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Original Article

A study on an M/G/1 retrial G-queue with unreliable server under variant working vacations policy and vacation interruption

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Abstract

We consider a single server retrial queueing system with variant working vacations and vacation interruption where the regular busy server is subjected to breakdown due to the arrival of negative customers. As soon as the orbit becomes empty at the time of regular service completion for a positive customer, the server takes at most J number working vacations until at least one customer is received in the orbit when the server returns from a working vacation. During the working vacation period, the server serves at a lower speed service rate ($\mu_v<\mu_b$). Using the method of supplementary variable technique, we determined the steady state probability generating function for the system and its orbit. We also obtained some analytical expressions for various performance measures such as system state probabilities, the mean orbit size, the mean system size, and reliability measures of this model. Finally, some numerical examples are presented.

Keywords: retrial queue, G-queue, variant working vacations, vacation interruption

1. Introduction

In queueing theory, vacation queues and retrial queues have been intensive research topics for a long time. We can find general models in vacation queues from Ke *et al.* (2010) and in retrial queues from Artalejo and Gomez-Corral (2008). In a retrial queueing system, retrial queues with repeated attempts are characterized by the fact that an arriving customer finds the server busy upon arrival and is requested to leave the service area and join a retrial queue called orbit. After some time the customer in the orbit can repeat their request for service. An arbitrary customer in the orbit who repeats the request for service is independent of the rest of the customers in the orbit. Such queues play a special role in computer and telecommunication systems.

Queues with negative customers (called *G*-queues) have huge interests of concern due to their extensive applications in computers, communication networks, neural networks and manufacturing systems (Chao *et al.,* 1999; Harrison, 2014). The name *G*-queue was adopted for a queue with negative customers in the acknowledgment of Gelenbe, who first introduced this type of queue in Gelenbe (1989, 1991). Harrison (2004) has studied the idea of compositional reversed Markov processes with applications to *G*-networks. The positive customer arrives into the system and gets service as ordinary queueing customers, but the negative customers enter into the system only at the service time of positive customers. This type of negative customer removes the positive customers who is in service from the system and causes the server breakdown and the service channel will fail for a short interval of time. When the server fails, it will be sent for repair immediately. After completion of repair, the server is again treated as good as new. Do (2011) has presented a survey on queueing systems with *G*-networks, negative customers and applications. Choudhury and Ke (2012) and Rajadurai *et al.* (2014, 2015a) have discussed the retrial queue with the concept of breakdown and repair. Recently, Krishnakumar *et al.* (2013), Do *et al.* (2014), Gao and Wang (2014), Peng *et al.* (2014), and Rajadurai *et al.* (2015b, 2016a, 2016b) have discussed different types of queueing models operating with the simultaneous presence of negative arrivals.

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In a vacation queueing system, the server completely stops the service and is unavailable for primary customers during a short period of time. This period of time is referred as a vacation. But in a working vacation period (WV), the server gives service to customers at a lower service rate. This queueing system has major applications in providing network service, web service, file transfer service, and mail service.

In 2002, an M/M/1 queueing system with working vacations was first introduced by Servi and Finn (2002). Later, Wu and Takagi (2006) extended the M/M/1/WV queue to an M/G/1/WV queue. Do (2010) have studied the concept of M/M/1 retrial queue with working vacations. Arivudainambi *et al*. (2014) introduced M/G/1 retrial queue with a single working vacation. Furthermore, during the working vacation period if there are customers at a lower service completion instant, the server can stop the vacation and come back to the normal busy state. This policy is called vacation interruption. Recently, Gao *et al*. (2014), Gao and Liu (2013), Rajadurai *et al*. (2016b, 2016c), Zhang and Hou (2012), Zhang and Liu (2015), analyzed a single server retrial queue with working vacations and vacation interruption.

1.1 The organization of the paper

In this paper, we consider a generalization of the well-known model discussed by Arivudainambi *et al.* (2014) and Gao *et al*. (2014) with concepts of an M/G/1 retrial *G*-queue with unreliable server under variant working vacations policy and vacation interruption. To the authors best of knowledge, there are reports available on the concept of a retrial queueing system with a working vacation using the method of matrix geometry analysis, but no work was published in the queueing literature with the combination of a single server retrial queueing system with general retrial times, negative customers, variant working vacations, vacation interruption, server breakdowns, and repair using the supplementary variable technique. The mathematical results and theory of queues of this model provide to serve a specific and convincing application in the computer processing system. This proposed model has potential practical real life application in production to order system to enhance the performance of the production facility and to stop the production facility from becoming overloaded, in computer processing system and telephone consultation of medical service systems. Our model is helpful to managers who design a system with economic management.

The rest of this paper is given as follows. The detailed mathematical model description and practical applications of this model are given is section 2. In section 3, the steady state joint distribution of the server state and the number of customers in the orbit/system are obtained. Some system performance measures and reliability measures are discussed in section 4. In section 5, important special cases are derived. In section 6, the effects of various parameters on the system performance are analyzed numerically. Conclusion and summary of the work are presented in section 7.

2. Description of the Model

This section, we consider a single server retrial queueing system with variant working vacations and vacation interruption, where the regular busy server is subjected to breakdown due to the arrival of negative customers.

- **The arrival process**: Customers arrive at the system according to a Poisson process with rate λ.
- **The retrial process:** If an arriving positive customer finds the server free, the customer begins his service immediately. Otherwise the server is busy, on working vacation or breakdown; the arrivals join the pool of blocked customers called an orbit in accordance with FCFS discipline. That is, only one customer at the head of the orbit queue is allowed access to the server. Inter-retrial times follow a general random variable R with an arbitrary distribution $R(t)$ having corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.
- **The variant working vacations policy:** The server begins a working vacation each time the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ. If any customer arrives in a vacation period, the server continues to work at a lower speed service rate ($\mu_v < \mu_b$). The working vacation period is an operational period at a lower speed. According to the vacation interruption rule, if any customer is in the system at the service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy state immediately. Otherwise, if no customers are in the system at the end of the vacation or at regular service completion instant, the server takes at most J number working vacations until at least one customer is received in the orbit and then the server returns from a working vacation. After completion of J^{th} working vacation, if there is no customer in the orbit, the server remains idle to serve a new customer. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. During the working vacation period, the service time follows a general random variable S_v with distribution function (d.f) $S_v(t)$ having LST $S_v^*(\vartheta)$ and the first moment is

$$S_v^{*'}(\theta) = \int_0^\infty x e^{-\theta x} dS_v(x), \text{ for } (i = 1, 2, \ldots J).$$

- **The regular service process:** Whenever a new positive customer or retry positive customer arrives at the server idle state, the server immediately starts normal service for the arrivals. The service time follows a general distribution and it is denoted by the random variable S_b with distribution function $S_b(t)$ having LST $S_b^*(\vartheta)$ and the first moment is given by $S_b^{*'}(\delta) = \int_0^\infty x e^{-\delta x} dS_b(x).$
- **The removal rule and the repair process:** The negative customers arrive from outside the system according to a Poisson arrival rate δ. These negative customers arrive only at the regular service time of the positive customers. Negative customers can not accumulate in a queue and do not receive service, will remove the positive customers being in service from the system. These types of negative customers cause server breakdown and the service channel will fail for a short interval of time. When the server fails, it is sent for

repair immediately. After completion of repair, the server is again treated as good as new. The repair time (denoted by G) of the server is assumed to be arbitrarily distributed with distribution function $G(t)$ having LST $G^*(\vartheta)$ and the first and second moments are denoted by $g^{(1)}$ and $g^{(2)}$.

- Various stochastic processes involved in the system are assumed to be independent of each other.

2.1 Practical application of the proposed model

Example 1: Our model has a potential practical application in the area of a computer processing system. In a computer processing system, the buffer size (orbit) used to store messages is finite and the messages (customers) arrive into the system one by one. The processor (server) is in charge of processing messages. A working mail server may be affected by a virus (negative customers) and the system may be subjected to electronic fails (breakdowns) during the service period and receive repair immediately. If the processor is available, indicating that it is not currently working on a task, a message is processed. The messages are temporarily stored in a buffer to be served some time later (retrial time) according to first-come-first-serve (FCFS) if the processor is unavailable. To enhance the computer performance, whenever all messages are processed and no new messages arrive, the processor will perform a sequence of maintenance jobs, such as virus scan (working vacations). During the maintenance period, the processor can deal with the messages at the slower rate to economize the cost (working vacation period). Upon completion of the each maintenance, the processor checks the messages and decides whether or not to resume the normal service rate (first working vacation). At this moment, if no message is in the system then the processor may decide to go for another maintenance activity (finite number of working vacations). This type of working vacation discipline is a good approximation of such computer processing system.

Example 2: The suggested model has another practical real life application in the telephone consultation of medical service systems. Nowadays, many doctors have opened telephone consultation services to patients (called positive customers). Here, we consider a telephone consultation service system staffed with a chief physician (main server) and a physician assistant (substitute server). The physician assistant only provides service to the patients when the chief physician is on vacation (working vacation) and the service rate of the physician assistant is usually slower than the chief physician. In generally, there is a phone operator who is responsible to establish communications between doctors and patients or notes down the order of the calls, corresponding to the 'orbit'. If the line is busy when a patient makes a call, he cannot queue but tries again sometime later (retrial), otherwise he is served immediately by the chief physician or the physician assistant. During the consultation time of patients, the telephone signal status is very low or no network coverage (negative customer), and the patient's call has lost service. Once the signal strength is full (repaired), then the system is again treated as good as new to serve. When the chief physician finds no patient call, he will need to rest from his work, i.e., go on a vacation. During the chief physician's vacation period, the physician assistant will serve the patients, if any, and after his service completion if there are patients in the system, the chief physician will come back from his vacation whether his vacation has ended or not, i.e., vacation interruption happens. Meanwhile, if there is no patient when a vacation ends, the chief physician begins a finite number of vacations (at most J working vacations), otherwise, the chief physician takes over as the physician assistant. To understand the patient's condition, the chief physician will restart his service no matter how long the physician assistant has served the patient. On the other hand, to minimize the idle time of the chief physician, immediately on a service completion, the phone operator will call (or search for) the customers who are in orbit under FCFS and the search time is assumed to be generally distributed, which is corresponding to the general retrial time policy.

3. Steady State Analysis of the System

In this section, we develop the steady state difference-differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times, the elapsed working vacation times, and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states and the PGF for the number of customers in the system and orbit.

3.1 The steady state equations

In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $S_v(0) = 0$, $S_v(\infty) = 1$, $G(0) = 0$, $G(\infty) = 1$ are continuous at $x = 0$. So that the function $a(x)$, $\mu_b(x)$, $\mu_v(x)$, and $\xi(y)$ are the conditional completion rates (hazard rate) for retrial, normal service, lower rate service, and repair, respectively.

$$i.e.,\ a(x)dx = \frac{dR(x)}{1-R(x)};\ \mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)};\ \mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)};\ \xi(x)dx = \frac{dG(x)}{1-G(x)}.$$

In addition, let $R^0(t)$, $S_b^0(t)$, $S_v^0(t)$ and $G^0(t)$ be the elapsed retrial time, elapsed normal service time, elapsed working vacation time, and elapsed repair time, respectively, at time t. Further, we introduce the random variable ($i = 1, 2,\ldots J$),

$$C(t) = \begin{cases} 0, & \text{if the server is free and in } i^{\text{th}} \text{ working vacation period,} \\ 1, & \text{if the server is free and in regular service period,} \\ 2, & \text{if the server is busy and in regular service period at time } t, \\ 3, & \text{if the server is busy and in } i^{\text{th}} \text{ working vacation period at time } t, \\ 4, & \text{if the server is under repair period at time } t. \end{cases}$$

Thus the supplementary variables $R^0(t)$, $S_b^0(t)$, $S_v^0(t)$ and $G^0(t)$ are introduced in order to obtain a bivariate Markov process $\{C(t), N(t);\ t \geq 0\}$, where $C(t)$ denotes the server states (0,1,2,3,4) depending on whether the server is free on both regular busy period and working vacation period, regular busy, on working vacation, or under repair. If $C(t) = 1$ and $N(t) > 0$, then $R^0(t)$ represents the elapsed retrial time and if $C(t) = 2$ and $N(t) \geq 0$, then

$S_b^0(t)$ corresponds to the elapsed time of the customer being served in a regular busy period. If $C(t) = 3$ and $N(t) \geq 0$, then $S_v^0(t)$ corresponds to the elapsed time the customer is being served at the lower rate service period on i^{th} stage. If $C(t) = 4$ and $N(t) \geq 0$, then $G^0(t)$ corresponds to the elapsed time of the server being repaired.

We analyze the ergodicity of the embedded Markov chain at departure, vacation or repair epochs. Let $\{t_n;\ n = 1,2,...\}$ be the sequence of epochs of either normal service completion times or working vacation completion times or the repair period ends. The sequence of random vectors $Z_n = \{X(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix that $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$ for our system is stable, where $\rho = (\lambda/\delta)(1 - S_b^*(\delta))(1 + \delta g^{(1)})$.

For the process, we define the limiting probabilities as $Q_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ and $P_0(t) = P\{C(t) = 1,\ N(t) = 0\}$ as the probability densities

$$P_n(x,t)dx = P\{C(t) = 1, N(t) = n,\ x \leq R^0(t) < x + dx\},\ \text{for } t \geq 0,\ x \geq 0 \text{ and } n \geq 1.$$

$$\Pi_{b,n}(x,t)dx = P\{C(t) = 2, N(t) = n,\ x \leq S_b^0(t) < x + dx\},\ \text{for } t \geq 0,\ x \geq 0,\ n \geq 0.$$

$$Q_{i,n}(x,t)dx = P\{C(t) = 3, N(t) = n,\ x \leq S_v^0(t) < x + dx\},\ \text{for } t \geq 0,\ x \geq 0 \text{ and } n \geq 0.$$

$$R_n(x,t)dx = P\{C(t) = 4,\ N(t) = n,\ x \leq G^0(t) < x + dx\},\ \text{for } t \geq 0,\ x \geq 0 \text{ and } n \geq 0.$$

The following probabilities are used in the subsequent sections:
$Q_{i,0}(t)$ is the probability that the system is idle at time t and the server is in i^{th} working vacation. $P_0(t)$ is the probability that the system is idle at time t and the server is in a regular busy period. $P_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed retrial time of the test customer undergoing retrial lying in between x and $x + dx$. $\Pi_{b,n}(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed regular service time of the test customer undergoing service lying in between x and $x + dx$. $Q_{i,n}(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed lower rate service time of the test customer undergoing service lying in between x and $x + dx$ on i^{th} stage. $R_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed repair time of server in between x and $x + dx$. We assume that the stability condition $(\rho < R^*(\lambda))$ is fulfilled in the sequel and so that we can set $Q_{i,0} = \lim_{t\to\infty} Q_{i,0}(t);\ P_0 = \lim_{t\to\infty} P_0(t)$ and limiting densities for $t \geq 0,\ x \geq 0,\ n \geq 1$ and (i=1,2,…J)

$$P_n(x) = \lim_{t\to\infty} P_n(x,t);\quad \Pi_{b,n}(x) = \lim_{t\to\infty} \Pi_{b,n}(x,t);\quad Q_{i,n}(x) = \lim_{t\to\infty} Q_{i,n}(x,t) \quad \text{and} \quad R_n(x) = \lim_{t\to\infty} R_n(x,t).$$

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$\lambda P_0 = \theta Q_{J,0}. \tag{1}$$

$$(\lambda + \theta)Q_{1,0} = \int_0^\infty \Pi_{b,0}(x)\mu_b(x)dx + \int_0^\infty Q_{1,0}(x)\mu_v(x)dx + \int_0^\infty R_0(x)\xi(x)dx. \tag{2}$$

$$(\lambda + \theta)Q_{i,0} = \theta Q_{i-1,0} + \int_0^\infty Q_{i,0}(x)\mu_v(x)dx,\ (i = 1,2,...J). \tag{3}$$

$$\frac{dP_n(x)}{dx} + (\lambda + a(x))P_n(x) = 0,\ n \geq 1. \tag{4}$$

$$\frac{d\Pi_{b,0}(x)}{dx} + (\lambda + \delta + \mu_b(x))\Pi_{b,0}(x) = 0,\ n = 0. \tag{5}$$

$$\frac{d\Pi_{b,n}(x)}{dx} + (\lambda + \delta + \mu_b(x))\Pi_{b,n}(x) = \lambda\Pi_{b,n-1}(x),\ n \geq 1. \tag{6}$$

$$\frac{dQ_{i,0}(x)}{dx}+\left(\lambda+\theta+\mu_v(x)\right)Q_{i,0}(x)=0,\ n=0,\ (i=1,2,...J). \tag{7}$$

$$\frac{dQ_{i,n}(x)}{dx}+\left(\lambda+\theta+\mu_v(x)\right)Q_{i,n}(x)=\lambda Q_{i,n-1}(x),\ n\geq 1,\ (i=1,2,...J). \tag{8}$$

$$\frac{dR_0(x)}{dx}+\left(\lambda+\xi(x)\right)R_0(x)=0,\ n=0. \tag{9}$$

$$\frac{dR_n(x)}{dx}+\left(\lambda+\xi(x)\right)R_n(x)=\lambda R_{n-1}(x),\ \ n\geq 1. \tag{10}$$

To solve Equations (4) through (10), the steady state boundary conditions at $x = 0$ are followed,

$$P_n(0)=\int_0^\infty \Pi_{b,n}(x)\mu_b(x)dx+\sum_{i=1}^{J}\int_0^\infty Q_{i,n}(x)\mu_v(x)dx+\int_0^\infty R_n(x)\xi(x)dx,\ n\geq 1. \tag{11}$$

$$\Pi_{b,0}(0)=\left(\int_0^\infty P_1(x)a(x)dx+\theta\sum_{i=1}^{J}\int_0^\infty Q_{i,0}(x)dx+\lambda P_0\right),\ n=0. \tag{12}$$

$$\Pi_{b,n}(0)=\left(\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\int_0^\infty P_n(x)dx+\theta\sum_{i=1}^{J}\int_0^\infty Q_{i,n}(x)dx\right),\ n\geq 1. \tag{13}$$

$$Q_{i,n}(0)=\begin{cases}\lambda Q_{i,0}, & n=0,\ (i=1,2,...J)\\ 0, & n\geq 1\end{cases}. \tag{14}$$

$$R_n(0)=\delta\int_0^\infty \Pi_n(x)dx,\ n\geq 0. \tag{15}$$

The normalizing condition is

$$P_0+\sum_{i=1}^{J}Q_{i,0}+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\left(\int_0^\infty \Pi_{b,n}(x)dx+\sum_{i=1}^{J}\int_0^\infty Q_{i,n}(x)dx+\int_0^\infty R_n(x)dx\right)=1. \tag{16}$$

3.2 Steady state solution

The steady state solution of the retrial queueing model is obtained by using the PGF function technique. To solve the above equations, the PGFs are defined for $|z| \leq 1$ as follows:

$$P(x,z)=\sum_{n=1}^{\infty}P_n(x)z^n;\ P(0,z)=\sum_{n=1}^{\infty}P_n(0)z^n;\ \Pi_b(x,z)=\sum_{n=0}^{\infty}\Pi_{b,n}(x)z^n;\ \ \Pi_b(0,z)=\sum_{n=0}^{\infty}\Pi_{b,n}(0)z^n;$$

$$Q_i(x,z)=\sum_{n=0}^{\infty}Q_{i,n}(x)z^n;\ Q_i(0,z)=\sum_{n=0}^{\infty}Q_{i,n}(0)z^n;\ \ R(x,z)=\sum_{n=0}^{\infty}R_n(x)z^n \text{ and } R(x,0)=\sum_{n=0}^{\infty}R_n(0)z^n.$$

On multiplying the Equations (4) through (10) by z^n and summing over n, (n = 0,1,2...) and solving the partial differential equations, we get

$$P(x,z)=P(0,z)[1-R(x)]e^{-\lambda x}. \tag{17}$$

$$\Pi_b(x,z)=\Pi_b(0,z)[1-S_b(x)]e^{-A_b(z)x}. \tag{18}$$

$$Q_i(x,z)=Q_i(0,z)[1-S_v(x)]e^{-A_v(z)x}. \tag{19}$$

$$R(x,z)=R(0,z)[1-G(x)]e^{-b(z)x}. \tag{20}$$

where $b(z)=\lambda(1-z),\ \ A_b(z)=\left(\delta+\lambda(1-z)\right)$ and $A_v(z)=\left(\theta+\lambda(1-z)\right)$.

From the Equations (11) through (15), we can obtain

$$P(0,z)=\int_0^\infty \Pi_b(x,z)\mu_b(x)dx+\sum_{i=1}^{J}\int_0^\infty Q_i(x,z)\mu_v(x)dx+\int_0^\infty R(x,z)\xi(x)dx-\lambda\left(\sum_{i=1}^{J}Q_{i,0}+P_0\right). \tag{21}$$

$$\Pi_b(0,z)=\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\lambda\int_0^\infty P(x,z)dx+\theta\sum_{i=1}^{J}\int_0^\infty Q_i(x,z)\mu_v(x)dx+\lambda P_0. \tag{22}$$

$$Q_i(0,z)=\lambda Q_{i,0}. \tag{23}$$

$$R(0,z)=\delta\int_0^\infty \Pi_b(x,z)dx. \tag{24}$$

Inserting the equations (21) through (24) in (18) and do some calculation, we get,

$$\Pi_b(0,z)=\frac{P(0,z)}{z}\left[R^*(\lambda)+z\left(1-R^*(\lambda)\right)\right]+\lambda P_0+\lambda V(z)\sum_{i=1}^{J}Q_{i,0}. \tag{25}$$

where $V(z)=\dfrac{\theta}{\theta+\lambda(1-z)}\left[1-S_v^*\left(A_v(z)\right)\right]$.

Using the equation (21)-(25) in (17), we get

$$P(0,z)=\Pi_b(0,z)S_b^*\left(A_b(z)\right)+\sum_{i=1}^{J}Q_i(0,z)S_v^*\left(A_v(z)\right)+R(0,z)G^*\left(b(z)\right)-\lambda\left(\sum_{i=1}^{J}Q_{i,0}+P_0\right). \tag{26}$$

Using equation (18) and (25) through (26) in (24), we get

$$R(0,z)=\delta\Pi_b(0,z)\left(\frac{1-S_b^*\left(A_b(z)\right)}{A_b(z)}\right). \tag{27}$$

Using equations (24) through (27), we get

$$P(0,z)=\frac{\lambda z\left\{P_0\left(S(z)-1\right)+\sum_{i=1}^{J}Q_{i,0}\left[V(z)S(z)+S_v^*\left(A_v(z)\right)-1\right]\right\}}{z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)}. \tag{28}$$

where $S(z)=\left(S_b^*\left(A_b(z)\right)+\dfrac{\delta G^*\left(b(z)\right)\left(1-S_b^*\left(A_b(z)\right)\right)}{A_b(z)}\right)$.

Using equation (28) in (25), we get

$$\Pi_b(0,z)=\frac{\lambda\left(P_0R^*(\lambda)(z-1)+\sum_{i=1}^{J}Q_{i,0}\left[zV(z)+S_v^*\left(A_v(z)\right)-1\right]\right)}{z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)}. \tag{29}$$

Using equations (23) and (27) through (29) in (17) through (20), then we get the results for the following PGFs $P(x,z)$, $\Pi_b(x,z)$, $Q_i(x,z)$ and $R(x,y,z)$. Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Corollary 1. Under the stability condition $\rho<R^*(\lambda)$, the stationary joint distributions of the number of customers in the orbit when server being idle, busy, on working vacation and under repairs is given by

$$P(z)=\int_0^\infty P(x,z)dx=\frac{z\left(1-R^*(\lambda)\right)\left\{P_0\left(S(z)-1\right)+\sum_{i=1}^{J}Q_{i,0}\left[V(z)S(z)+S_v^*\left(A_v(z)\right)-1\right]\right\}}{z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)}. \tag{30}$$

$$\Pi_b(z)=\int_0^\infty \Pi_b(x,z)dx=\frac{\lambda\left(1-S_b^*\left(A_b(z)\right)\right)\left(P_0R^*(\lambda)(z-1)+\sum_{i=1}^{J}Q_{i,0}\left[zV(z)+S_v^*\left(A_v(z)\right)-1\right]\right)}{A_b(z)\times\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)\right)}. \tag{31}$$

$$Q_i(z)=\int_0^\infty Q_i(x,z)dx=\left\{\lambda Q_{i,0}V(z)/\theta\right\},\ for\ (i=1,2,...J). \tag{32}$$

$$R(z)=\int_0^{\infty}R(x,z)dx=\frac{\delta\lambda\left(1-S_b^*\left(A_b(z)\right)\right)\left(1-G^*\left(b(z)\right)\right)\left(P_0R^*(\lambda)(z-1)+\sum_{i=1}^{J}Q_{i,0}\left[zV(z)+S_v^*\left(A_v(z)\right)-1\right]\right)}{A_b(z)\times b(z)\times\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)\right)}. \quad (33)$$

Applying the normalizing condition $P_0+\sum_{i=1}^{J}Q_{i,0}+P(1)+\Pi_b(1)+\sum_{i=1}^{J}Q_i(1)+R(1)=1$ and using the equations by setting $z=1$ in (43)-(48), we get

$$P_0R^*(\lambda)+\sum_{i=1}^{J}Q_{i,0}\begin{bmatrix}R^*(\lambda)+(\lambda/\theta)\left(1-S_v^*(\theta)\right)\\+(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)\left(\left(1-S_v^*(\theta)\right)\left(2-R^*(\lambda)\right)-1\right)\end{bmatrix}=R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right). \quad (34)$$

where $\rho=(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)$; $A_b(z)=\left(\delta+\lambda(1-z)\right)$, $A_v(z)=\left(\theta+\lambda(1-z)\right)$; $b(z)=\lambda(1-z)$.

and $S(z)=\left(S_b^*\left(A_b(z)\right)+\frac{\delta G^*\left(b(z)\right)\left(1-S_b^*\left(A_b(z)\right)\right)}{A_b(z)}\right)$.

Corollary 2. The probability generating function of number of customers in the system and orbit size distribution at stationary point of time is

$$K_s(z)=P_0+\sum_{i=1}^{J}Q_{i,0}+P(z)++R(z)+z\left(\Pi_b(z)+\sum_{i=1}^{J}Q_i(z)\right)=\frac{Nr1(z)}{Dr1(z)}+\frac{Nr2(z)}{(1-z)Dr1(z)}. \quad (35)$$

where

$$Nr1(z)=P_0R^*(\lambda)(z-1)S(z)$$

$$Nr2(z)=\sum_{i=1}^{J}Q_{i,0}\begin{Bmatrix}(1-z)\left(1+\left(\lambda zV(z)/\theta\right)\right)\times Dr1(z)+z\left(1-R^*(\lambda)\right)\left[V(z)S(z)+S_v^*\left(A_v(z)\right)-1\right]\\+\left(1-S_b^*\left(A_b(z)\right)\right)\left(zb(z)+\delta\left(1-G^*\left(b(z)\right)\right)\right)\left[zV(z)+S_v^*\left(A_v(z)\right)-1\right]\end{Bmatrix}$$

$$Dr1(z)=\left\{z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S(z)\right\}$$

$$S(z)=\left(S_b^*\left(A_b(z)\right)+\frac{\delta G^*\left(b(z)\right)\left(1-S_b^*\left(A_b(z)\right)\right)}{A_b(z)}\right)$$

$$K_o(z)=P_0+\sum_{i=1}^{J}Q_{i,0}+P(z)+R(z)+\left(\Pi_b(z)+\sum_{i=1}^{J}Q_i(z)\right)=\frac{Nr3(z)}{Dr1(z)}+\frac{Nr4(z)}{(1-z)Dr1(z)}. \quad (36)$$

where

$$Nr3(z)=P_0R^*(\lambda)(z-1)$$

$$Nr4(z)=\sum_{i=1}^{J}Q_{i,0}\begin{Bmatrix}(1-z)\left(1+\left(\lambda V(z)/\theta\right)\right)\times Dr1(z)+z\left(1-R^*(\lambda)\right)\left[V(z)S(z)+S_v^*\left(A_v(z)\right)-1\right]\\+\left(1-S_b^*\left(A_b(z)\right)\right)\left(zb(z)+\delta\left(1-G^*\left(b(z)\right)\right)\right)\left[zV(z)+S_v^*\left(A_v(z)\right)-1\right]\end{Bmatrix}$$

Where P_0 and $Q_{i,0}$ is given in Equation (34).

4. System Performance Measures

In this section, we derive some system probabilities, mean number of customers in the orbit/system and reliability measures of the model.

4.1 System state probabilities

From Equations (30) through (33), by setting $z\to 1$ and applying L-Hospital's rule whenever necessary, then we get the following results,

(i) The probability that the server is idle during the retrial, is given by,

$$P = P(1) = \frac{\left(1-R^*(\lambda)\right)\left[P_0(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)+\left(\lambda\left(1-S_v^*(\theta)\right)\sum_{i=1}^{J}Q_{i,0}\left((1/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)+(1/\theta)\right)\right)\right]}{R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)}.$$

(ii) The probability that the server is regular busy, is given by,

$$\Pi_b = \Pi_b(1) = \frac{\lambda\left(\frac{1-S_b^*(\delta)}{\delta}\right)\left(P_0R^*(\lambda)+\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\left(1+(\lambda/\theta)\right)\right)}{R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)}.$$

(iii) The probability that the server is on working vacation, is given by,

$$\sum_{i=1}^{J}Q_i = \sum_{i=1}^{J}Q_i(1) = \left\{\lambda\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\Big/\theta\right\}.$$

(iv) The probability that the server is under repair, is given by,

$$R = R(1) = \frac{\lambda\left(1-S_b^*(\delta)\right)g^{(1)}\left(P_0R^*(\lambda)+\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\left(1+(\lambda/\theta)\right)\right)}{R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)}.$$

4.2 Mean system size and orbit size

If the system is in the steady state condition,

(i) The mean number of customers in the orbit (L_q) is obtained by differentiating (35) with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z) = Q_0\left[\frac{Nr_q'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

$$Nr_q'' = 2\delta P_0R^*(\lambda)+2\sum_{i=1}^{J}Q_{i,0}\begin{bmatrix}\left(1+\left(\lambda\left(1-S_v^*(\theta)\right)/\theta\right)\right)\left(Dr_q''\right)+2\rho\left(1+\lambda/\theta\right)\left(1-S_v^*(\theta)\right)\\-2\left(1-R^*(\lambda)\right)\left(1-S_v^*(\theta)\right)\left((\lambda/\theta)(\theta+\delta)+\delta\lambda g^{(1)}-\lambda S_b^*(\delta)(1+\delta g^{(1)})\right)\end{bmatrix}$$

$$Nr_q''' = 6\lambda P_0R^*(\lambda)-3\sum_{i=1}^{J}Q_{i,0}\begin{bmatrix}\left(1+\left(\lambda\left(1-S_v^*(\theta)\right)/\theta\right)\right)\left(Dr_q''\right)+2V'(1)(\lambda/\theta)\left(Dr_q''\right)\\+2\left(1-R^*(\lambda)\right)\left(\tau\left(1-S_v^*(\theta)\right)+2\lambda^2S_v^{*'}(\theta)+2V'(1)\left((\lambda\delta/\theta)+\delta g^{(1)}-\lambda S_b^*(\delta)(1+\delta g^{(1)})\right)\right)\\+\left(1-S_b^*(\delta)\right)(1+\lambda/\theta)\left[2V'(1)\lambda(1+\delta g^{(1)})+\delta\lambda^2g^{(2)}\left(1-S_v^*(\theta)\right)\right]\end{bmatrix}$$

$$Dr_q''(1) = -2\delta\left(R^*(\lambda)-\rho\right)$$

$$Dr_q'''(1) = 3\left(\tau+2\lambda\left(1-R^*(\lambda)\right)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)\right)$$

where $\rho = (\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)$; $S_v^{*'}(\theta) = \int_0^\infty xe^{-\theta x}dS_v(x)$; $V'(1) = \frac{\lambda}{\theta}\left(1-S_v^*(\theta)+\theta S_v^{*'}(\theta)\right)$;

$\tau = \lambda^2\left(2S_b^{*'}(\delta)\left(1+\delta g^{(1)}\right)+\delta g^{(2)}\left(1-S_b^*(\delta)\right)\right)$;

(ii) The mean number of customers in the system (L_s) is obtained by differentiating (34) with respect to z and evaluating at $z = 1$

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z) = Q_0\left[\frac{Nr_s''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

where $Nr_s''(1) = Nr_q''(1)+P_0R^*(\lambda)\left(6\lambda+(1+2\delta)\left(\delta g^{(1)}-\lambda S_b^*(\delta)(1+\delta g^{(1)})\right)\right)-3\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\left[(\lambda/\theta)\left(Dr_q''\right)+\lambda^2\left(2+\delta g^{(1)}\right)\right]$

4.3 Reliability measures

To justify and validate the analytical results of the model, the availability measure (A_v) and failure frequency (F_f) are obtained as follows:

(i) The steady state availability A_v, which is the probability that the server is either busy, on a working vacation or in an idle period such that the steady state availability of the server is given by

$$\begin{aligned} A_v &= 1-\lim_{z\to 1}\left(R(z)\right) \\ &= 1-R(1) \\ &= 1-\frac{\lambda\left(1-S_b^*(\delta)\right)g^{(1)}\left(P_0R^*(\lambda)+\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\left(1+(\lambda/\theta)\right)\right)}{R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)}. \end{aligned}$$

(ii) The steady state failure frequency is obtained as

$$F_f = \delta\times\Pi_b = \frac{\lambda\left(1-S_b^*(\delta)\right)\left(P_0R^*(\lambda)+\sum_{i=1}^{J}Q_{i,0}\left(1-S_v^*(\theta)\right)\left(1+(\lambda/\theta)\right)\right)}{R^*(\lambda)-(\lambda/\delta)\left(1-S_b^*(\delta)\right)\left(1+\delta g^{(1)}\right)}.$$

5. Special Cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

Case (i): No negative arrival, no breakdown, no vacation interruption, and single working vacation

In this case, our model becomes an $M/G/1$ retrial queue with a single working vacation. We assume that (δ, θ, J) → (0, 0, 1) in the main result is obtained as follows,

$$K_s(z)=\left\{\frac{S_v^*(\lambda)\left(R^*(\lambda)-\lambda\beta^{(1)}\right)}{\lambda E(S_v)-R^*(\lambda)S_v^*(\lambda)}\right\}\left\{\frac{\left\{\left[\left(S_v^*(\lambda-\lambda z)-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+(1-z)R^*(\lambda)S_v^*(\lambda)\right]\right\}S_b^*(\lambda-\lambda z)}{S_v^*(\lambda)\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(\lambda-\lambda z)\right)}\right\}.$$

This coincides with the result of Arivudainambi *et al*. (2014).

Case (ii): No negative arrival, no breakdown, and multiple working vacations

In this case, our model becomes a single server retrial queueing system with working vacations. We assume that $\delta = 0$ and $J = \infty$ and the main result of Ks(z) can be as follows:

$$K_s(z)=P_0\frac{\left\{\begin{array}{l}(1-z)\left[\begin{array}{l}\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(A_b(z))\right)\left((\lambda V(z)/\theta)+1\right)\\ \quad+z\left(1-R^*(\lambda)\right)\left[\left(S_v^*(A_v(z))+V(z)S_b^*(A_b(z))\right)-1\right]\end{array}\right]\\ +\left(1-S_b^*(A_b(z))\right)\left[\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+zV(z)\right]\end{array}\right\}}{(1-z)\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(A_b(z))\right)}.$$

This coincides with the result of Gao *et al*. (2014).

Case (iii): Single working vacation

When $J = 1$, our model can be reduced to a single server retrial queueing system with negative customers, single working vacation, and vacation interruption where the server is subjected to breakdown and repair.

Case (iv): Multiple working vacations

When $J = \infty$, our model can be reduced to a single server retrial queueing system with negative customers, multiple working vacations, and vacation interruption where the server is subjected to breakdown and repair.

6. Numerical Examples

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. For the purpose of a numerical illustration, we assume that all distribution functions like retrial, regular service, working vacation, and repair are exponentially, Erlangian, and hyper-exponentially distributed. All parameter values are selected with the aim of satisfying the steady state condition $\rho < R^*(\lambda)$, where the exponential distribution is $f(x)=\upsilon e^{-\upsilon x}, x>0$, Erlang–2 stage distribution is $f(x)=\upsilon^2 x e^{-\upsilon x}, x>0$,and hyper-exponential distribution is $f(x)=c\upsilon e^{-\upsilon x}+(1-c)\upsilon^2 e^{-\upsilon^2 x}, x>0$. The interpretation of the results based on numerical illustration carried out for the different performance measures is shown in Tables 1-3.

Table 1. Effects of retrial rates (a) on $Q_{1,0}$, L_q and P.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
a	$Q_{1,0}$	L_q	P	$Q_{1,0}$	L_q	P	$Q_{1,0}$	L_q	P
retrial rate									
2.00	0.1053	0.2736	0.1228	0.0048	1.5899	0.5394	0.0829	0.3967	0.2011
3.00	0.1115	0.1702	0.0827	0.0321	0.4576	0.3532	0.0930	0.2643	0.1354
4.00	0.1147	0.1180	0.0623	0.0455	0.3268	0.2617	0.0981	0.1977	0.1020
5.00	0.1167	0.0865	0.0500	0.0534	0.2615	0.2076	0.1013	0.1576	0.0818
6.00	0.1180	0.0654	0.0417	0.0586	0.2211	0.1720	0.1033	0.1309	0.0682

Table 2. Effects of negative arrival rates (δ) on L_q, A_v and F_f.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
δ	L_q	A_v	F_f	L_q	A_v	F_f	L_q	A_v	F_f
negative arrival rate									
0.20	0.3282	0.9980	0.0020	0.7140	0.9870	0.0065	0.4330	0.9976	0.0031
0.30	0.3535	0.9970	0.0044	0.8093	0.9808	0.0144	0.4667	0.9965	0.0069
0.40	0.3784	0.9961	0.0078	0.9048	0.9747	0.0253	0.5002	0.9954	0.0121
0.50	0.4031	0.9951	0.0121	1.0005	0.9687	0.0391	0.5334	0.9943	0.0187
0.60	0.4274	0.9942	0.0173	1.0964	0.9628	0.0557	0.5665	0.9933	0.0266

Table 3. Effects of lower speed service rates (μ_v) on P_0, L_q and Q_1.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
μ_v	P_0	L_q	Q_1	P_0	L_q	Q_1	P_0	L_q	Q_1
Lower speed service rate									
2.00	0.2334	0.0788	0.0729	0.1633	0.1539	0.0766	0.2178	0.1056	0.0783
3.00	0.2405	0.0757	0.0601	0.1720	0.1536	0.0688	0.2256	0.1033	0.0665
4.00	0.2455	0.0738	0.0512	0.1793	0.1514	0.0622	0.2313	0.1025	0.0578
5.00	0.2493	0.0726	0.0445	0.1854	0.1489	0.0568	0.2357	0.1028	0.0511

Table 1 shows that when retrial rate (a) increases, then the probability that server is idle in working vacation ($Q_{1,0}$) increases, the mean orbit size (L_q) decreases and probability that server is idle during retrial time (P) also decreases for the values of λ = 1; θ = 2; μb = 5; δ = 0.2; μv = 3; $\xi = 3$; $J = \infty$; $c = 0.7$. Table 2 shows that when the negative arrival rate (δ) increases, the mean orbit size (L_q) increases, the servers availability (A_v) decreases, and the servers failure frequency (F_f) also increases for the values of $\lambda = 1$; $\theta = 3$; $\mu_b = 10$; $J = \infty$; $\xi = 5$; $a = 2$; $\mu_v = 4$; $c = 0.7$. Table 3 shows that when the lower speed service rate (μ_v) increases, the probability that the server is idle (P_0) increases, then the mean orbit size (L_q) decreases and the probability that the server is on working vacation (Q1) also decrease for the values of λ = 0.5; θ = 2; a = 2; μb = 5; δ = 0.3; $\xi = 3$; $J = 1$; $c = 0.7$. The above results facilitate an insight into the system performance measures of the unreliable M/G/1 retrial G-queue with variant working vacations.

For the effect of the parameters λ, a, δ, ξ, μb, and μv on the system performance measures, three dimensional graphs are illustrated in Figures 1-3. In Figure 1, the surface displays an upward trend as expected when increasing the value of the arrival rate (λ) and negative arrival rate (δ) against the mean orbit size (L_q), that is suppose the number of arriving messages and the number of viruses affecting time increases, the average number of packets in the buffer increases. Figure 2 shows that the server's availability probability (A_v) increases when increasing the value of the lower service rate (μ_v) and regular service rate (μ_b) that is, if the systems availability increases when increasing the values of the processing time and the virus scan processing time. In Figure 3, we demonstrate the effect of variation of the mean orbit size (L_q) decreases for increasing the value of repair rate (ξ) and retrial rate (a), that is the average number of packets in the buffer increases which increases the values of retransmission time and recovering time of the system.

From the above numerical examples, we can find the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

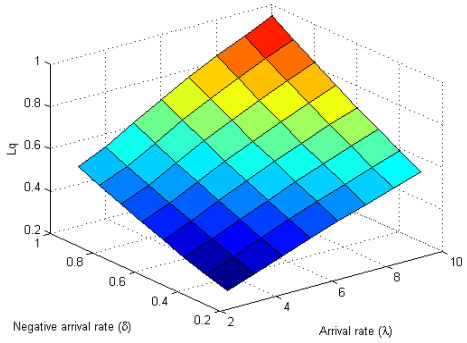

Figure 1. L_q versus λ and δ.

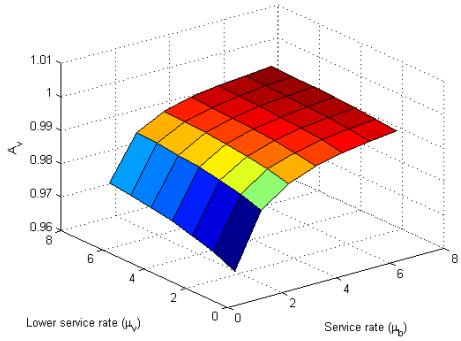

Figure 2. A_v versus μ_b and μ_v.

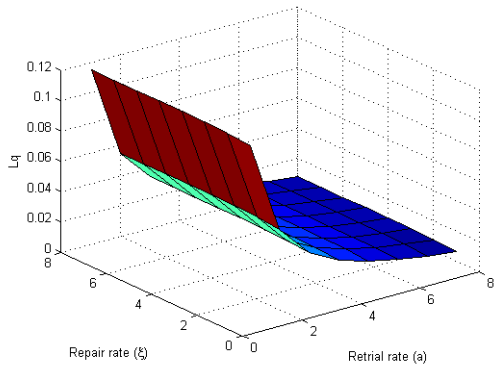

Figure 3. L_q versus μ_v and a.

7. Conclusions

In this paper, we have studied a single server retrial queueing system with variant working vacations and vacation interruption, where the regular busy server subjected to breakdown and repair due to the arrival of negative customers. The analytical results, which are validated with the help of numerical illustrations, may be useful in various real life situations to design the outputs. The probability generating functions for the numbers of customers in the system and its orbit when it is free, busy, on working vacation or under repair are found by using the supplementary variable technique. Some varieties of performance measures of the system are calculated. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples are presented to study the impact of the system parameters. The novelty of this investigation is the introduction of both single working vacation (J=1) and multiple working vacations (J=∞) in presence of retrial G-queues and server breakdown. This proposed model has potential practical real life application in production and order systems to enhance the performance of the production facility and to prevent the production facility from overload in a computer processing system or in telephone consultation of medical service systems. Hopefully, this investigation will be a great help to the system managers who can design a system with economic management and make decisions regarding the size of the system and other factors in a well-to-do manner.

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Appendix

The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$, for our system to be stable, where $\rho = (\lambda/\delta)\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)$.

Proof: To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (Pakes, 1969), which states that the chain $\{Z_n;\ n \in N\}$ is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E\left[f(z_{n+1}) - f(z_n)/z_n = j\right]$ is finite for all $j \in N$ and $\psi_j \leq -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} (\lambda/\delta)\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right) - 1, & \text{if } j = 0, \\ (\lambda/\delta)\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right) - R^*(\lambda), & \text{if } j = 1, 2... \end{cases}$$

Clearly the inequality $\rho < R^*(\lambda)$ is a sufficient condition for ergodicity.

To prove the necessary condition, As noted in Sennott *et al.* (1983), if the Markov chain $\{Z_n;\ n \geq 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \geq 0$ and there exits $j_0 \in$ N such that $\psi_j \geq 0$ for $j \geq j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n;\ n \in N\}$. Then $\rho \geq R^*(\lambda)$ implies the non-Ergodicity of the Markov chain.