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*Original Article*

# The $\chi^2$ sequence space over p-metric spaces defined by Musielak modulus

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**Abstract**

In this paper, we introduce bonacci numbers of $\chi^2(F)$ sequence space over $p$–metric spaces defined by Musielak function and examine some topological properties of the resulting these spaces.

**Keywords:** analytic sequence, double sequences, $\chi^2$ space, fibonacci number, Musielak - modulus function, $p$–metric space

## 1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate-wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich(1965). Later on it was investigated by Hardy (1917); Moricz (1991); Moricz and Rhoades (1988); Basarir and Solankan (1999); Tripathy *et al.* (2003, 2004, 2006, 2007, 2008, 2009, 2010, 2011 and 2013); Turkmenoglu (1999); Raj (2010, 2011, 2012, 2013) and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t)$ and $\mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t)$\$;

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document's related to the double sequence spaces. Gökhan *et al.* (2005) have proved that $M_u(t)$ and $C_p(t)$, $C_{bp}(t)$ are complete paranormed spaces of double sequences and obtained the $\alpha$–, $\beta$–, $\gamma$– duals of the spaces $M_u(t)$ and $C_{bp}(t)$ : Quite recently, in her PhD thesis, Zeltser (2001) has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen *et al.* (2003, 2004, 2013, 2014) and Tripathy *et al.* (2003, 2004, 2006, 2007, 2008, 2009, 2010, 2011 and 2013) have independently introduced the statistical convergence and Cauchy for double sequences and established the relation between statistical convergent and strongly Cesaro summable double sequences. Altay and Ba591sar (2005) have defined the spaces $BS$; $BS(t)$, $CS_p$, $CS_{bp}$, $CS_r$ and $BV$ of

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double sequences consisting of all double series whose sequence of partial sums are in the spaces $M_u$, $M_u(t)$, $C_p$, $C_{bp}$, $C_r$ and $L_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha$– duals of the spaces $BS$, $BV$, $CS_{bp}$ and the $\beta(\vartheta)$ – duals of the spaces $CS_{bp}$ and $CS_r$ of double series. Ba592sar and Sever (2009) have introduced the Banach space $L_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $\mathcal{L}_q$. Recently Subramanian and Misra (2010) have studied the space $\chi_M^2(p,q,u)$ of double sequences and proved some inclusion relations.

The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox (1986) as an extension of the definition of strongly Cesaro summable sequences. Connor (1989) further extended this definition to a definition of strong $A$– summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A$– summability, strong $A$– summability with respect to a modulus, and $A$– statistical convergence. In Pringsheim (1900) the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$; we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi$= {all finite sequences} .

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{i,j} \Im_{ij}$ for all $m,n \in \mathbb{N};$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)$X$ is said to have AK property if $(\Im_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Let $M$ and $\Phi$ be mutually complementary modulus functions. Then, we have

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{ (Young's in equality)}$$

[See Kampthan *et al.* (1981)] (1.2)

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{1.4}$$

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space $\ell_M$ with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\, u - f_{mn}(u) : u \geq 0\}, m, n = 1, 2, ...$$

is called the complementary function of a Musielak-modulus function $f$. For a given Musielak modulus function $f$, the Musielak-modulus sequence space $t_f$ is defined by

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 asm, n \to \infty \right\},$$

where $t_f$ is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider $t_f$ equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left( \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left( \frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [16] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\},$$

for $Z=c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k\in\mathbb{N}$

Here $c$, $c_0$ and $\ell_\infty$ denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference sequence space $bv_p$ of the classical space $\ell_p$ is introduced and studied in the case $1\le p\le\infty$ by Basar and Altay and in the case $0<p<1$ by Altay *et al.* (2005). The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and $bv_p$ are Banach spaces normed by

$$\|x\|=|x_1|+sup_{k\ge1}|\Delta x_k| \$and\$ \|x\|_{bv_p}=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},(1\le p<\infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\},$$

where $Z=\Lambda^2, \chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})$ $=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn}=\Delta^{m-1}x_{mn}-\Delta^{m-1}x_{mn+1}-\Delta^{m-1}x_{m+1n}+\Delta^{m-1}x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn}=$

$$\sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}.$$

## 2. Definition and Preliminaries

Let $n\in\mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n\le w$. A real valued function $d_p(x_1, ..., x_n) = \|(d_1(x_1, 0), ..., d_n(x_n, 0))k_p$ on $X$ satisfying the following four conditions:

(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p=0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p=|\alpha|\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha\in\mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n))=$

$\left(d_X(x_1,x_2,\cdots x_n)^p+d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p}$ for $1\le p<\infty$;

(or)

(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)):=$

$\sup\{d_X(x_1,x_2,\cdots x_n),d_Y(y_1,y_2,\cdots y_n)\}$, for $x_1,x_2,\cdots x_n\in X$, $y_1,y_2,\cdots y_n\in Y$ is called the $p$–product metric of the Cartesian product of $n$–metric spaces is the $p$–norm of the $n$–vector of the norms of the $n$–sub spaces.

A trivial example of $p$–product metric of $n$–metric space is the $p$–norm space is $X=\mathbb{R}$ equipped with the following Euclidean metric in the product space is the $p$–norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E=sup(|det(d_{mn}(x_{mn},0))|)=$$

$$sup\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0)\\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0)\\ \vdots & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0)\end{vmatrix}$$

where $x_i=(x_{i1},\cdots x_{in})\in\mathbb{R}^n$ for each $i=1,2,\cdots n$.

If every Cauchy sequence in $X$ converges to some $L\in X$, then $X$ is said to be complete with respect to the $p$– metric. Any complete $p$– metric space is said to be $p$–Banach metric space.

### 2.1 Defintion.

Let $A=(a_{k,\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $k,\ell-th$ th term of $Ax$ is as follows:

$$(Ax)_{k\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}^{mn}x_{mn}$$

such transformation is said to be non-negative if $a_{k\ell}^{mn}$ is non-negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which both added an adiditional assumption of boundedness. This assumption was made since a double sequence which is $P$– convergent is not necessarily bounded.

Let $\lambda$ and $\mu$ be two sequence spaces and $A=(a_{k,\ell}^{mn})$ be a four dimensional infinite matrix of real numbers $(a_{k,\ell}^{mn})$, where $m,n,k,\ell\in\mathbb{N}$. Then, we say $A$ defines a matrix mapping from $\lambda$ into $\mu$ and we denote it by writing $A:\lambda\to\mu$. if for every sequence $x=(x_{mn})\in\lambda$ the sequence $Ax=\{(Ax)_{k\ell}\}$, the $A$– transform of $x$, is in $\mu$. By $(\lambda:\mu)$, we denote the class of all matrices $A$ such that $A:\lambda\to\mu$. Thus $A\in(\lambda:\mu)$ if and only if the series converges for each $k,\ell\in\mathbb{N}$. A

sequence $x$ is said to be $A-$ summable to $\alpha$ if $Ax$ converges to $\alpha$ which is called as the $A-$ limit of $x$.

**2.2 Lemma. [See Maddox (1986)]**

Matrix $A=\left(a_{k,\ell}^{mn}\right)$ is regular if and only if the following three conditions hold:

(1) There exists $M>0$ such that for every $k,\ell=1,2,\cdots$ the following inequality holds: $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|a_{k\ell}^{mn}\right|\le M$;

(2) $\lim_{k,\ell\to\infty}a_{k\ell}^{mn}=0$ for every $k,\ell=1,2,\cdots$

(3) $\lim_{k,\ell\to\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}^{mn}=1.$

Let $(q_{mn})$ be a sequence of positive numbers and

$$Q_{k\ell}=\sum_{m=0}^{k}\sum_{n=0}^{\ell}q_{mn}\left(k,\ell\in\mathbb{N}\right). \tag{2.1}$$

Then, the matrix $R^{q}=\left(r_{k\ell}^{mn}\right)^{q}$ of the Riesz mean is given by

$$\left(r_{k\ell}^{mn}\right)^{q}=\begin{cases}\dfrac{q_{mn}}{\%\,Q_{k\ell}} & \text{if } 0\le m,n\le k,\ell\\ 0 & \text{if } (m,n)>k\ell\end{cases} \tag{2.2}$$

The fibonacci numbers are the sequence of numbers $f_{k\ell}^{mn}\left(k,\ell,m,n\in\mathbb{N}\right)$ defined by the linear recurrence equations $f_{00}=1$ and $f_{11}=1, f_{mn}=f_{m-1n-1}+f_{m-2n-2}; m,n\ge 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are the following.

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n}f_{mn}=f_{m+2n+2}-1; m,n\ge 1,$$

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n}f_{mn}^{2}=f_{mn}f_{m+1n+1}; m,n\ge 1,$$

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{f_{k\ell}^{mn}}\ \text{converges.}$$

In this paper, we define the fibonacci matrix $F=\left(f_{k\ell}^{mn}\right)_{m,n=1}^{\infty}$, which differs from existing Fibonacci matrix by using Fibonacci numbers $f_{k\ell}$ and introduce some new sequence spaces $\chi^{2}$ and $\Lambda^{2}$. Now, we define the Fibonacci matrix $F=\left(f_{k\ell}^{mn}\right)_{m,n=1}^{\infty}$, by

$$\left(f_{k\ell}^{mn}\right)=\begin{cases}\dfrac{f_{k\ell}}{f_{(k+2)(\ell+2)}-1} & \text{if } 0\le k\le m; 0\le\ell\le n\\ 0 & \text{if } (m,n)>k\ell\end{cases}$$

that is,

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0\ldots & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\ldots & \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0\ldots & \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0\ldots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

It is obvious that the four dimensional infinite matrix $F$ is triangular matrix. Also it follows from lemma 2.2 that the method $F$ is regular.

Let $M$ be an Musielak modulus function. We introduce the following sequence spaces based on the four dimensinoal infinite matrix $F$:

$$\left[\Lambda_{M}^{2F},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right]=F_{\eta}\left(x\right)$$

$$=sup_{k\ell}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f_{k\ell}^{mn}\left|x_{mn}\right|^{1/m+n},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right)\right]<\infty\right\}$$

$$=sup_{k\ell}$$

$$\left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f_{k\ell}^{mn}\left|x_{mn}\right|^{1/m+n},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right)\right]<\infty\right\},$$

$\left(k,\ell\in\mathbb{N}\right)$.

Consider the metric space

$\left[\Lambda_{M}^{2F},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right]$ with the metric

$$d\left(x,y\right)=sup_{k\ell}\left\{M\left(F_{\eta}\left(x\right)-F_{\eta}\left(y\right)\right):m,n=1,2,3,\cdots\right\}. \tag{2.3}$$

$$\left[\chi_{M}^{2F},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right]=F_{\mu}\left(x\right)$$

$$=lim_{k,\ell\to\infty}$$

$$\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f_{k\ell}^{mn}\left(\left(m+n\right)!\left|x_{mn}\right|\right)^{1/m+n},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right)\right]=0\right\}$$

$$=lim_{m,n\to\infty}$$

$$\left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(f_{k\ell}^{mn}\left(\left(m+n\right)!\left|x_{mn}\right|\right)^{1/m+n},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right)\right]\right.$$

$\left.,\left(k,\ell\in\mathbb{N}\right)\right.$.

Consider the metric space

$\left[\chi_{M}^{2F},\left\|\left(d\left(x_{1},0\right),d\left(x_{2},0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_{p}\right]$ with the metric

$$d\left(x,y\right)=sup_{k\ell}\left\{M\left(F_{\mu}\left(x\right)-F_{\mu}\left(y\right)\right):m,n=1,2,3,\cdots\right\}. \tag{2.4}$$

## 3. Main Results

### 3.1 Theorem.

The spaces $\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ and $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ are BK spaces with the metric (2.4) and (2.5)

**Proof:** By Theorem 4.3.12 of Wilansky (1984) and since the four dimensional infinite matrix $F$ is triangular, we have the result.

### 3.2. Theorem.

The spaces $\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ and $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ are isomorphic to the spaces $\left[\Lambda_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ and $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$, respectively

(i.e) $\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \cong$ $\left[\Lambda_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ and

$\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \cong$ $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$

**Proof:** Let us consider the spaceof $\chi^2$, since the four dimensional infinite matrix $F$ is triangular, it has a unique inverse, which is also triangular [see Tripathy *et al.* (2009)]. Therefore the linear operator

$L_F : \left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \to$ $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$, defined by $L_F\left(x\right) = F\left(x\right)$ for all

$x \in \left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \cong$ $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$,

is bijective and is metric preserving by (2.5) in Theorem 3.1. Hence

$\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \cong$ $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

Similarly the proof for the other space can be established.

### 3.3 Theorem.

The inclusion

$\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \subset$ $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ and

$\left[\Lambda_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \subset$ $\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ holds.

**Proof:** As $F$ is a regular four dimensional infinite matrix, so the inclusion

$\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right] \subset$ $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ is obvious.

Now, let $x = \left(x_{mn}\right) \in \left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$. Then there is a constant $M > 0$ such that $\left|x_{mn}\right|^{1/m+n} \le M$ for all $m,n \in \mathbb{N}$. Thus for each $k,\ell \in \mathbb{N}$.

$\left|F_\eta\left(x\right)\right| \le$

$$\left\{\frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}\left[M\left(f_{k\ell}^{mn}\left|x_{mn}\right|^{1/m+n}, \|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\|_p\right)\right]\right\}$$

$$\le\left\{\frac{M}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}\left[M\left(f_{k\ell}^{mn}\left|x_{mn}\right|^{1/m+n}, \|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\|_p\right)\right]\right\}$$

$< \infty$ which shows that $FX \in \left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

Thus we conclude that $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ $\subset \left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

**Example:** Consider the sequence $x = \left(x_{mn}\right) =$

$$\begin{pmatrix}1&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix},\begin{pmatrix}-1&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix},\begin{pmatrix}0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix},\begin{pmatrix}0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\0&0&0&0&0\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}\cdots.$$

Then we have for every $k,\ell \in \mathbb{N}$, $F_\mu\left(x\right) = \frac{1}{f_{(k+2)(\ell+2)}-1}\sum_{m=1}^{k}\sum_{n=1}^{\ell}$

$$\left[M\left(f_{k\ell}^{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}, \|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\|_p\right)\right] \ne 0.$$

This shows that $FX \in \chi^2$ but $x$ is not in $\chi^2$. Thus the

sequence $x$ is in $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

Hence the inclusion $\left[\chi_M^{2}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ $\subseteq\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ is strictly holds.

**3.4 Theorem.**

The sequence $x=\left(x_{mn}\right)\notin$ $\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$ but in $\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$

**Proof:** Consider the sequence $x=\left(x_{mn}\right)$

$$=\begin{pmatrix}1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix},$$ for all $k,\ell\in\mathbb{N}$. Then we have for every

$$k,\ell\in\mathbb{N}, F_\mu\left(x\right)=\begin{pmatrix}1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\1&1&1&1&1\ldots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}.$$ This shows that

$FX\notin\left[\chi_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

Again, consider the sequence $x=\left(x_{mn}\right)=\frac{\left(-1\right)^{mn}\left(f_{m+2n+2+m+1n+1}\right)-1}{f_{mn}}$,

for all $k,\ell\in\mathbb{N}$. Then we have for every $k,\ell\in\mathbb{N}, F_\eta\left(x\right)=$

$$\begin{pmatrix}\left(-1\right)^{mn}&\left(-1\right)^{mn}\ldots\\\left(-1\right)^{mn}&\left(-1\right)^{mn}\ldots\\\left(-1\right)^{mn}&\left(-1\right)^{mn}\ldots\\\left(-1\right)^{mn}&\left(-1\right)^{mn}\ldots\\\vdots&\vdots\end{pmatrix}.$$ This shows that

$FX\in\left[\Lambda_M^{2F}, \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right]$.

**Competing Interests**

Author has declared that no competing interests exist.

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