Songklanakarin J. Sci. Technol.
42 (2), 263-273, Mar. - Apr. 2020

http://www.sjst.psu.ac.th

Original Article

Orlicz arithmetic convergence defined by matrix transformation and lacunary sequence

Kuldip Raj* and Anu Choudhary

School of Mathematics, Shri Mata Vaishno Devi University, Katra, Jammu and Kashmir, 182320 India

Received: 22 February 2018; Revised: 12 October 2018; Accepted: 5 December 2018

Abstract

In this paper we introduce and study some spaces of Orlicz arithmetic convergence sequences with respect to matrix transformation and lacunary sequence. We make an effort to examine some algebraic and topological properties of these sequence spaces. Some inclusion relation between these sequence spaces have been established.

Keywords: Orlicz function, infinite matrix, lacunary sequence, arithmetic convergence, matrix transformation

1. Introduction

Ruckle (2012) introduced the idea of arithmetic convergence. A sequence $x=(x_k)$ is said to be arithmetically convergent, if for each $\varepsilon>0$ there is an integer l such that we have $|x_k - x_{\langle k,l\rangle}|<\varepsilon$ for every integer k, where $\langle k,l\rangle$ denotes the greatest common divisor of two integers k and l. The sequence space of all arithmetic convergent sequences is denoted by $\mathcal{A}C$. Subsequently, the arithmetic convergence has been discussed in (Yaying & Hazarika, 2017a, 2017b).

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of natural, real, and complex numbers, respectively. We write

$$w=\{x=(x_k): x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

as the space of all real or complex sequences. An increasing non-negative integer sequence $\theta=(k_r)$ with $k_0=0$ and $k_r-k_{r-1}\to\infty$, as $r\to\infty$ is known as a lacunary sequence. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$. We write $h_r=k_r-k_{r-1}$ and q_r denotes the ratio $\frac{h_r}{h_{r-1}}$. The space of lacunary strongly convergent sequence was defined by Freedman, Sember, and Raphael (1978) as follows:

$$N_\theta=\left\{x=(x_k): \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\to I_r}|x_k-L|=0 \text{ for some L}\right\}.$$

The space N_θ is a BK-space with the norm

$$\|x\|=\sup_r \frac{1}{h_r}\sum_{k\in I_r}|x_k|.$$

The concept of lacunary convergence has been studied by many authors. Some of them are Çolak, Tripathy, and Et (2006), Kara and İlkhan (2016), Raj and Pandoh (2016),

*Corresponding author
Email address: kuldipraj68@gmail.com

Savaş and Patterson (2007), Tripathy and Baruah (2010), Tripathy and Dutta (2012), Tripathy and Et (2005), Tripathy, Hazarika, and Choudhary (2012), Tripathy and Mahanta (2004). Freedman, Sember, and Raphael (1978) also introduced the concept of lacunary refinement. A lacunary refinement of a lacunary sequence θ is a lacunary sequence $\theta' = (k_r')$ satisfying $(k_r) \subseteq (k_r')$. For more details about sequence spaces see (Et, Lee, & Tripathy, 2006; Raj & Kilicman, 2015; Raj & Sharma, 2016).

Let $A = (a_{kj})$ be an infinite matrix of real or complex numbers a_{kj}, where $k, j \in \mathbb{N}$. We write the A transform of $x = (x_k)$ as Ax and $Ax = (A_k(x))$ if $A_k(x) = \sum_{j=1}^{\infty} a_{kj} x_j$ converges for each $k \in \mathbb{N}$.

A function $M : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function, if it is convex, continuous and non-decreasing with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $R > 0$ such that $M(2u) \le RM(u), u \ge 0.$

The idea of Orlicz function was used by Lindenstrauss and Tzafriri (1971) to define the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

known as Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \mu(t)dt,$$

where μ is known as the kernel of M, is right differentiable for t $\ge$ 0, $\mu(0) = 0$, $\mu(t)>0$, μ is non-decreasing and $\mu(t) \to \infty, t \to \infty$. A sequence $M = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (Maligranda, 1989; Musielak, 1983). The complementary function of a Musielak-Orlicz function is denoted by $N = N_k$ and is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u > 0\}, k = 1, 2, \dots .$$

Let $p = (p_k)$ be a positive sequence with sup $p_k = H$, $C = \max(1, 2^{H-1})$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1.1}$$

Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ a sequence of positive real numbers. Let $A = (a_{kj})$ be an infinite matrix of real or complex numbers a_{kj}, where $k, j \in \mathbb{N}$. In the present paper we define the Orlicz arithmetic sequences using matrix transformations and lacunary sequence as follows:

$$[\mathcal{A}C_{\theta}, u, p, M, A] = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k\left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} = 0, \text{for some integer } l \text{ and } \rho > 0 \right\}$$

and

$$[\mathcal{A}C_{\sigma_1}, u, p, M, A] = \left\{ \begin{array}{l} x=(x_k): \text{there exist an integer } l \text{ such that } \frac{1}{q}\sum_{k=1}^{q} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} \to 0 \\ \text{as } q\to\infty \text{ for some } \rho>0 \end{array}\right\}.$$

For $p_k = 1$

$$[\mathcal{A}C_{\theta}, u, M, A] = \left\{ x=(x_k): \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right) = 0, \text{for some integer } l \text{ and } \rho>0 \right\}.$$

The arithmetic convergent sequence space $[\mathcal{A}C, u, p, M, A]$ is defined

$$[\mathcal{A}C, u, p, M, A] = \left\{ x=(x_k): \lim_{k\to\infty} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} = 0, \text{for some integer } l \text{ and } \rho>0 \right\}.$$

The main aim of the paper is to introduce the sequence spaces $[\mathcal{A}C_{\theta}, u, p, M, A]$ and $[\mathcal{A}C_{\sigma_1}, u, p, M, A]$. We shall investigate some algebraic properties, topological properties and inclusion relation between these sequence spaces.

2. Main Results

Theorem 2.1. Let $M=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. Then the space $[\mathcal{A}C_{\theta}, u, p, M, A]$ is linear over the complex field $\mathbb{C}$.

Proof. Let $x=(x_k)$ and $y=(y_k) \in [\mathcal{A}C_{\theta}, u, p, M, A]$. Then for an integer l

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} = 0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(y)-y_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} = 0.$$

Let γ and η be two scalars in $\mathbb{C}$, then there exist integers A_γ and B_η such that $|\gamma| \le A_\gamma$ and $|\eta| \le B_\eta$. Therefore

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|\gamma A_k(x)+\eta A_k(y)-(\gamma x_{\langle k,l\rangle}+\eta y_{\langle k,l\rangle})|}{\rho}\right)\right]^{p_k}$$

$$\le A_\gamma \frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} + B_\eta \frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(y)-y_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}$$

$$\Rightarrow \gamma A_k(x)+\eta A_k(y) \to \gamma x_{\langle k,l\rangle}+\eta y_{\langle k,l\rangle}.$$

Therefore, $[\mathcal{A}C_{\theta}, u, p, M, A]$ is a linear space.

Theorem 2.2. Let $M=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. Then, we have

$[\mathcal{A}C, u, p, M, A] \subseteq [\mathcal{A}C_\theta, u, p, M, A]$.

Proof. Let $x = (x_k) \in [\mathcal{A}C, u, p, M, A]$. Then there is an integer l such that

$$u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} < \varepsilon, \text{ for } \varepsilon > 0.$$

Thus, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}$$

$$= \frac{1}{h_r}\left[\sum_{k=1}^{k_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} - \sum_{k=1}^{k_{r-1}} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\right]$$

$$< \frac{1}{h_r}(h_r\varepsilon) = \varepsilon.$$

Therefore, $x = (x_k) \in [\mathcal{A}C_\theta, u, p, M, A]$.

Theorem 2.3. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of positive real numbers. Then, $[\mathcal{A}C_\theta, u, p, M, A]$ is a topological space paranormed by

$$g(x) = \inf\left\{\rho^{p_k/H} : \left(\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\right)^{1/H} \le 1, \text{for some } \rho > 0\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. (i) Clearly $g(x) \ge 0$, for $x = (x_k) \in [\mathcal{A}C_\theta, u, p, M, A]$. Since $M_k(0) = 0$, we get $g(0) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_k), y = (y_k) \in [\mathcal{A}C_\theta, u, p, M, A]$, then there exist $\rho_1, \rho_2 > 0$ such that

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho_1}\right)\right]^{p_k} \le 1$$

and

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(y) - y_{\langle k,l\rangle}|}{\rho_2}\right)\right]^{p_k} \le 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|(A_k(x) - x_{\langle k,l\rangle}) + (A_k(y) - y_{\langle k,l\rangle})|}{\rho}\right)\right]^{p_k}$$

$$\le \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho_1}\right)\right]^{p_k}$$

$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|A_k(y)-y_{\langle k,l\rangle}|}{\rho_2}\right)\right]^{p_k}$$

and thus

$$g(x+y)$$

$$=\inf\left\{(\rho_1+\rho_2)^{p_k/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|(A_k(x)-x_{\langle k,l\rangle})+(A_k(y)-y_{\langle k,l\rangle})|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1\right\}$$

$$\le g(x)+g(y).$$

(iv) Finally, we show that scalar multiplication is continuous. Let α be any complex number, then

$$g(\alpha x)=\inf\left\{\rho^{p_k/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{\alpha|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1\right\}$$

$$=\inf\left\{(|\alpha|s)^{p_k/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{s}\right)\right]^{p_k}\right)^{1/H}\le 1\right\},$$

where $s=\frac{\rho}{|\alpha|}$. Hence $[\mathcal{A}C_\theta, u, p, M, A]$ is a paranormed space.

Theorem 2.4. Let $0\le p_k\le q_k$ for all k and let (q_k/p_k) be bounded. Then

$$[\mathcal{A}C_\theta, u, q, M, A]\subseteq[\mathcal{A}C_\theta, u, p, M, A].$$

Proof. Let $x=(x_k)\in[\mathcal{A}C_\theta, u, q, M, A]$ and

$$d_k=\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{q_k}$$

and $\nu_k=p_k/q_k$ for all $k\in\mathbb{N}$. Then $0<\nu_k\le 1$ for all $k\in\mathbb{N}$. Take $0<\nu\le\nu_k$ for $k\in\mathbb{N}$.

Define sequences (y_k) and (z_k) as follows:

For $d_k\ge 1$, let $y_k=d_k$ and $z_k=0$ and for $d_k<1$, let $y_k=0$ and $z_k=d_k$. Then clearly,

for all $k\in\mathbb{N}$, we have

$$d_k=y_k+z_k,\ d_k^{\nu_k}=y_k^{\nu_k}+z_k^{\nu_k}.$$

Now it follows that $y_k^{\nu_k}\le y_k\le d_k$ and $z_k^{\nu_k}\le z_k^{\nu}$. Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r}u_kd_k^{\nu_k}=\frac{1}{h_r}\sum_{k\in I_r}u_k\quad(y_k^{\nu_k}+z_k^{\nu_k})$$

$$\frac{1}{h_r}\sum_{k\in I_r} u_k d_k^{\nu_k} = \frac{1}{h_r}\sum_{k\in I_r} u_k\,(y_k^{\nu_k} + z_k^{\nu_k})$$

$$\leq \frac{1}{h_r}\sum_{k\in I_r} u_k d_k + \frac{1}{h_r}\sum_{k\in I_r} u_k z_k^{\nu}.$$

Now for each k,

$$\frac{1}{h_r}\sum_{k\in I_r} u_k z_k^{\nu} = \sum_{k\in I_r}\left(\frac{1}{h_r}u_k z_k\right)^{\nu}\left(\frac{1}{h_r}u_k\right)^{1-\nu}$$

$$= \left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}u_k z_k\right)^{\nu}\right]^{1/\nu}\right)^{\nu} \times \left(\sum_{k\in I_r}\left[\left(\frac{1}{h_r}u_k\right)^{1-\nu}\right]^{1/1-\nu}\right)^{1-\nu}$$

$$= \left(\frac{1}{h_r}\sum_{k\in I_r} u_k z_k\right)^{\nu}$$

and hence

$$\frac{1}{h_r}\sum_{k\in I_r} u_k z_k{}^{\nu} \leq \frac{1}{h_r}\sum_{k\in I_r} u_k d_k + \left(\frac{1}{h_r}\sum_{k\in I_r} u_k z_k\right)^{\nu}.$$

Therefore, $x = (x_k) \in [\mathcal{A}C_{\theta}, u, p, M, A]$.

Theorem 2.5. (i) If $0 < \inf\, p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, then

$$[\mathcal{A}C_{\theta}, u, p, M, A] \subseteq [\mathcal{A}C_{\theta}, u, M, A].$$

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$ for all $k \in \mathbb{N}$, then

$$[\mathcal{A}C_{\theta}, u, M, A] \subseteq [\mathcal{A}C_{\theta}, u, p, M, A].$$

Proof. (i) Let $x = (x_k) \in [\mathcal{A}C_{\theta}, u, p, M, A]$, then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} = 0.$$

Since $0 < \inf\, p_k \leq p_k \leq 1$, implies

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right] \leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}.$$

Thus, $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right] = 0.$ Hence, $[\mathcal{A}C_{\theta}, u, p, M, A] \subseteq [\mathcal{A}C_{\theta}, u, M, A]$.

(ii) Let $1 \leq p_k \leq \sup p_k$ and $x = (x_k) \in [\mathcal{A}C_{\theta}, u, M, A]$, then for each $\rho > 0$, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x) - x_{\langle k,l\rangle}|}{\rho}\right)\right] = 0.$$

For given conditions we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} \leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right] = 0.$$

Therefore, $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M, A]$.Hence, $[\mathcal{A}C_\theta, u, M, A]\subseteq[\mathcal{A}C_\theta, u, p, M, A]$.

Theorem 2.6. If $0<\inf p_k<\sup p_k=H<\infty$, for all $k\in\mathbb{N}$, then

$$[\mathcal{A}C_\theta, u, p, M, A]=[\mathcal{A}C_\theta, u, M, A].$$

Proof. We omit the details because it is easy to prove.

Theorem 2.7. Let $0<\inf p_k=h\leq p_k\leq\sup p_k=H<\infty$ and $M=(M_k), M'=(M'_k)$ be two Musielak Orlicz functions satisfying Δ_2-condition, then we have

$$[\mathcal{A}C_\theta, u, p, M', A]\subset[\mathcal{A}C_\theta, u, p, M\circ M', A].$$

Proof. Suppose $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M', A]$, then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M'_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} = 0.$$

Now choose β with $0<\beta<1$ and $\varepsilon>0$ such that $M_k(w)<\varepsilon$ for $0\leq w\leq\beta$.

Let $z_k=M'_k\left[\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right]$ for all $k\in\mathbb{N}$, then we have

$$\frac{1}{h_r}\sum_{k\in I_r}u_k\left(M_k[z_k]\right)^{p_k}=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\leq\beta}}u_k\left(M_k[z_k]\right)^{p_k}+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\geq\beta}}u_k\left(M_k[z_k]\right)^{p_k}.$$

Therefore,

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\leq\beta}}u_k\left(M_k[z_k]\right)^{p_k}\leq\left[M_k(1)\right]^H\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\leq\beta}}u_k\left(M_k[z_k]\right)^{p_k}$$

$$\leq\left[M_k(2)\right]^H\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\leq\beta}}u_k\left(M_k[z_k]\right)^{p_k}. \tag{2.1}$$

Since $(M_k)'$ are convex and non-decreasing and for $z_k>\beta, z_k<\frac{z_k}{\beta}<1+\frac{z_k}{\beta}$, we have

$$M_k(z_k)<M_k\left(1+\frac{z_k}{\beta}\right)<\frac{1}{2}M_k(2)+\frac{1}{2}M_k\left(\frac{2z_k}{\beta}\right)$$

and as $M=(M_k)$ satisfies Δ_2 condition, so we have

$$M_k(z_k)<\frac{1}{2}T\frac{z_k}{\beta}M_k(2)+\frac{1}{2}T\frac{z_k}{\beta}M_k(2)=T\frac{z_k}{\beta}M_k(2).$$

Thus,

$$\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\geq\beta}} u_k\left(M_k\left[z_k\right]\right)^{p_k} \leq \max\left(1,\left(T\frac{M_k(2)}{\beta}\right)^H\right)\frac{1}{h_r}\sum_{\substack{k\in I_r\\ z_k\leq\beta}} u_k\left(M_k\left[z_k\right]\right)^{p_k}. \tag{2.2}$$

From (2.1) and (2.2), we have $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M\circ M', A]$. This completes the proof.

Theorem 2.8. Let $0<h=\inf p_k = p_k < \sup p_k = H < \infty$. Then for Musielak-Orlicz function $M=(M_k)$ which satisfies Δ_2-condition, we have

$$[\mathcal{A}C_\theta, u, p, A]\subset[\mathcal{A}C_\theta, u, p, M, A].$$

Proof. We omit it because it is easy to prove.

Theorem 2.9. Let $M=(M_k)$ and $M'=(M'_k)$ be two Musielak-Orlicz functions. Then we have

$$[\mathcal{A}C_\theta, u, p, M, A]\cap[\mathcal{A}C_\theta, u, p, M', A]\subset[\mathcal{A}C_\theta, u, p, M+M', A].$$

Proof. Let $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M, A]\cap[\mathcal{A}C_\theta, u, p, M', A]$, then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M'_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}=0.$$

Also by using (1.1) we have

$$\left[\left(M_k+M'_k\right)\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}$$

$$\leq C\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}+C\left[M'_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}.$$

Now multiplying both sides of the above inequality by $\frac{1}{h_r}, u_k$ and applying $\sum_{k\in I_r}$, we get

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\left[\left(M_k+M'_k\right)\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{C}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}+\frac{C}{h_r}\sum_{k\in I_r} u_k\left[M'_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}.$$

This completes the proof.

Theorem 2.10. Let $M=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. Then, $[\mathcal{A}C_{\theta'}, u, p, M, A]\subseteq[\mathcal{A}C_\theta, u, p, M, A]$, where θ' is the lacunary refinement of a lacunary sequence θ.

Proof. Let each I_r of θ contains the point $(k_{r,c})_{c=1}^{\delta(r)}$ of θ', such that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \cdots < k'_{r,\delta(r)} = k_r.$$

So $r, \delta(r) \geq 1$ as $k_r \subseteq (k'_r)$.

Suppose $\left(I^*\right)_{e=1}^{\infty}$ be the sequence of interval $\left(I^*_{r,t}\right)$ ordered by increasing right end points. Since $(x_k) \in [\mathcal{A}C_{\theta'}, u, p, M, A]$, then for each $\varepsilon > 0$,

$$\frac{1}{h_e^*} \sum_{I_e^* \subset I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} < \varepsilon$$

and $h'_{r,t} = k'_{r,t} - k'_{r,t-1}$ in view of $k_r \subseteq (k'_r)$. Now for each $\varepsilon > 0$,

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} \leq \frac{1}{h_e^*} \sum_{I_e^* \subset I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} < \varepsilon.$$

Hence, $(x_k) \in [\mathcal{A}C_{\theta}, u, p, M, A]$.

Theorem 2.11. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of positive real numbers. Then the space $[\mathcal{A}C_{\sigma_1}, u, p, M, A]$ is closed under addition and scalar multiplication.

Since the proof can be established using standard techniques, it is omitted.

Theorem 2.12. Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of positive real numbers. If $\liminf q_r > 1$, then $[\mathcal{A}C_{\sigma_1}, u, p, M, A] \subseteq [\mathcal{A}C_{\theta}, u, p, M, A]$.

Proof. Suppose $\liminf q_r > 1$ and $x = (x_k) \in [\mathcal{A}C_{\sigma_1}, u, p, M, A]$. Then there exists $\alpha > 0$, such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \alpha$ and $\frac{k_r}{h_r} \leq \frac{1+\alpha}{\alpha}$ for sufficiently large r. Then

$$\begin{aligned}
\frac{1}{k_r} \sum_{k=1}^{k_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} &\geq \frac{1}{k_r} \sum_{k \in I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} \\
&= \frac{h_r}{k_r} \left(\frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} \right) \\
&\geq \frac{\alpha}{1+\alpha} \left(\frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k \left(\frac{|A_k(x) - x_{\langle k,l \rangle}|}{\rho} \right) \right]^{p_k} \right).
\end{aligned}$$

Hence, $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M, A]$.

Theorem 2.13. Let $M=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. If $\limsup q_r<\infty$, then $[\mathcal{A}C_\theta, u, p, M, A]\subseteq[\mathcal{A}C_{\sigma_1}, u, p, M, A]$.

Proof . Suppose $\limsup q_r<\infty$, then there exists $N>0$ such that $q_r<N$ for every r.

For $x=(x_k)\in[\mathcal{A}C_\theta, u, p, M, A]$ and $\varepsilon>0$, there exists P such that for every $r\geq P$,

$$\phi(r)=\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}<\varepsilon.$$

For $Q>0$, $\phi(r)\leq Q, \forall r$. Let q be an integer with $k_r\geq q\geq k_{r-1}$. Then

$$\begin{aligned}
\frac{1}{q}\sum_{k=1}^{q}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k} &\leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{k_{r-1}}\sum_{j=1}^{P}\sum_{k\in I_j}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\\
&+\frac{1}{k_{r-1}}\sum_{j=P+1}^{k_r}\sum_{k\in I_j}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\\
&\leq\frac{1}{k_{r-1}}\sum_{j=1}^{P}\sum_{k\in I_j}u_k\left[M_k\left(\frac{|A_k(x)-x_{\langle k,l\rangle}|}{\rho}\right)\right]^{p_k}\\
&+\frac{1}{k_{r-1}}\left(\varepsilon\left(k_r-k_P\right)\right)\\
&\leq\frac{1}{k_{r-1}}\sum_{j=1}^{P}h_j\phi_j+\frac{1}{k_{r-1}}\left(\varepsilon\left(k_r-k_P\right)\right)\\
&\leq\frac{1}{k_{r-1}}\left(\sup_{j\leq P}\phi_j k_P\right)+\varepsilon N<\frac{k_P}{k_{r-1}}Q+\varepsilon N.
\end{aligned}$$

From here we conclude that $x=(x_k)\in[\mathcal{A}C_{\sigma_1}, u, p, M, A]$.

Theorem 2.14. Let $M=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. If $\liminf q_r<\limsup q_r<\infty$, then $[\mathcal{A}C_\theta, u, p, M, A]=[\mathcal{A}C_{\sigma_1}, u, p, M, A]$.

Proof. The proof is easy and follows from Theorem 2.12 and Theorem 2.13.

Acknowledgements

The authors would like to express their sincere thanks to referees for a careful reading and several constructive comments that have improved the presentation of the paper.

References

Çolak, R., Tripathy, B. C., & Et, M. (2006). Lacunary strongly summable sequences and q-lacunary almost statistical convergence. *Vietnam Journal of Mathematics,* *34,* 129-138.

Et., M., Lee, P. Y., & Tripathy, B. C. (2006). Strongly almost $(V,\lambda)(\Delta^r)$-summable sequences defined by Orlicz function. *Hokkaido Mathematical Journal, 35,* 197-213.

Freedman, A. R., Sember, J. J., & Raphael, M. (1978). Some Cesàro-type summability spaces. *Proceedings of the London Mathematical Society 37,* 508-520.

Kara, E. E., & İlkhan, M. (2016). Lacunary I-convergent and lacunary I-bounded sequence spaces defined by an Orlicz function. *Electronic Journal of Mathematical Analysis and Applications*, *4,* 150-159.

Lindenstrauss, J., & Tzafriri, L. (1971). On Orlicz sequence spaces. *Israel Journal of Mathematics, 10,* 379-390.

Maligranda, L. (1989). *Orlicz spaces and interpolation (Seminars in Mathematics 5),* São Paulo, Brazil: Department of Mathematics, The University of Campinas.

Musielak, J. (1983). *Orlicz spaces and modular spaces*. Berlin, Poland: Springer-Verlag Berlin Heidelberg.

Raj, K., & Sharma, C. (2016). Applications of strongly convergent sequences to Fourier series by means of modulus functions. *Acta Mathematica Hungarica, 150*, 396-411.

Raj, K., & Kilicman, A. (2015). On certain generalized paranormed spaces. *Journal of Inequalities and Applications, 2015*.

Raj, K., & Pandoh, S. (2016). Some generalized double lacunary Zweier convergent sequence spaces. *Commentationes Mathematicae, 56*, 185-207.

Ruckle, W. H. (2012). Arithmetical Summability. *Journal of Mathematical Analysis and Applications, 396*, 741-748.

Savaş, E., & Patterson, R. F. (2007). Double sequence spaces characterized by lacunary sequences. *Applied Mathematics Letters, 20,* 964-970.

Tripathy, B. C., & Baruah, A. (2010). Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers. *Kyungpook Mathematical Journal, 50,* 565-574.

Tripathy, B. C., & Dutta, H. (2012). On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary Δ_m^n-statistical convergence. *Analele Stiintifice ale Universitatii Ovidius, Seria Mathematica, 20,* 417-430.

Tripathy, B. C., & Et, M. (2005). On generalized difference lacunary statistical convergence. *Studia Universitatis Babes-Bolyai Mathematica, 50,* 119-130.

Tripathy, B. C., Hazarika, B., & Choudhary, B. (2012). Lacunary I-convergent sequences. *Kyungpook Mathematical Journal, 52,* 473-482.

Tripathy, B. C., & Mahanta, S. (2004). On a class of generalized lacunary difference sequence spaces defined by Orlicz function. *Acta Mathematicae Applicatae Sinica (English Series), 20,* 231-238.

Yaying, T., & Hazarika, B. (2017). On arithmetical summability and multiplier sequences. *National Academy Science Letters, 40,* 43-46.

Yaying, T., & Hazarika, B. (2017). On arithmetic continuity. *Boletim da Socieda de Paranaense de Matemática, 35,* 139-145.