dx}\frac{dU_{m-1-z}(x)}{dx}\right)$$

(63)

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases},$$

(64)

The first, second and third terms of the solution's series are as follows:

$$U_0(x) = 0,$$

(65)

$$U_1(x) = \frac{0.5hMILs(s-L)}{E}$$

(66)

$$U_2(x) = \frac{0.5hMILs(s-L+hs-hL)}{E}$$

The series solution is developed to the 10th order of approximation.

2.8 Convergence of HAM solution

The analytical solution should converge. It should be noted that the auxiliary parameter $\hbar$ (Fooladi *et al.*, 2009), controls the convergence and accuracy of the solution series. In order to define a region such that the solution series is independent on $\hbar$, a multiple of -curves are plotted. The region where the distribution of versus is a horizontal line is known as the convergence region for the corresponding function, see Figure 2.

3. Results and Discussions

In this section the results of perturbation, HPM, HAM and VIM methods are compared with each other for $0 < l < 0.5$. The same results can be obtained for the other half of the beam due to symmetry (Table 1 and Figure 3).

4. Conclusion

In this paper, the homotopy perturbation method, perturbation method, homotopy analysis method and variational iteration method have been successfully applied to governing differential equation of beam deformation with specified boundary conditions. These methods enable to convert a difficult problem into a simple problem which can easily be solved. The comparisons of the results obtained here provide more realistic solutions, reinforcing the conclusions pointed out by many researchers about the efficiency of these methods. Therefore the HPM, HAM, perturbation and variational iteration methods are powerful mathematical tools and can be widely applied to structural engineering such as beam problems.

Table 1.   Comparison between different results

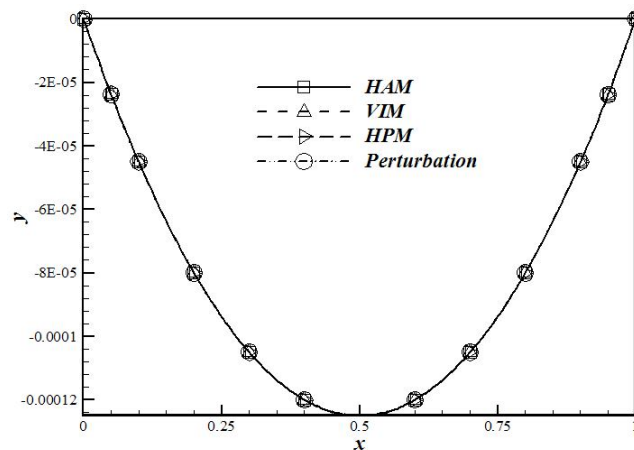| X	VIM	Perturbation	HPM	HAM
0	0	0	0	0
0.1	-6.16666667E-06	-6.16666668E-06	-6.16666668E-06	-6.16666668E-06
0.2	-1.18333333E-05	-1.18333334E-05	-1.18333334E-05	-1.18333334E-05
0.3	-1.65000000E-05	-1.65000000E-05	-1.65000000E-05	-1.65000000E-05
0.4	-1.96666667E-05	-1.96666667E-05	-1.96666667E-05	-1.96666667E-05
0.5	-2.08333334E-05	-2.08333334E-05	-2.08333334E-05	-2.08333334E-05

Figure 3.   comparison between different results

References

Adomian, G.1983. Stochastic Systems. Academic Press Inc. New York.

Adomian, G. 1986. Nonlinear Stochastic Operator Equations. Academic Press Inc. New York.

Adomian, G. 1994. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer. Boston.

Aude, P., Beghein, C., Depecher, P. and Inard, C. 1998. Perturbation of the input data of models used for prediction of turbulent air flow in an enclosure, Numerical Methods in Heat Transfer, Part B. 139-164.

Barari, A., Omidvar, M., Ganji, D.D. and Tahmasebi Poor, A. 2008a. An Approximate solution for boundary value problems in structural engineering and fluid mechanics. Journal of Mathematical Problems in Engineering. Article ID 394103. 1-13.

Barari, A., Ghotbi, Abdoul R., Farrokhzad, F. and Ganji, D.D.2008b. Variational iteration method and Homotopy-perturbation method for solving different types of wave equations. Journal of Applied Sciences. 8, 120-126.

Barari, A., Omidvar, M., Ghotbi, Abdoul R. and Ganji, D.D. 2008c. Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations. Acta Applicanda Mathematicae. 104, 161-171

Choobbasti, A.J., Barari, A., Farrokhzad, F. and Ganji, D.D. 2008a. Analytical investigation of a fourth- order boundary value problem in deformation of beams and plate deflection theory. Journal of Applied sciences. 8, 2148-2152.

Choobbasti, A.J., Farrokhzad, F., Barari, A. and Ganji, D.D. 2008b. Application of wave equation in pile foundation using homotopy perturbation method and variational iteration method. Journal of Applied Sciences. 8, 2158-2162.

Fillo, J. and Geer, J. 1996a. Hybrid perturbation-Galerkin technique for nonlinear heat condition problems 2: partial differential equations. Numerical Methods in Heat Transfer. Part B Fundam. 75-89

Fillo, J. and Geer, J. 1996b. Hybrid perturbation-Galerkin technique for nonlinear heat condition problems 1: Steady problem. Numerical Methods in Heat Transfer. Part B Funmdam. 61-73.

Fooladi, M., Abaspour, S.R., Kimiaeifar, A. and Rahimpour, M. 2009. On the analytical solution of nonlinear normal mode for continuous systems by means of HAM. World Applied Sciences Journal. 6, 297-302.

Fouladi, F., Hosseinzadeh, E., Barari, A. and Domairry, G. 2010. Highly nonlinear temperature dependent fin analysis by variational iteration method, Journal of Heat Transfer Research. 41, 155-165.

Ganji, D.D., Jannatabadi, M. and Mohseni, E. 2007. Application of He's variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM. Journal of Computational and Applied Mathematics. 207, 35-45.

Ganji, D.D. and Sadighi, A. 2007. Application of homotopy–perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. Journal of Computational and Applied Mathematics. 207, 24-34.

Ganji, D.D. and Sadighi, A. 2006. Application of He's Homotopy–prturbation Method to Nonlinear Coupled Systems of Reaction–diffusion Equations, International Journal of Nonlinear Sciences and Numerical Simulation. 7, 411–418.

Ghasempour, M., Rokni, E., kimiaeifar, A. and Rahimpour, M. 2009. Assessment of HAM and PEM to find analytical solution for calculating displacement functions of geometrically nonlinear prestressed cable structures with concentrated mass. International Journal of World Applied Science. 9, 2264-2271.

Ghotbi, Abdoul R., Barari, A. and Ganji, D.D. 2008. Solving ratio-dependent predator–prey system with constant effort harvesting using Homotopy perturbation method. Journal of Mathematical Problems in Engineering. Article ID 945420, 1-8.

He,J.H. 1997. A new approach to nonlinear partial differential equations. Communication in Nonlinear Sciences and Numerical Simulation, 2, 230–235.

He,J.H. 1999a. Variational iteration method – a kind of nonlinear analytical technique: some examples. International Journal of Non-Linear Mechanics, 34, 699–708.

He, J.H. 2006a. Some asymptotic methods for strongly nonlinear equations. International Journal of Modern Phys. B. 20, 1141–1199.

He, J.H. and Wu. X.H.2007. Variational iteration method: New development and applications. Computers Mathematics with Applications. 54, 881-894.

He, J.H. 2007. Variational iteration method - Some recent results and new interpretations. Journal of Computational and Applied Mathematics. 207, 3-17.

He, J.H. 2008a. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. International Journal Modern Phys. B., 22 (2008), 3487-3578.

He, J.H. 2003. Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation. 135, 73–79.

He, J.H. 2006b. Addendum: new interpretation of homotopy perturbation method. International Journal of Modern Phys. 20, 2561–2568.

He, J.H. 1999b. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering. 178, 257–262.

He, J.H. 2008b. Recent development of the homotopy perturbation method. Topological Methods in Nonlinear Analysis. 31, 205-209.

Hosein Nia, S.H., Ranjbar, A.N., Ganji, D.D., Soltani, H. and Ghasemi, J. 2008. Maintaining the stability of nonlinear differential equations by the enhancement of HPM. Physics Letters A. 372, 2855-2861.

Jamshidi, N. and Ganji, D.D. 2010. Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire. Current Applied Physics.10, 484-486.

Kimiaeifar, A., Saidi, A.R., Bagheri, G.H. and Rahimpour, M. DOmairry, D.G. 2009a.Analytical solution for Van der Pol –Duffing oscillators. Chaos, Solitons and Fractals. 42, 2660-2666.

Kimiaeifar, A., Bagheri, G.H., Rahimpour, M. and Mehrabian, M.A. 2009b.Analytical solution of two-dimensional stagnation flow towards a shrinking sheet by means of Homotopy Analysis Method. Journal of Process Mechanical Engineering. 223, 133-143.

Kimiaeifar, A. 2010. An analytical approach to investigate the response and stability of Van der Pol-Mathieu-Duffing oscillators under different excitation functions. Journal of Mathematical Methods in the Applied Sciences. DOI 10.1002/mma.1269.

Mirmoradi, H., Mazaheripour, H., Ghanbarpour, S. and Barari, A. 2009. International Journal of Research and Reviews in Applied Sciences. 1, 163-173.

Moghimi, M., Kimiaeifar, A., Rahimpour, M. and Bagheri, G.H. 2010. Homotopy analysis method for marangoni mixed convection boundary layer flow. Journal of Mechanical Engineering Science. DOI 10.1243/ 09544062JMES1945.

Momeni, M., Jamshidi, N., Barari, A. and Ganji, D.D. 2010, Application of He's energy balance method to Duffing harmonic oscillators, International Journal of Computer Mathematics. DOI: 10.1080/00207160903337239.

Omidvar, M., Barari, A., Momeni, M. and Ganji, D.D. 2010, New class of solutions for water infiltration problems in unsaturated soils. Geomechanics and Geoengineering: An International Journal. 5, 127-135.

Sajid, M. and hayat, T. 2009. Comparison of HAM and HPM solutions in heat radiation equations. International Communications in Heat and Mass Transfer. 36, 59-62.

Sohouli, A.R., Famouri, M., Kimiaeifar, A. and Domairry, G. 2010.Application of homotopy analysis method for natural convection of Darcian fluid about a vertical full cone embedded in pours media prescribed surface heat flux. Communication in Nonlinear Science and Numerical Simulation. 15, 1691-1699.