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On fuzzy soft bi-ideals over semigroups

Manoj Siripitukdet* and Peerapong Suebsan

*Research Center for Academic Excellence in Mathematics,
Department of Mathematics, Faculty of Science,
Naresuan University, Mueang, Phitsanulok, 65000 Thailand.*

Abstract

The aim of this paper is to study fuzzy soft bi-ideals over semigroups and give some properties of prime, strongly prime and semiprime fuzzy soft bi-ideals over semigroups. We also show that a semigroup is both regular and intra-regular if and only if every fuzzy soft bi-ideal over the semigroup is a semiprime.

Keyword: prime fuzzy soft bi-ideal, strongly prime fuzzy soft bi-ideal, semiprime fuzzy soft bi-ideal

1. Introduction

The real life problems in environment, engineering, economics, medical science and many other fields are numerous and complicated. These problems are solved by the classical methods but the uncertainties are present in these problems. There are theories such as theory of fuzzy sets (Zadeh, 1965), theory of rough sets (Pawlak, 1982), and theory of vague sets (Gau and Bueher, 1993), which these can be considered as mathematical methods for dealing with uncertainties. Later, Molodtsov (1999) introduced the first concept of soft set theory as a new mathematical tool to solve such problems. He discussed how soft set theory is free from the parameterization inadequacy condition of rough set theory, fuzzy set theory, probability theorem etc. Maji *et al.* (2002, 2003) presented soft subset, equality of two soft sets and supported examples. They discussed an application of soft sets in decision making problems. Maji *et al.* (2001) introduced the concept of fuzzy soft set and defined the union and intersection of fuzzy soft set over common universe. Roy and Maji (2007) presented an application of fuzzy soft sets in a decision making problems. They used comparison table from resultant fuzzy soft set in a decision

making problem. Neog (2011) studied the union and intersection of fuzzy soft sets and presented some properties such as associative property, commutative property, distributive property of the union and intersection of fuzzy soft set over common universe. Rehman *et al.* (2013) studied some operations of fuzzy soft sets and discussed properties of fuzzy soft sets and their interrelation with respect to different operations such as union, intersection, restricted union and extended intersection. They also proved that certain De Morgan's laws hold in fuzzy soft set theory with respect to different operations on fuzzy soft sets. Yaqoob *et al.* (2013) introduced the notion of intuitionistic fuzzy soft groups and studied (l, q)-level set, union and intersection. Furthermore, they provided a definition of direct product of intuitionistic fuzzy soft group under soft function.

In 2011, Yang introduced the first notions of the fuzzy soft semigroups and fuzzy soft (left, right) ideals over semigroup. Moreover, he showed that fuzzy soft image and fuzzy soft inverse image of fuzzy semigroups are fuzzy soft semigroups. Naz *et al.* (2013) defined a product of two fuzzy soft semigroups and studied some properties like union, intersection of fuzzy soft bi-ideals, generalized bi-ideals, quasi-ideals, and interior ideals over semigroups under certain conditions. Akram *et al.* (2013) introduced the notion of intuitionistic fuzzy soft ideals over an ordered ternary semigroup and studied their basic properties. Now, the theory of fuzzy soft semigroups is a new mathematical tool. It is

* Corresponding author.
  Email address: manojs@nu.ac.th

necessary to develop the fuzzy soft semigroup theory for solving real life problems.

In this paper, we study fuzzy soft bi-ideals over semigroups and discuss prime, strongly prime and semiprime fuzzy soft bi-ideals over semigroups and give some properties with supported examples. We also show that a semigroup is both regular and intra-regular under sufficient and necessary conditions for every fuzzy soft bi-ideal over the semigroups.

2. Preliminaries

In this section, we shall give some of basic definitions and results that will be used.

Let S be a semigroup. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$ (Mordeson *et al.*, 2010). A bi-ideal B of a semigroup S is called *prime* (*strongly prime*) *bi-ideal* if $B_1B_2 \subseteq B$ (or $B_1B_2 \cap B_2B_1 \subseteq B$, resp.) implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S. A bi-ideal B of a semigroup S is called *semiprime bi-ideal* if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal $B_1 \in S$ (Shabir and Kanwal, 2007). Every strongly prime bi-ideal is a prime bi-ideal and every prime bi-ideal is a semiprime bi-ideal but the converse is not true. A semigroup S is called *regular* if for every element $a \in S$, there exists an element $x \in S$ such that $a = axa$, and it is called *intra-regular* if for every element aS, there exists an element $x, y \in S$ such that $a = xa^2y$ (Howie, 1976). Let T be a non-empty subset of a semigroup S, the *characteristic function* on T is the function $\chi_T : S \to [0, 1]$ defined by $\chi_T(x) = 1$ if $x \in T$ and $\chi_T(x) = 0$ if $x \notin T$ (Kuroki, 1981).

The following theorem is due to Shabir (Shabir and Kanwal, 2007).

Theorem 2.1: For a semigroup S, the following assertions are equivalent.
     (i) S is both regular and intra-regular.
     (ii) $B^2 = B$ for every bi-ideal B of S.
     (iii) $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for all bi-ideals B_1 and B_2 of S.
     (iv) Each bi-ideal of S is a semiprime.
     (v) Each proper bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contain it.
Proof : See Shabir and Kanwal (2007) (Theorem 2.10).

Moreover, Chinram and Jirojkul (2007) studied bi-ideals of semigroups extended to bi- Γ-ideals in Γ-semigroups.

The concept of fuzzy semigroups are introduced by Kuroki (Kuroki, 1981, 1991). For any $a, b \in [0, 1]$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Then $a \wedge b$ and $a \vee b$ are elements in $[0, 1]$. A function f from a semigroup S to the unit interval $[0, 1]$ is called a *fuzzy set* on S. A fuzzy set f on S is called a *fuzzy subsemigroup* on S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy set f on a semigroup S is called a *fuzzy left* [*right*] *ideal* on S if $f(xy) \geq f(y) [f(xy) \geq f(x)]$ for all $x, y \in S$. A fuzzy set f on a semigroup S is called a *fuzzy ideal* on S if it is both a fuzzy left and a fuzzy right ideal on S. We note that any fuzzy left [right] ideal on S is a fuzzy subsemigruop on S. A fuzzy subsemigroup f on a semigroup S is called a *fuzzy bi-ideal* on S if $f(xyz) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$.

If f and g are fuzzy sets on a semigroup S then $f \leq g$, $f \vee g$, $f \wedge g$ (some authors Mordeson *et al.* (2010) and Kuroki (1991) use notations $f \subseteq g, f \cup g, f \cap g$, respectively) and $f \circ g$, respectively, are defined as follows:

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \in S,$$
$$(f \vee g)(x) = f(x) \vee g(x) \text{ for all } x \in S,$$
$$(f \wedge g)(x) = f(x) \wedge g(x) \text{ for all } x \in S,$$

and

$$(f \circ g)(x) \begin{cases} \bigvee_{x=yz} [f(y) \wedge g(z)] & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise,} \end{cases}$$

where $\bigvee_{x=yz} [f(y) \wedge g(z)] = \sup\{f(y) \wedge g(z) | x = yz\}$.

The following lemmas are used in this paper.

Lemma 2.2: Let f be a fuzzy set on a semigroup S. Then the following statements holds:
     (i) f is a fuzzy subsemigroup on S if and only if $f \circ f \leq f$.
     (ii) f is a fuzzy bi-ideal on S if and only if $f \circ f \leq f$ and $f \circ \mathbb{S} \circ f \leq f$ where $\mathbb{S}$ is the fuzzy set on S which maps every element of S into 1.
Proof : See Mordeson *et al.* (2010) (Lemma 2.3.2(1), Lemma 2.4.2).

Lemma 2.3: Let T be a nonempty subset of a semigroup S, then the characteristic function on T is the fuzzy bi-ideal on S if and only if T is a bi-ideal of S.
Proof : See Mordeson *et al.* (2010) (Lemma 2.4.1).

The concept of prime fuzzy bi-ideals of semigroups are studied by Shabir *et al.* (2010).

A fuzzy bi-ideal f on a semigroup S is called a *prime fuzzy bi-ideal* on S if for any fuzzy bi-ideals g, h on S, $g \circ h \leq f$ implies $g \leq f$ or $h \leq f$. A fuzzy bi-ideal f on a semigroup S is called a *strongly prime fuzzy bi-ideal* if for any fuzzy bi-ideals g and h on S, $g \circ h \wedge h \circ g \leq f$ implies either $g \leq f$ or $h \leq f$. A fuzzy bi-ideal g on S is said to be *idempotent* if $g = g^2 = g \circ g$. A fuzzy bi-ideal f on a semigroup S is called a *semiprime fuzzy bi-ideal* if for every fuzzy bi-ideal g on S, $g \circ g = g^2 \leq f$ implies $g \leq f$.

The following theorems are results of prime fuzzy bi-ideals of semigroups, which have been proved by Shabir *et al.* (2010).

Theorem 2.4: Let I be a subset of a semigroup S. Then I is a prime bi-ideal of S if and only if the characteristic function on I is a prime fuzzy bi-ideal on S.
Proof : See Shabir *et al.* (2010) (Theorem 3.2).

Theorem 2.5: Let I be a subset of a semigroup S. Then I is a strongly prime bi-ideal of S if and only if the characteristic function on I is a strongly prime fuzzy bi-ideal on S.
Proof : See Shabir *et al.* (2010) (Theorem 3.4).

Theorem 2.6: Let I be a subset of a semigroup S. Then I is a semiprime bi-ideal of S if and only if the characteristic function on I is a semiprime fuzzy bi-ideal on S.
Proof : See Shabir *et al.* (2010) (Theorem 3.7).

Lemma 2.7: The intersection of any family of prime fuzzy bi-ideals on a semigroup S is a semiprime fuzzy bi-ideal on S.
Proof : See Shabir *et al.* (2010) (Theorem 3.8).

Theorem 2.8: For a semigroup S, the following assertions are equivalent.
    (i) S is both regular and intra-regular.
    (ii) $f \circ f = f$ for every fuzzy bi-ideal f on S.
    (iii) $g \wedge h = g \circ h \wedge h \circ g$ for all fuzzy bi-ideals g and h on S.
    (iv) Each fuzzy bi-ideal on S is a fuzzy semiprime.
    (v) Each proper fuzzy bi-ideal on S is the intersection of irreducible semiprime fuzzy bi-ideals on S which contain it.
Proof : See Shabir *et al.* (2010) (Theorem 4.2).

    Now, we give some definitions and lemmas of soft sets and fuzzy soft sets over semigroups.

Definition 2.9: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $\varnothing \neq A \subseteq E$. A pair (F, A) is called a *soft set* (Molodtsov, 1999) over U, where F is a mapping given by $F : A \rightarrow P(U)$.

Definition 2.10: Let E be a set of parameters and $\varnothing \neq A \subseteq E$. A pair (F, A) is called a *fuzzy soft set* (Maji *et al.*, 2001) over a semigroup S, where $F : A \rightarrow Fuz(S)$ and $Fuz(S)$ is the set of all fuzzy sets on S.
    Note that let (F, A) be a fuzzy soft set over a semigroup S.
    For $p \in A$, $F(p) \in Fuz(S)$. Set $F_p := F(p)$. Then $F_p \in Fuz(S)$.
    We now present the following example satisfying above definition.

Example 2.11: Let $S = \{c_1(13\text{km/L}), c_2(15\text{km/L}), c_3(16\text{km/L}), c_4(18\text{km/L}), c_5(20\text{km/L})\}$ be the set of five cars with rate of fuel consumption under consideration, where km/L denotes kilometer per liter. Let $\bullet$ be a binary operation, by choosing the car that get the maximum fuel saving, which is defined as follows:

Table 1. Multiplication table of a semigroup S

$\bullet$	c_1	c_2	c_3	c_4	c_5
c_1	c_1	c_2	c_3	c_4	c_5
c_2	c_2	c_2	c_3	c_4	c_5
c_3	c_3	c_3	c_3	c_4	c_5
c_4	c_4	c_4	c_4	c_4	c_5
c_5	c_5	c_5	c_5	c_5	c_5

Let $E = \{e_1\{\text{beautiful}\}, e_2\{\text{modern}\}, e_3\{\text{sport}\}, e_4\{\text{deluxe}\}, e_5\{\text{expensive}\}\}$ be the set of parameters and $A = \{e_1, e_3, e_5\}$. Then

$$(F, A) = \{F_{e_1} = \{c_1/0.6, c_2/0.9, c_3/0.1, c_4/0.5, c_5/0.4\},$$

$$F_{e_3} = \{c_1/0.7, c_2/0.5, c_3/0.1, c_4/0.4, c_5/0.3\},$$

$$F_{e_5} = \{c_1/0.1, c_2/0.1, c_3/0.6, c_4/0.2, c_5/0.2\}\}$$

is a fuzzy soft set over S representing the "attractiveness of the car" which Mr. X is going to buy.
    For other examples of fuzzy soft sets over semigroups, see Yang (2011).

Definition 2.12: Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) \leq (G, B)$ if (i) $A \subseteq B$ and (ii) $F_p \leq G_p$ for all $p \in A$ (Maji *et al.*, 2001).
    The following example shows that $(F, A) \leq (G, B)$.

Example 2.13: Let $S = \{c_1(13\text{km/L}), c_2(15\text{km/L}), c_3(16\text{km/L}), c_4(18\text{km/L}), c_5(20\text{km/L})\}$ be a semigroup as Example 2.11. Let $E = \{e_1\{\text{beautiful}\}, e_2\{\text{modern}\}, e_3\{\text{sport}\}, e_4\{\text{deluxe}\}, e_5\{\text{expensive}\}\}$ be the set of parameters and $A = \{e_1, e_4\}$ and $B = \{e_1, e_4, e_5\}$. Then

$$(F, A) = \{F_{e_1} = \{c_1/0.6, c_2/0.9, c_3/0.1, c_4/0.5, c_5/0.4\},$$

$$F_{e_4} = \{c_1/0.7, c_2/0.5, c_3/0.1, c_4/0.4, c_5/0.3\}\}$$

is a fuzzy soft set over S representing the "attractiveness of the car" which Mr.X is going to buy and

$$(G, B) = \{G_{e_1} = \{c_1/0.7, c_2/1.0, c_3/0.4, c_4/0.6, c_5/0.5\},$$

$$G_{e_4} = \{c_1/0.8, c_2/0.6, c_3/0.1, c_4/0.6, c_5/0.3\},$$

$$G_{e_5} = \{c_1/0.3, c_2/0.4, c_3/0.5, c_4/0.6, c_5/0.7\}\}$$

is a fuzzy soft set over S representing the "attractiveness of the car" which Mr.Y is going to buy. We then have $A \subseteq B$, and $F_p \leq G_p$ for all $p \in A$. Hence $(F, A) \leq (G, B)$.

Definition 2.14: Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S with $A \cap B \neq \varnothing$. Define $(F, A) \tilde{\Lambda} (G, B) := (F \wedge G, A \cap B)$, where for each $p \in A \cap B$

$$(F \wedge G)_p = F_p \wedge G_p.$$

Then $(F, A) \tilde{\Lambda} (G, B)$ is a fuzzy soft set over S. (Maji *et al.*, 2001).

Remark: Some authors called $(F, A) \tilde{\Lambda} (G, B)$, the intersection of (F, A) and (G, B).

Definition 2.15: Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) \tilde{V} (G, B) := (F \vee G, A \cup B)$, where for each $p \in A \cup B$

$$(F \vee G)_p = \begin{cases} F_p & \text{if} \quad p \in A - B \\ G_p & \text{if} \quad p \in B - A \\ F_p \vee G_p & \text{if} \quad p \in A \cap B. \end{cases}$$

Then $(F, A) \tilde{V} (G, B)$ is a fuzzy soft set over S.

Remark: Some authors called $(F, A) \tilde{V} (G, B)$, the union of (F, A) and (G, B) (Maji *et al.*, 2001).

Definition 2.16: The *product* $\odot$ of two fuzzy soft sets (F, A) and (G, B) over a semigroup S is defined by $(F, A) \odot (G, B) := (F \circ G, A \cup B)$, where for all $p \in A \cup B$

$$(F \circ G)_p = \begin{cases} F_p & \text{if} \quad p \in A - B \\ G_p & \text{if} \quad p \in B - A \\ F_p \circ G_p & \text{if} \quad p \in A \cap B \end{cases}$$

(Naz *et al.*, 2013).
We note here that the product $(F, A) \odot (G, B)$ is a fuzzy soft set over S.

If (F, A), (G, B) and (H, C) be fuzzy soft sets over a semigroup S then

$$((F, A) \odot (G, B)) \odot (H, C) = (F, A) \odot ((G, B) \odot (H, C))$$

(See Naz *et al.* (2013)(Proposition 2)).

Definition 2.17: A fuzzy soft set (F, A) over a semigroup S is called a *fuzzy soft subsemigroup* (Yang, 2011) if F_p is a fuzzy subsemigroup on S for each $p \in A$.

Definition 2.18: A fuzzy soft set (F, A) over a semigroup S is called a *fuzzy soft left [right] ideal* (Yang, 2011) if F_p is a fuzzy left [right] ideal on S for each $p \in A$.

Definition 2.19: A fuzzy soft set (F, A) over a semigroup S is called a *fuzzy soft ideal* (Yang, 2011) if (F, A) is both a fuzzy soft left and a fuzzy soft right ideal on S.

Definition 2.20: A fuzzy soft set (F, A) over a semigroup S is said to be a *fuzzy soft bi-ideal* (Naz *et al.*, 2013) over S if F_p is a fuzzy bi-ideal on S for each $p \in A$.

The following example is an example of a fuzzy soft bi-ideal over a semigroup, and we show that a fuzzy soft set over a semigroup is not a fuzzy soft left, right ideals and bi-ideal over a semigroup.

Example 2.21: Let $S = \{c_1(19\text{km/L}), c_2(17\text{km/L}), c_3(15\text{km/L})\}$ be the set of five cars with rate of fuel consumption under consideration, where km/L denoted kilometer per liter. Let $\bullet$ be a binary operation, by choosing the car that get the maximum fuel saving, defined as follows :

Table 2. Multiplication table of a semigroup S

$\bullet$	c_1	c_2	c_3
c_1	c_1	c_1	c_1
c_2	c_1	c_2	c_2
c_3	c_1	c_2	c_3

Let $E = \{e_1\{\text{beautiful}\}, e_2\{\text{modern}\}, e_3\{\text{sport}\}, e_4\{\text{deluxe}\}, e_5\{\text{expensive}\}\}$ be the set of parameters, $A = \{e_1, e_3\}$ and $B = \{e_1, e_2\}$.
We consider a fuzzy soft set over S,

$$(F, A) = \{F_{e_1} = \{c_1/0.6, c_2/0.4, c_3/0.1\}, F_{e_3} = \{c_1/0.9, c_2/0.6, c_3/0.2\}\}.$$

It is easy to verify that (F, A) is a fuzzy soft bi-ideal over S. Next, consider a fuzzy soft set over S

$$(G, B) = \{G_{e_1} = \{c_1/0.3, c_2/0.1, c_3/0.8\}, G_{e_2} = \{c_1/0.4, c_2/0.2, c_3/0.2\}\}.$$

We then have (G, B) is a fuzzy soft subsemigroup over S. Moreover,

$$G_{e_1}(c_1 \bullet c_3) = G_{e_1}(c_1) = 0.3 < 0.8 = G_{e_1}(c_3),$$

$$G_{e_1}(c_3 \bullet c_1) = G_{e_1}(c_1) = 0.3 < 0.8 = G_{e_1}(c_3)$$

and $\quad G_{e_1}(c_3 \bullet c_1 \bullet c_3) = G_{e_1}(c_1) = 0.3 < 0.8 = \min\{G_{e_1}(c_3),$

$$G_{e_1}(c_3)\}.$$

Hence (G, B) is not fuzzy soft left and right ideals over S, and it is not fuzzy soft bi-ideal over S.

Definition 2.22: Let $\{(F_i, A_i) | i \in I\}$ be family of fuzzy soft sets over a semigroup S and $\underset{i \in I}{\cap} A_i \neq \varnothing$. Define $\tilde{\Lambda}_{i \in I} (F_i, A_i) = (\underset{i \in I}{\wedge} F_i, \underset{i \in I}{\cap} A_i)$ (Naz *et al.*, 2013).

Lemma 2.23: If $\{(F_i, A_i) | i \in I\}$ is a family of fuzzy soft bi-ideals over a semigroup S, then $\tilde{\Lambda}_{i \in I} (F_i, A_i)$ is a fuzzy soft bi-ideal over S.

Proof: See Naz *et al.* (2013) (Proposition 1).

3. Results

In this section, we discuss fuzzy soft bi-ideals over semigroups and obtain properties of prime, strongly prime and semiprime fuzzy soft bi-ideals over semigroups. We present a semigroup which is both regular and intra-regular if and only if every fuzzy soft bi-ideal over the semigroup is a semiprime.

Proposition 3.1: Let (F, A), (G, B) and (H, C) be fuzzy soft sets over a semigroup S such that $(F, A) \leq (G, B)$. If $B \cap C \neq \varnothing$ or $C \subseteq A$ then $(F \circ H, A \cup C) \leq (G \circ H, B \cup C)$ and $(H \circ F, C \cup A)(H \circ G, C \cup B)$.

Proof: We note first that $A \cup C \subseteq B \cup C$. Suppose that $B \cap C \neq \varnothing$. Let $p \in A \cup C$.

Case 1. $p \in A - C$. We then have $p \in B \cup C$. But $p \notin C$, so $p \in B$.

Thus $(F \circ H)_p = F_p \leq G_p = (G \circ H)_p$. Hence $(F \circ H)_p \leq (G \circ H)_p$.

Case 2. $p \in C - A$. Thus $(F \circ H)_p = H_p$ and $(G \circ H)_p = H_p$. Hence $(F \circ H)_p \leq (G \circ H)_p$.

Assume that $C \subseteq A$. Case 1. $p \in A \cap C$.

Then $(F \circ H)_p = F_p \circ H_p \leq G_p \circ H_p = (G \circ H)_p$. Hence $(F \circ H)_p \leq (G \circ H)_p$.

Case 3. $p \in A - C$. We then have $p \in B \cup C$. But $p \notin C$, so $p \in B$.

Thus $(F \circ H)_p = F_p \leq G_p = (G \circ H)_p$. Hence $(F \circ H)_p \leq (G \circ H)_p$.

Therefore $(F \circ H, A \cup C) \leq (G \circ H, B \cup C)$.

Similarly, we can show that $(H \circ F, C \cup A) \leq (H \circ G, C \cup B)$.

The following example shows that the converse of Proposition 3.1 is not true.

Example 3.2: Let $S = \{c_1(19\text{km/L}), c_2(17\text{km/L}), c_3(15\text{km/L})\}$ be a semigroup as Example 2.21. Let $E = \{e_1\{\text{beautiful}\}, e_2\{\text{modern}\}, e_3\{\text{sport}\}, e_4\{\text{deluxe}\}, e_5\{\text{expensive}\}\}$ be the set of parameters and $A = \{e_2, e_3\}$, $B = \{e_1, e_2, e_3\}$ and $C = \{e_1\}$. Thus $B \cap C \neq \varnothing$ and $C \subseteq A$. We consider fuzzy soft sets over S,

$(F, A) = \{ F_{e_2} = \{c_1/0.3, c_2/0.5, c_3/0.2\}, F_{e_3} = \{c_1/0.9, c_2/0.3, c_3/0.7\}\}$,

$(G, B) = \{ G_{e_1} = \{c_1/0.4, c_2/0.3, c_3/0.2\}, G_{e_2} = \{c_1/0.4, c_2/0.6, c_3/0.3\}$,

$\quad\quad G_{e_3} = \{c_1/0.9, c_2/0.5, c_3/0.8\}\}$ and

$(H, C) = H_{e_1} = \{c_1/0.5, c_2/0.6, c_3/0.7\}\}$.

Consider $(F, A) \circledcirc (H, C) = (F \circ H, A \cup C)$. Then

$(F \circ H, A \cup C) = \{ (F \circ H)_{e_1} = \{c_1/0.5, c_2/0.6, c_3/0.7\} = (H \circ F)_{e_1}$,

$\quad\quad (F \circ H)_{e_2} = \{c_1/0.3, c_2/0.5, c_3/0.2\} = (H \circ F)_{e_2}$,

$\quad\quad (F \circ H)_{e_3} = \{c_1/0.9, c_2/0.3, c_3/0.7\} = (H \circ F)_{e_3}\}$.

Next, consider $(G, B) \circledcirc (H, C) = (G \circ H, B \cup C)$. Then

$(G \circ H, B \cup C) = \{ (G \circ H)_{e_1} = \{c_1/0.4, c_2/0.3, c_3/0.2\} = (H \circ G)_{e_1}$,

$(G \circ H)_{e_2} = \{c_1/0.4, c_2/0.6, c_3/0.3\} = (H \circ G)_{e_2}$,

$(G \circ H)_{e_3} = \{c_1/0.9, c_2/0.5, c_3/0.8\} = (H \circ G)_{e_3}\}$.

Since $(F \circ H)_{e_1} \nleq (G \circ H)_{e_1}$ and $(H \circ F)_{e_1} \nleq (H \circ G)_{e_1}$, it follows that

$(F \circ H, A \cup C) \nleq (G \circ H, B \cup C)$ and $(H \circ F, C \cup A) \nleq (H \circ G, C \cup B)$.

Definition 3.3: Let T be a nonempty subset of a semigroup S and let A be a set of parameters. A *fuzzy soft characteristic function* (α^T, A) over S is defined by $(\alpha^T)_p = \chi_T$ for all $p \in A$.

The following lemma is used in the last theorem.

Lemma 3.4: Let (F, A) and (α^S, A) be two fuzzy soft sets over a semigroup S. Then the following holds.

(i) (F, A) is a fuzzy soft subsemigroup over S if and only if $(F, A) \circledcirc (F, A) \leq (F, A)$ and $(F \circ F)_p \leq F_p$ for all $p \in A$.

(ii) (F, A) is a fuzzy soft bi-ideal over S if and only if $(F, A) \circledcirc (F, A) \leq (F, A)$ and $(F \circ F)_p \leq F_p$, and $(F \circ \alpha^S \circ F)_p \leq F_p$ for all $p \in A$.

Proof: (i) It follows from Lemma 2.2(i).

(ii) It follows from Lemma 2.2(ii).

The following lemma shows that a nonempty subset of a semigroup is a bi-ideal of a semigroup if and only if the fuzzy soft characteristic function over the semigroup is the fuzzy soft bi-ideal over the semigroup.

Lemma 3.5: Let T be a nonempty subset of a semigroup S. Then T is a bi-ideal of S if and only if (α^T, A) is the fuzzy soft bi-ideal over S.

Proof: Suppose that T is a bi-ideal of S. By Lemma 2.3, χ_T is a fuzzy bi-ideal on S.

Let $p \in A$. Since $(\alpha^T)_p = \chi_T$, we get $(\alpha^T)_p$ is a fuzzy bi-ideal on S.

Therefore (α^T, A) is the fuzzy soft bi-ideal over S.

On the other hand, assume that (α^T, A) is the fuzzy soft bi-ideal over S.

Let $p \in A$. Then $(\alpha^T)_p$ is a fuzzy bi-ideal on S. Since $(\alpha^T)_p = \chi_T$, it implies that χ_T is a fuzzy bi-ideal on S. By Lemma 2.3, T is a bi-ideal of S.

Definition 3.6: A fuzzy soft set (F, A) over a semigroup S is said to be a *prime fuzzy soft bi-ideal* on A over S if F_p is a prime fuzzy bi-ideal on S for all $p \in A$.

Remark: If (F, A) is a prime fuzzy soft bi-ideal over a semigroup S, we mean (F, A) is a prime fuzzy soft bi-ideal on A over S.

The following theorem shows that a subset of a semigroup is a prime bi-ideal of a semigroup if and only if the fuzzy soft characteristic function over the semigroup is the prime fuzzy soft bi-ideal over the semigroup.

Theorem 3.7: Let T be a subset of a semigroup S and let (α^T, A) be a fuzzy soft characteristic function over S. Then T is a prime bi-ideal of S if and only if (α^T, A) is a prime fuzzy soft bi-ideal over S.
Proof: It follows from Theorem 2.4.

Theorem 3.8: Let (F, A), (G, B) and (H, C) be fuzzy soft bi-ideals over a semigroup S. Suppose that (F, A) is a prime fuzzy soft bi-ideal with $(G \circ H, B \cup C) \leq (F, A)$. Then the following statements hold.

(i) If $B \cap C = \varnothing$ then $(G, B) \leq (F, A)$ and $(H, C) \leq (F, A)$.

(ii) If $|B \cap C| = 1$ then $(G, B) \leq (F, A)$ or $(H, C) \leq (F, A)$.
Proof: Suppose that $(G \circ H, B \cup C) \leq (F, A)$. Then $B \cup C \subseteq A$ and $(G \circ H)_p \leq F_p$ for all $p \in B \cup C$. Since $B \cup C \subseteq A$, we then have $B \subseteq A$ and $C \subseteq A$.

(i) Assume that $B \cap C = \varnothing$. Let $p \in B$. Then $G_p = (G \circ H)_p \leq F_p$. Thus $(G, B) \leq (F, A)$. Similarly, let $q \in C$. Then $H_p = (G \circ H)_p \leq F_p$. Hence $(H, C) \leq (F, A)$.

(ii) Assume that $|B \cap C| = 1$. Suppose that $(G, B) \nleq (F, A)$. Then there exists $q \in B$ such that $G_q \nleq F_q$. If $q \notin C$ then $G_q = (G \circ H)_q \leq F_q$. Hence $q \in B \cap C$. Let $r \in C$.

We must show that $H_r \leq F_r$.
Case 1. $r \in C - B$. Then $H_r = (G \circ H)_r \leq F_r$.
Case 2. $r \in B \cap C$. Then $r = q$. Thus $G_r \circ H_r = (G \circ H)_r \leq F_r$. Since (F, A) is a prime fuzzy soft bi-ideal over a semigroup S, we have $G_r \leq F_r$ or $H_r \leq F_r$. But $G_r \nleq F_r$, so $H_r \leq F_r$. Therefore $(H, C) \leq (F, A)$.

We now present the following example satisfying Theorem 3.8.

Example 3.9: Let $S = \{c_1(3.90), c_2(3.75), c_3(3.50)\}$ be the set of three candidates with grade point average (GPA) under consideration. Let $\bullet$ be a binary operation, by choosing the candidate who has the maximum GPA, defined as follows:

Table 3. Multiplication table of a semigroup S

$\bullet$	c_1	c_2	c_3
c_1	c_1	c_1	c_1
c_2	c_1	c_2	c_2
c_3	c_1	c_2	c_3

The committee consider a set of parameters.
Let $E = \{e_1\{$good speaking$\}$, $e_2\{$good computer knowledge$\}$, $e_3\{$good English skill$\}$, $e_4\{$experience$\}\}$,

$A = \{e_1, e_2, e_4\}$, $B = \{e_2, e_4\}$, $C = \{e_1, e_2\}$
and let $(F, A) = \{F_{e_1} = \{c_1/1.0, c_2/0.9, c_3/0.8\}$,

$F_{e_2} = \{c_1/1.0, c_2/0.9, c_3/0.8\}$,

$F_{e_4} = \{c_1/0.9, c_2/0.8, c_3/0.7\}\}$

be a fuzzy soft bi-ideal over S representing the "attractiveness of the candidate" which committee Mr. X is going to choose. Let

$(G, B) = \{G_{e_2} = \{c_1/0.8, c_2/0.7, c_3/0.7\}$,

$G_{e_4} = \{c_1/0.9, c_2/0.8, c_3/0.6\}\}$

be a fuzzy soft bi-ideal over S representing the "attractiveness of the candidate" which committee Mr. Y is going to choose. Let

$(H, C) = \{H_{e_1} = \{c_1/0.9, c_2/0.7, c_3/0.7\}$,

$H_{e_2} = \{c_1/1.0, c_2/1.0, c_3/0.9\}\}$

be a fuzzy soft bi-ideal over S representing the "attractiveness of the candidate" which committee Mr. Z is going to choose. Consider

$(G \circ H, B \cup C) = \{(G \circ H)_{e_1} = \{c_1/0.9, c_2/0.7, c_3/0.7\}$,
$\qquad\qquad (G \circ H)_{e_2} = \{c_1/0.8, c_2/0.7, c_3/0.7\}$,
$\qquad\qquad (G \circ H)_{e_4} = \{c_1/0.9, c_2/0.8, c_3/0.6\}\}$.

Thus $(G \circ H, B \cup C) \leq (F, A)$. Since $|B \cap C| = 1$ and $(H, C) \nleq (F, A)$, it follows that $(G, B) \leq (F, A)$.

Definition 3.10: A fuzzy soft set (F, A) over a semigroup S is said to be a *strongly prime fuzzy soft bi-ideal* on A over S if F_p is a strongly prime fuzzy bi-ideal on S for all $p \in A$.

Remark: If (F, A) is a strongly prime fuzzy soft bi-ideal over a semigroup S, we mean (F, A) is a strongly prime fuzzy soft bi-ideal on A over S.

The next theorem shows that a nonempty subset of a semigroup is a strongly prime bi-ideal of the semigroup if and only if the fuzzy soft characteristic function over the semigroup is the strongly prime fuzzy soft bi-ideal over the semigroup.

Theorem 3.11: Let T be a subset of a semigroup S and let (α^T, A) be a fuzzy soft characteristic function over S. Then T is a strongly prime bi-ideal of S if and only if (α^T, A) is a strongly prime fuzzy soft bi-ideal over S.
Proof: It follows from Theorem 2.5.

Theorem 3.12: Let (F, A), (G, B) and (H, C) be fuzzy soft bi-ideals over a semigroup S. Suppose that (F, A) is a strongly prime fuzzy soft bi-ideal with $|B \cap C| = 1$ and

$(G \circ H, B \cup C) \ \tilde{\Lambda} \ (H \circ G, C \cup B) \leq (F, A)$.

Then $(G, B) \leq (F, A)$ or $(H, C) \leq (F, A)$.

Proof: By the assumption, we have $B \cup C \subseteq A$ and
$$(G \circ H)_p \wedge (H \circ G)_p \leq F_p \text{ for all } p \in B \cup C.$$
Since $B \cup C \subseteq A$, we then have $B \subseteq A$ and $C \subseteq A$. Suppose that $(G, B) \nleq (F, A)$. Then there exists $q \in B$ such that $G_q \nleq F_q$.
If $q \notin C$ then $G_q = (G \circ H)_q \wedge (H \circ G)_q \leq F_q$.
Hence $q \in B \cap C$.
Let $r \in C$. We must show that $H_r \leq F_r$.
Case 1. $r \in C - B$. Then $H_r = (G \circ H)_r \wedge (H \circ G)_r \leq F_r$.
Case 2. $r \in B \cap C$. Then $r = q$. Thus $G_r \circ H_r = (G \circ H)_r \wedge (H \circ G)_r \leq F_r$.
Since (F, A) is a strongly prime fuzzy soft bi-ideal over a semigroup S, we have $G_r \leq F_r$ or $H_r \leq F_r$. But $G_r \nleq F_r$, so $H_r \leq F_r$. Therefore $(H, C) \leq (F, A)$.

Definition 3.13: A fuzzy soft subsemigroup (G, B) over a semigroup S is said to be *fuzzy soft idempotent* if G_p is idempotent for all $p \in B$.
Note that (G, B) is a fuzzy soft idempotent if and only if $(G, B) = (G, B)^2 = (G \circ G, B)$.

Definition 3.14: A fuzzy soft set (F, A) over a semigroup S is said to be a *semiprime fuzzy soft bi-ideal* on A over S if F_p is a semiprime fuzzy bi-ideal on S for all $p \in A$.

Remark: If (F, A) is a semiprime fuzzy soft bi-ideal over a semigroup S, we mean (F, A) is a semiprime fuzzy soft bi-ideal on A over S.

The following theorem shows that a subset of a semigroup is a semiprime bi-ideal of a semigroup if and only if the fuzzy soft characteristic function over the semigroup is the semiprime fuzzy soft bi-ideal over the semigroup.

Theorem 3.15: Let T be a subset of a semigroup S and let (α^T, A) be a fuzzy soft characteristic function over S. Then T is a semiprime bi-ideal of S if and only if (α^T, A) is a semiprime fuzzy soft bi-ideal over S.
Proof: It follows from Theorem 2.6.

Theorem 3.16: Let (F, A) be fuzzy soft bi-ideal over a semigroup S. Then (F, A) is a semiprime if and only if for every fuzzy soft bi-ideal (G, B) over S,
$$(G \circ G, B) = (G, B)^2 \leq (F, A) \text{ implies } (G, B) \leq (F, A).$$
Proof: Let (G, B) be a fuzzy soft bi-ideal over S. Let $p \in B$. Suppose that $(G \circ G, B) = (G, B)^2 \leq (F, A)$. We then have $B \subseteq A$ and $(G \circ G)_p \leq F_p$.
Thus $G_p = (G \circ G)_p \leq F_p$. Hence $(G, B) \leq (F, A)$.
Conversely, let $p \in A$ and $(G_p)^2 \leq F_p$. By the assumption, $(G \circ G, A) \leq (F, A)$ then $(G, B) \leq (F, A)$. Thus $G_p \leq F_p$. Hence (F, A) is a semiprime fuzzy soft bi-ideal over a semigroup S.

Theorem 3.17: The intersection of any family of prime fuzzy soft bi-ideals over a semigroup S is a semiprime fuzzy soft bi-ideal over S.
Proof: Let $\{(F_i, A_i) \mid i \in I\}$ be a nonempty family of prime fuzzy soft bi-ideals over S. By Lemma 2.23, $\tilde{\Lambda}_{i \in I} (F_i, A_i)$ is a

fuzzy soft bi-ideal over S. Let $p \in \cap_{i \in I} A_i \neq \varnothing$. Then we have $\{(F_i)_p \mid i \in I\}$ be prime fuzzy bi-ideals on S. By Lemma 2.7, $\wedge_{i \in I} (F_i)_p$ is a semiprime fuzzy bi-ideal on S. Hence $\tilde{\Lambda}_{i \in I} (F_i, A_i)$ is a semiprime fuzzy soft bi-ideal over S.

The next theorem shows that a semigroup is both regular and intra-regular if and only if every fuzzy soft bi-ideal over the semigroup is a semiprime.

Theorem 3.18: Let S be a semigroup. Then the following statements are equivalent.
  (i) S is both regular and intra-regular.
  (ii) $(F, A) \tilde{\Lambda} (G, B) \leq (F, A) \odot (G, B) \tilde{\Lambda} (G, B) \odot (F, A)$ for every fuzzy soft bi-ideals (F, A) and (G, B) over S with $A \cap B \neq \varnothing$.
  (iii) For any fuzzy soft bi-ideal (F, A) over S, $(F, A) \leq (F, A) \odot (F, A)$.
  (iv) Every fuzzy soft bi-ideal over S is a fuzzy soft idempotent.
  (v)   Each fuzzy soft bi-ideal over S is a semiprime.
  (vi) $(F, A) \tilde{\Lambda} (G, B) \leq (F, A) \odot (G, B)$ for every fuzzy soft bi-ideals (F, A) and (G, B) over S with $A \cap B \neq \varnothing$.
Proof: $(i) \Rightarrow (ii)$ Suppose that (i) holds. Let (F, A) and (G, B) be fuzzy soft bi-ideals over S with $A \cap B \neq \varnothing$. Let $p \in A \cap B$. Then we have $A \cap B \subseteq A \cup B$. By Theorem 3.3.14(12) (Mordeson *et al.*, 2010),
$$(F \wedge G)_p = F_p \wedge G_p \leq (F_p \circ G_p) \wedge (G_p \circ F_p) = (F \circ G)_p \wedge (G \circ F)_p.$$
Thus $(F, A) \tilde{\Lambda} (G, B) \leq (F, A) \odot (G, B) \tilde{\Lambda} (G, B) \odot (F, A)$.

$(ii) \Rightarrow (iii)$ Suppose that (ii) holds. Let (F, A) be a fuzzy soft bi-ideal over S.
Then $(F, A) = (F, A) \tilde{\Lambda} (F, A)$. By the assumption,
$$(F, A) = [(F, A) \odot (F, A)] \tilde{\Lambda} [(F, A) \odot (F, A)]$$
$$= [(F \circ F, A)] \tilde{\Lambda} [(F \circ F, A)] = (F \circ F, A) = (F, A) \odot (F, A)$$
Hence $(F, A) \leq (F, A) \odot (F, A)$.

$(iii) \Rightarrow (iv)$ Assume that (iii) holds. By Lemma 3.4(ii), $(F, A) \odot (F, A) \leq (F, A)$. By the assumption, $(F, A) \leq (F, A) \odot (F, A)$. Hence (F, A) is a fuzzy soft idempotent.

$(iv) \Rightarrow (v)$ Assume that (iv) holds. Let (F, A) and (G, B) be fuzzy soft bi-ideals over S with $(G \circ G, B) = (G, B)^2 \leq (F, A)$. We then have $B \subseteq A$. By the assumption,
$$(G \circ G, B) = (G, B)^2 = (G, B). \text{ Thus } (G, B) \leq (F, A).$$
By Theorem 3.16, (F, A) is a semiprime fuzzy soft bi-ideal over S.

$(v) \Rightarrow (i)$ Suppose that (v) holds. Let T be a bi-ideal of S.
By Lemma 3.5, (α^T, A) is a fuzzy soft bi-ideal over S. By the assumption, (α^T, A) is a semiprime fuzzy soft bi-ideal over S. By Theorem 3.15, T is a semiprime bi-ideal of S. By Theorem 2.1, S is both regular and intra-regular.

$(i) \Rightarrow (vi)$ Assume that (i) holds. Let (F, A) and (G, B) be fuzzy soft bi-ideals over S with $A \cap B \neq \varnothing$. Let $p \in A \cap B$.

Then $A \cap B \subseteq A \cup B$. By Theorem 3.3.13(8) (Mordeson *et al.*, 2010), we have

$$(F \wedge G)_p = F_p \wedge G_p \leq F_p \circ G_p = (F \circ G)_p.$$

Thus $(F, A) \tilde{\Lambda} (G, B) \leq (F, A) \odot (G, B)$.

$(vi) \Rightarrow (i)$ Suppose that (vi) holds. Let f and g be fuzzy bi-ideals on S. Let A be a nonempty set. Define $F_p = f$ and $G_p = g$ for all $p \in A$. Thus (F, A) and (G, B) are fuzzy soft bi-ideals over S. By the assumption, we have

$$(F, A) \tilde{\Lambda} (G, B) \leq (F, A) \odot (G, B).$$

Fix $p \in A$, we have

$$F_p \wedge G_p = (F \wedge G)_p \leq (F \circ G)_p = F_p \circ G_p.$$

Hence $f \wedge g \leq f \circ g$. By Theorem 3.3.13 (Mordeson *et al.*, 2010), S is both regular and intra-regular.

4. Conclusions

In this paper, we gave some properties of fuzzy soft bi-ideals over semigroups and introduced prime, strongly prime and semiprime fuzzy soft bi-ideals over semigroups and obtained properties and supported examples. We also proved that a semigroup which is both regular and intra-regular if and only if every fuzzy soft bi-ideal over the semigroup is a semiprime.

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