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Original Article

Linear inequations with 5-digit and regularity conditions on ordered ternary semigroups

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Abstract

We investigate linear inequations with 5-digit of regular elements of ordered ternary semigroups, define the notions of ordered weakly left ideals, ordered weakly right ideals and ordered strong bi-ideals of ordered ternary semigroups, give some characterizations of all regular elements of ordered ternary semigroups using many kinds of ordered ideals. Finally, we investigate connections among all types of linear inequations with 5-digit of regular elements of ordered ternary semigroups.

Keywords: ordered ternary semigroup, ordered left ideal, ordered right ideal, ordered lateral ideal, ordered quasi-ideal, ordered bi-ideal, regular ordered ternary semigroup

1. Introduction

The concept of a regular semigroup has been widely studied for a long time. Many notable kinds of regularities which are bases of many researches about regularities on (partially) ordered semigroups and related algebraic structures were introduced by Kehayopulu (1993, 1998). The notion of a ternary algebraic structure was introduced first by Lehmer (1932). Later, Los (1995) studied the concept of a ternary semigroup and showed that every semigroup can be considered as a ternary semigroup but there is an example of a ternary semigroup which does not reduce to a semigroup. In other words, a ternary semigroup is a generalization of a semigroup. Sioson (1965) defined the concept of a regular ternary semigroup and studied its prime ideals and semiprime ideals. Afterwards, Santiago (1990) and Sri Bala (2010) studied more properties of regular ternary semigroups. Then, Dutta, Kar and Maity (2008) presented more kinds of regularities of ternary semigroups including left regular, right regular, completely regular and intra-regular ternary semigroups. As a generalization of a ternary semigroup, Iampan (2009) defined an ordered ternary semigroup which is a ternary semigroup together with a partially ordered relation on the ternary semigroup connected by the compatibility property. By the way, Daddi and Pawar (2012) defined the notions of ordered ideals, ordered left ideals, ordered right ideals, ordered quasi-ideals and ordered bi-ideals of ordered ternary semigroups. S. Lekkoksung and N. Lekkoksung (2012) gave the notion of an intra-regular ordered ternary semigroup and characterize it using its ordered ideals. Later, Lekkoksung and Jampachon (2014) defined the notions of left weakly regular and right weakly regular ordered ternary semigroups and gave some of their characterizations in terms of ordered ideals. In 2019, more types of regularities of ordered ternary semigroups such as left regular, right regular, left lightly regular, right lightly regular, left generalized regular, right generalize regular and completely regular ordered ternary semigroups were introduced and studied by Pornsurat and Pibaljommee (2019). Moreover, they characterized regularities of ordered ternary semigroups using many kinds of ordered ideals.

In this work, we investigate linear inequations with 3-digit and 5-digit of regular elements of ordered ternary semigroups, define the notions of ordered weakly left ideals, ordered weakly right ideals and ordered strong bi-ideals of

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ordered ternary semigroups, give some characterizations of all regular elements of ordered ternary semigroups using many kinds of ordered ideals and finally, we investigate connections among all types of linear inequations with 3-digit and 5-digit of regular elements of ordered ternary semigroups.

2. Preliminaries

A nonempty set S is called a *ternary semigroup* if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that

$$\big[[x_1x_2x_3]x_4x_5\big] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

An *ordered ternary semigroup* $(S, [\,], \leq)$ is a ternary semigroup $(S, [\,])$ together with a partial order relation $\leq$ on S which is compatible with ternary operation $[\,]$, i.e.,

$$x \leq y \Rightarrow [xuv] \leq [yuv], [uxv] \leq [uyv], [uvx] \leq [uvy]$$

for all $x, y, u, v \in S$.

For any x, y, z in a ternary semigroup S, we write xyz for $[xyz]$. Throughout this paper, we write S for an ordered ternary semigroup, unless specified otherwise.

Let A be a nonempty subset of S. We denote

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Lemma 2.1. Let A, B and C be nonempty subsets of S. Then the following statements hold:

(i) $A \subseteq (A]$;
(ii) $(A] = ((A]]$;
(iii) $A \subseteq B$ implies $(A] \subseteq (B]$;
(iv) $(A](B](C] \subseteq (ABC]$;
(v) $(A]BC \subseteq (ABC]$, $A(B]C \subseteq (ABC]$ and $AB(C] \subseteq (ABC]$;
(vi) $(A \cup B] = (A] \cup (B]$.

A nonempty subset I of S is called an *ordered left* (resp. *right*, *lateral*) *ideal* of S if $SSI \subseteq I$ (resp. $ISS \subseteq I$, $SIS \subseteq I$) and $(I] = I$.

If I is an *ordered left*, an *ordered right* and an *ordered lateral ideal* of S, then it is called an *ordered ideal* of S.

A nonempty subset Q of S is called an *ordered quasi-ideal* of S if $(QSS] \cap (SQS] \cap (SSQ] \subseteq Q$, $(QSS] \cap (SSQSS] \cap (SSQ] \subseteq (Q]$ and $(Q] = Q$.

A nonempty subset B of S is called an *ordered generalized bi-ideal* of S if $BSBSB \subseteq B$ and $(B] = B$.

Definition 2.2. A nonempty subset I of S is called an *ordered weakly left* (resp. *right*) *ideal* of S if $SII \subseteq I$ (resp. $IIS \subseteq I$) and $(I] = I$.

Definition 2.3. A nonempty subset I of S is called an *ordered strong bi-ideal* of S if $ISI \subseteq I$ and $(I] = I$.

We note that every ordered left (resp. right) ideal of S is an ordered weakly left (resp. right) ideal of S and every ordered strong bi-ideal is an ordered generalized bi-ideal of S.

For a nonempty set A of S, we denote by $L(A)$, $R(A)$, $M(A)$, $I(A)$, $Q(A)$, $B(A)$, $L_w(A)$, $R_w(A)$ and $B_s(A)$ the ordered left ideal, the ordered right ideal, the ordered lateral ideal, the ordered ideal, the ordered quasi-ideal, the ordered generalized bi-ideal, the ordered weakly left ideal, the ordered weakly right ideal and the ordered strong bi-ideal of S generated by A, respectively. Daddi and Pawar (2012) gave some of their constructions as the following lemma.

Lemma 2.4. Let A be a nonempty subset of S. Then the following assertions hold.

(i) $L(A) = (A \cup SSA]$.
(ii) $R(A) = (A \cup ASS]$.
(iii) $M(A) = (A \cup SAS \cup SSASS]$.
(iv) $I(A) = (A \cup SSA \cup ASS \cup SAS \cup SSASS]$.
(v) $Q(A) = (A \cup SSA] \cap (A \cup SAS \cup SSASS] \cap (A \cup ASS]$.
(vi) $B(A) = (A \cup ASASA]$.

Lemma 2.5. Let A be a nonempty subset of S. Then the following assertions hold.

(i) $L_w(A) = (A \cup SAA]$.
(ii) $R_w(A) = (A \cup AAS]$.
(iii) $B_s(A) = (A \cup ASA]$.

Proof. We show that $L_w(A) = (A \cup SAA]$.

Since $A \subseteq (A \cup SAA]$and $S(A \cup SAA](A \cup SAA] \subseteq (SAA \cup SASAA \cup SSAAA \cup SSAASAA] \subseteq (A \cup SAA]$, $(A \cup SAA]$ is an ordered weakly left ideal of S containing A. Then $L_w(A) \subseteq (A \cup SAA]$. Let L be an ordered weakly left ideal of S containing A. We have

$$(A \cup SAA] \subseteq (L \cup SLL] \subseteq (L \cup L] = (L] = L.$$

Hence, $L_w(A) = (A \cup SAA]$. Similarly, we can show that $R_w(A) = (A \cup AAS]$ and $B_s(A) = (A \cup ASA]$.

In particular, for $a \in S$, we write $L(a), R(a), M(a), I(a), Q(a), B(a), L_w(a), R_w(a)$ and $B_s(a)$ instead of $L(\{a\}), R(\{a\}), M(\{a\}), I(\{a\}), Q(\{a\}), B(\{a\})$, $L_w(\{a\}), R_w(\{a\})$ and $B_s(\{a\})$, respectively. Using Lemma 2.4 and Lemma 2.5, we easily obtain that for any $a \in S$,

(i) $L(a) = (a \cup SSa]$;
(ii) $R(a) = (a \cup aSS]$;
(iii) $M(a) = (a \cup SaS \cup SSaSS]$;
(iv) $I(a) = (a \cup SSa \cup aSS \cup SaS \cup SSaSS]$;
(v) $Q(a) = (a \cup SSa] \cap (a \cup SaS \cup SSaSS] \cap (a \cup aSS]$;
(vi) $B(a) = (a \cup aSaSa]$;
(vii) $L_w(a) = (a \cup Saa]$;
(viii) $R_w(a) = (a \cup aaS]$;
(ix) $B_s(a) = (a \cup aSa]$.

3. Linear Inequations with 5-Digit of Regularities

Let X be a countable set, $x \in X$ be called a variable and $c \notin X$ be called a constant. An element u in the ternary semigroup generated by $X \cup \{c\}$ is called *linear* with respect to variables if it satisfies the following conditions:

(i) the constant c appears at least once in u;
(ii) at least one of the variables appears in u;
(iii) any variable appears at most once in u.

With u mention above, we call $c \leq u$ a *linear inequation* or *regularity inequation*.

An ordered ternary semigroup S is said to be regular (Daddi & Pawar, 2012) if for all $a \in S$, the inequation $a \leq axaya$ with the variables x, y is solvable in S. This is equivalent to $a \in (aSaSa]$ for all $a \in S$.

In this section, we consider the regularity conditions determined by 5-digit elements of ordered ternary semigroups, as follows.

Let a be an element of S. Then we classify the regularity conditions determined by linear inequations with 5-digit, as follows.

(1) $a \leq aaxaa$, for some $x \in S$.
(2) $a \leq aaaxa$, for some $x \in S$.
(3) $a \leq aaxya$, for some $x, y \in S$.
(4) $a \leq axaaa$, for some $x \in S$.
(5) $a \leq axyaa$, for some $x, y \in S$.
(6) $a \leq xayaa$, for some $x, y \in S$.
(7) $a \leq xyzaa$, for some $x, z, y \in S$.
(8) $a \leq xyaaa$, for some $x, y \in S$.
(9) $a \leq xaaaa$, for some $x \in S$.
(10) $a \leq aaxay$, for some $x, y \in S$.
(11) $a \leq aaxyz$, for some $x, y, z \in S$.
(12) $a \leq aaaxy$, for some $x, y \in S$.
(13) $a \leq aaaax$, for some $x \in S$.
(14) $a \leq xaaay$, for some $x, y \in S$.
(15) $a \leq xaaya$, for some $x, y \in S$.
(16) $a \leq xyaaz$, for some $x, y, z \in S$,
(17) $a \leq xaayz$, for some $x, y, z \in S$.
(18) $a \leq axaay$, for some $x, y \in S$.
(19) $a \leq xayza$, for some $x, y, z \in S$.
(20) $a \leq xayaz$, for some $x, y, z \in S$.
(21) $a \leq axyaz$, for some $x, y, z \in S$.
(22) $a \leq waxyz$, for some $w, x, y, z \in S$.
(23) $a \leq wxyaz$, for some $w, x, y, z \in S$.
(24) $a \leq xyaza$, for some $x, y, z \in S$.
(25) $a \leq wxayz$, for some $w, x, y, z \in S$.
(26) $a \leq axayz$, for some $x, y, z \in S$.
(27) $a \leq axyza$, for some $x, y, z \in S$.
(28) $a \leq axaya$, for some $x, y \in S$.
(29) $a \leq wxyza$, for some $w, x, y, z \in S$.
(30) $a \leq awxyz$, for some $w, x, y, z \in S$.

Remark 3.1. Let S be an ordered ternary semigroup and $a \in S$.

(i) Inequations (2) and (3) are equivalent.

(ii) Inequations (4) and (5) are equivalent.

(iii) Inequations (6), (7), (8) and (9) are equivalent. These inequations are also equivalent to the inequation $a \leq xaa$ for some $x \in S$. An element a of S satisfying the inequation $a \leq xaa$ for some $x \in S$ is called *left regular* (Pornsurat & Pibaljommee, 2019).

(iv) Inequations (10), (11), (12) and (13) are equivalent. These inequations are also equivalent to the inequation $a \leq aax$ for some $x \in S$. An element a of S satisfying the inequation $a \leq aax$ for some $x \in S$ is called *right regular* (Pornsurat & Pibaljommee, 2019).

(v) Inequations (16) and (17) are equivalent.

(vi) Inequations (22) and (23). These inequations are also equivalent to the inequation $a \leq xay$ for some $x, y \in S$.

(vii) Inequations (27) and (28) are equivalent. These inequations are also equivalent to the inequation $a \leq axa$ for some $x \in S$. An element a of S satisfying the inequation $a \leq axa$ for some $x \in S$ is called *regular* (Daddi & Pawar, 2012).

(viii) Inequation (29) is equivalent to $a \leq xya$ for some $x, y \in S$ and inequation (30) is equivalent to $a \leq axy$ for some $x, y \in S$.

Proof. (i): Let a be an element in an ordered ternary semigroup S such that $a \leq aaaxa$ for some $x \in S$. Then $a \in (aaaSa]$. Since $a \in (aaaSa] \subseteq (aaSSa]$, $a \leq aaxya$ for some $x, y \in S$. Conversely, assume that $a \leq aaxya$ for some $x, y \in S$. Then $a \leq aaxya \leq a(aaxya)xya = aaa(xyaxy)a \in aaaSa$. So, $a \in (aaaSa]$. The statements

(ii) to (vii) can be proved similarly.

(viii): Assume that $a \leq wxyza$ for some $x, y, z, w \in S$. Then $a \leq w(xyz)a$. Conversely, $a \leq xy(xya) \leq xyxy(xya) = xyx(yxy)a$. Inequation (30) is equivalent to $a \leq axy$ can be proved similarly.

An element a of S is called *right weakly regular* and *left weakly regular* (Lekkoksung & Jampachon, 2014) if it satisfies the inequations (19) and (21), respectively.

An element a of S is called *completely regular*, *intra-regular*, *left lightly regular*, *right lightly regular* and *generalized regular* (Pornsurat & Pibaljommee, 2019) if it satisfies the inequations (1), (14), (24), (26) and (25), respectively.

Now, we give many characterizations of regularities of ordered ternary semigroups using many types of their ordered ideals.

Theorem 3.2. Let a be an element of S. The following statements hold.

(i) The following statements are equivalent.

(a) $a \in (aaSaa]$ for all $a \in S$.

(b) $R_w \cap B \cap L_w \subseteq (R_w B L_w]$ for every ordered weakly right ideal R_w, ordered generalized bi-ideal B and ordered weakly left ideal L_w of S.

(c) $R_w \cap B_s \cap L_w \subseteq (R_w B_s L_w]$ for every ordered weakly right ideal R_w, ordered strong bi-ideal B_s and ordered weakly left ideal L_w of S.

(ii) The following statements are equivalent.

(a) $a \in (aaaSa]$ for all $a \in S$.

(b) $R_w \cap L_w \subseteq (R_w R_w L_w]$ for every ordered weakly right ideal R_w and ordered weakly left ideal L_w of S.

(c) $R_w \cap L = (R_w R_w L]$ for every ordered weakly right ideal R_w and ordered left ideal L of S.

(d) $R_w \cap R \cap L_w \subseteq (R_w R L_w]$ for every ordered weakly right ideal R_w, ordered right ideal R and ordered weakly left ideal L_w of S.

(e) $R_w \cap R \cap L \subseteq (R_w R L]$ for every ordered weakly right ideal R, ordered right ideal R and ordered left ideal L of S.

(iii) The following statements are equivalent.

(a) $a \in (aSaaa]$ for all $a \in S$.

(b) $R_w \cap L_w \subseteq (R_w L_w L_w]$ for every ordered weakly right ideal R_w and ordered weakly left ideal L_w of S.

(c) $R \cap L_w = (R L_w L_w]$ for every ordered right ideal R, ordered weakly left ideal L_w of S.

(d) $R_w \cap L \cap L_w \subseteq (R_w L L_w]$ for every ordered right ideal R, ordered left ideal L and ordered weakly left ideal L_w of S.

(e) $R \cap L \cap L_w \subseteq (RLL_w]$ for every ordered right ideal R, ordered left ideal L and ordered weakly left ideal L_w of S.

(iv) The following statements are equivalent.

(a) $a \in (Saa]$ for all $a \in S$.

(b) $L_w = (L_wL_wL_w]$ for every ordered weakly left ideal L_w of S.

(v) The following statements are equivalent.

(a) $a \in (aaS]$ for all $a \in S$.

(b) $R_w = (R_wR_wR_w]$ for every ordered weakly right ideal R_w of S.

(vi) The following statements are equivalent.

(a) $a \in (SaaaS]$ for all $a \in S$.

(b) $L_w \cap R \subseteq (L_wL_wR]$ for every ordered weakly left ideal L_w and ordered right ideal R of S.

(c) $L \cap R_w \subseteq (LR_wR_w]$ for every ordered left ideal L and ordered weakly right ideal R_w of S.

(vii) The following statements are equivalent.

(a) $a \in (SaaSa]$ for all $a \in S$.

(b) $L \cap B \subseteq (LBB]$ for every ordered left ideal L and ordered generalized bi-ideal B of S.

(c) $L \cap B_s \subseteq (LB_sB_s]$ for every ordered left ideal L and ordered strong bi-ideal B_s of S.

(viii) The following statements are equivalent.

(a) $a \in (aSaaS]$ for all $a \in S$.

(b) $B \cap R \subseteq (BBR]$ for every ordered generalized bi-ideal B and ordered right ideal R of S.

(c) $B_s \cap R \subseteq (B_sB_sR]$ for every ordered strong bi-ideal B_s and ordered right ideal R of S.

(ix) The following statements are equivalent.

(a) $a \in (SSaaS]$ for all $a \in S$.

(b) $L \cap R \subseteq (LLR]$ for every ordered left ideal L and ordered right ideal R of S.

(c) $L \cap R \subseteq (LRR]$ for every ordered right ideal R and ordered left ideal L of S.

(x) The following statements are equivalent.

(a) $a \in (SaSSa]$ for all $a \in S$.

(b) $L = (LLL]$ for every ordered left ideal L of S.

(xi) The following statements are equivalent.

(a) $a \in (aSSaS]$ for all $a \in S$.

(b) $R = (RRR]$ for every ordered right ideal R of S.

(xii) The following statements are equivalent.

(a) $a \in (SaSaS]$ for all $a \in S$.

(b) $B \subseteq (SBSBS]$ for every ordered generalized bi-ideal B of S.

(c) $B_s \subseteq (SB_sSB_sS]$ for every ordered strong bi-ideal B_s of S.

(xiii) The following statements are equivalent.

(a) $a \in (SaS]$ for all $a \in S$.

(b) $L_w \subseteq (SL_wS]$ for every ordered weakly left ideal L_w of S.

(c) $L \subseteq (SLS]$ for every ordered left ideal L of S.

(d) $R_w \subseteq (SR_wS]$ for every ordered weakly right ideal R_w of S.

(e) $R \subseteq (SRS]$ for every ordered right ideal R of S.

(xiv) The following statements are equivalent.

(a) $a \in (SSaSa]$ for all $a \in S$.

(b) $R \cap M \cap L \subseteq (SSRML]$ for every ordered right ideal R ordered lateral ideal M and ordered left ideal L of S.

(c) $L \cap M = (LML]$ for every ordered left ideal L and ordered lateral ideal M of S.

(xv) The following statements are equivalent.

(a) $a \in (SSaSS]$ for all $a \in S$.

(b) $L \subseteq (SSLSS]$ for every ordered left ideal L of S.

(c) $R \subseteq (SSRSS]$ for every ordered right ideal R of S.

(d) $M = (SSMSS]$ for every ordered lateral ideal M of S.

(xvi) The following statements are equivalent.

(a) $a \in (aSaSS]$ for all $a \in S$.

(b) $L \cap M \cap R \subseteq (LMRSS]$ for every ordered left ideal L, ordered lateral ideal M and ordered right ideal R of S.

(c) $M \cap R = (RMR]$ for every ordered right ideal R and ordered lateral ideal M of S.

(xvii) The following statements are equivalent.

(a) $a \in (aSa]$ for all a $\in S$.

(b) $R \cap M \cap L \subseteq (RML]$ for every ordered right ideal R, ordered lateral ideal M and ordered left ideal L of S.

(xviii) The following statements are equivalent.

(a) $a \in (SSa]$ for all $a \in S$.

(b) $L = (SSL]$ for every ordered left ideal L of S.

(xix) The following statements are equivalent.

(a) $a \in (aSS]$ for all $a \in S$.

(b) $R = (RSS]$ for every ordered right ideal R of S.

Proof. We refer to the proofs of (x) and (xi) by (Lekkoksung & Jampachon, 2014), (xiv), (xv) and (xvi) by (Pornsurat & Pibaljommee, 2019) and (xvii) by (Daddi & Pawar, 2012).

(i): (a) ⇒ (b): Let $b \in R_w \cap B \cap L_w$. By assumption, there exists $x \in S$ such that $b \leq bbxbb$. Since $bb(xbb) \in R_wBL_w, b \in (R_wBL_w]$.

(b) ⇒ (c): It is clear because every ordered strong bi-ideal is an ordered generalized bi-ideal.

(c) ⇒ (a): Let $a \in S$. We have

$$\begin{aligned} a \in\ & R_w(a) \cap B_s(a) \cap L_w(a) \\ \subseteq\ & (R_w(a)B_s(a)L_w(a)] \\ =\ & \big((a \cup aaS](a \cup aSa](a \cup Saa]\big] \\ \subseteq\ & (aaa \cup aaSaa \cup aaSaa \cup aaSaSaa \cup aaSaa \cup aaSaSaa \cup aaSaSaa \cup aaSaSaSaa] \\ =\ & (aaa] \cup (aaSaa]. \end{aligned}$$

If $a \in (aaa]$, then $a \leq aaa \leq (aaa)aa \in (aaSaa]$. Now, we have $a \in (aaSaa]$ for all $a \in S$.

(ii): (a) ⇒ (b): Let $b \in R_w \cap L_w$. By assumption, there exists $x \in S$ such that $b \leq bbbxb$. Since $b(bbx)b \in R_wR_wL_w, b \in (R_wR_wL_w]$.

(b) ⇒ (c): It is clear because every ordered left ideal is an ordered weakly left ideal.

(b) ⇒ (d): It is clear because every ordered right ideal is an ordered weakly right ideal.

(c) ⇒ (e): It is clear because every ordered right ideal is an ordered weakly right ideal.

(d) ⇒ (e): We can similarly prove as (c) ⇒ (e).
(e) ⇒ (a): Let $a \in S$. We have
$a \in R_w(a) \cap R(a) \cap L(a)$
$\subseteq (R_w(a)R(a)L(a)]$
$= \big((a \cup aaS](a \cup aSS](a \cup SSa]\big]$
$\subseteq (aaa \cup aaSSa \cup aaSSa \cup aaSSSSa \cup aaSaa \cup aaSaSSa \cup aaSaSSa \cup aaSaSSSSa]$
$= (aaa] \cup (aaSSa] \cup (aaSSSSa] \cup (aaSaa] \cup (aaSaSSa] \cup (aaSaSSSSa]$
$\subseteq (aaa] \cup (aaSSa] \cup (aaSaa]$.
If $a \in (aaa], then\ a \leq aaa \leq (aaa)aa \in (aaaSa]$. If $a \in (aaSaa]$, then there exists $x \in S$ such that $a \leq aaxaa \leq a(aaxaa)xaa \in (aaaSa]$. By Remark 3.1 (i), $a \in (aaSSa]$ is equivalent to $a \in (aaaSa]$. Now, we have $a \in (aaaSa]$ for all $a \in S$.
(iii): We can similarly prove as (ii).
(iv): Clearly, $L_w \supseteq (L_wL_wL_w]$. Let $b \in L_w$. By assumption, there exists $x \in S$ such that $b \leq xbb$. Since $xbb \leq x(xbb)b \leq xx(xbb)bb \in L_wL_wL_w,\ b \in (L_wL_wL_w]$.
So, $L_w \subseteq (L_wL_wL_w]$. Conversely, let $a \in S$. We have
$a \in L_w(a)$
$= (L_w(a)L_w(a)L_w(a)]$
$= \big((a \cup Saa](a \cup Saa](a \cup Saa]\big]$
$\subseteq (aaa \cup aaSaa \cup aSaaa \cup aSaaSaa \cup Saaaa \cup SaaaSaa \cup SaaSaaa \cup SaaSaaSaa]$
$= (aaa] \cup (aaSaa] \cup (aSaaa] \cup (aSaaSaa] \cup (Saaaa] \cup (SaaaSaa] \cup (SaaSaaa](SaaSaaSaa]$
$\subseteq (Saa]$.
(v): We can similarly prove as (iv).
(vi): (a) ⇒ (b): Let $b \in L_w \cap R$. By assumption, there exist $x, y \in S$ such that $b \leq xbbby$. Since $b \leq xbbby \leq xbb(xbbby)y \in L_wL_wR$, $b \in (L_wL_wR]$.
(b) ⇒ (a): Let $a \in S$. We have
$a \in L_w(a) \cap R(a)$
$\subseteq (L_w(a)L_w(a)R(a)]$
$= \big((a \cup Saa](a \cup Saa](a \cup aSS]\big]$
$\subseteq (aaa \cup aaaSS \cup aSaaa \cup aSaaaSS \cup Saaaa \cup SaaaaSS \cup SaaSaaa \cup SaaSaaaSS]$
$= (aaa] \cup (aaaSS] \cup (aSaaa] \cup (aSaaaSS] \cup (Saaaa] \cup (SaaaaSS] \cup (SaaSaaa] \cup (SaaSaaaSS]$
$\subseteq (aaa] \cup (aaaSS] \cup (SSaaa] \cup (SSaaaSS]$.
If $a \in (aaa]$, then $a \leq aaa \leq (aaa)aa \in (SaaaS]$. If $a \in (aaaSS]$, then there exist $x, y \in S$ such that $a \leq aaaxy \leq aa(aaaxy)xy \in (SaaaS]$. Similarly, we can show that if $a \in (SSaaa], then\ a \in (SaaaS]$. If $a \in (SSaaaSS]$, then there exist$w, x, y, z \in S$ such that $a \leq wxaaayz \leq wxa(wxaaayz)ayz \in (SaaaS]$. Now, we have $a \in (SaaaS]$ for all $a \in S$.
(a) ⇔ (c): It can be proved similarly.
(vii) (a) ⇒ (b): Let $b \in L \cap B$. By assumption, there exist $x, y \in S$ such that $b \leq xbbyb$.
Since $b \leq xbbyb \leq x(xbbyb)byb \leq xxbbyb(xbbyb)yb \in LBB$, $b \in (LBB]$.
(b) ⇒ (c): It is clear because every ordered strong bi-ideal is an ordered generalized bi-ideal.
(c) ⇒ (a): Let $a \in S$. We have
$a \in L(a) \cap B_s(a)$
$\subseteq (L(a)B_s(a)B_s(a)]$
$= \big((a \cup SSa](a \cup aSa](a \cup aSa]\big]$
$\subseteq (aaa \cup aaaSa \cup aaSaa \cup aaSaaSa \cup SSaaa \cup SSaaaSa \cup SSaaSaa \cup SSaaSaaSa]$
$= (aaa] \cup (aaaSa] \cup (aaSaa] \cup (aaSaaSa] \cup (SSaaa] \cup (SSaaaSa] \cup (SSaaSaa] \cup (SSaaSaaSa]$
$\subseteq (aaa] \cup (SaaSa] \cup (aaSaa] \cup (SSaaa] \cup (SSaaSaa]$.
If $a \in (aaa]$, then $a \leq aaa \leq (aaa)aa \in (SaaSa]$. If $a \in (aaSaa]$, then there exists $x \in S$ such that $a \leq aaxaa \leq a(aaxaa)xaa \in (SaaSa]$. If $a \in (SSaaa]$ then there exist $x, y \in S$ such that $a \leq xyaaa \leq xy(xyaaa)aa \in (SaaSa]$. If $a \in (SSaaSaa]$, then there exist $x, y, z \in S$ such that $a \leq xyaazaa \leq xy(xyaazaa)azaa \in (SaaSa]$.
Now, we have $a \in (SaaSa]$ for all $a \in S$.
(viii): We can similarly prove as (vii).
(ix): (a) ⇒ (b): Let $b \in L \cap R$. There exist $x, y, z \in S$ such that $b \leq xybbz$.
Since $b \leq xyb(xybbz)z \in LLR$, $b \in (LLR]$.
(b) ⇒ (a): Let $a \in S$. We have
$a \in L(a) \cap R(a)$
$\subseteq (L(a)L(a)R(a)]$
$= \big((a \cup SSa](a \cup SSa](a \cup aSS]\big]$
$\subseteq (aaa \cup aaaSS \cup aSSaa \cup aSSaaSS \cup SSaaa \cup SSaaaSS \cup SSaSSaa \cup SSaSSaaSS]$
$= (aaa] \cup (aaaSS] \cup (aSSaa] \cup (aSSaaSS] \cup (SSaaa] \cup (SSaaaSS] \cup (SSaSSaa] \cup (SSaSSaaSS]$
$\subseteq (aaa] \cup (SSaaS] \cup (SaaSS] \cup (Saa]$.
If $a \in (aaa]$, then $a \leq aaa \leq (aaa)aa \in (SSaaS]$. By Remark 3.1(v), $a \in (SaaSS]$ is equivalent to $a \in (SSaaS]$. If $a \in (Saa]$, then there exists $x \in S$ such that $a \leq xaa \leq x(xaa)a \in (SSaaS]$. Now, we have $a \in (SSaaS]$ for all $a \in S$.
(a)⇔(c): It can be proved similarly.
(xii): (a) ⇒ (b): Let $b \in B$. By assumption, there exist $x, y, z \in S$ such that $b \leq xbybz$.
It is obvious that $b \in (SBSBS]$.
(b) ⇒ (c): It is clear because every ordered strong bi-ideal is an ordered generalized bi-ideal.
(c) ⇒ (a): Let $a \in S$. We have
$a \in B_s(a)$
$\subseteq (SB_s(a)SB_s(a)S]$
$= (S(a \cup aSa]S(a \cup aSa]S]$
$\subseteq (SaSaS \cup SaSaSaS \cup SaSaSaS \cup SaSaSaSaS]$
$\subseteq (SaSaS]$.
(xiii): (a) ⇒ (b): Let $b \in L_w$. By assumption, there exist $x, y \in S$ such that $b \leq xby$. It is obvious that $b \in (SL_wS]$.
(b) ⇒ (c): It is clear because every ordered left ideal is an ordered weakly left ideal.
(c) ⇒ (a): Let $a \in S$. We have $a \in L(a) \subseteq (SL(a)S] = (S(a \cup SSa]S] \subseteq (SaS \cup SSSaS] \subseteq (SaS]$.
(a) ⇔ (d) ⇔ (e): It can be proved similarly.
(xviii): Cleary, $L \supseteq (SSL]$. Let $b \in L$. By assumption, there exist $x, y \in S$ such that $b \leq xyb$. It is obvious that $b \in (SSL]$.
So, $L \subseteq (SSL]$. Conversely, let $a \in S$. We have
$a \in L(a)$
$= (SSL(a)]$
$= \big(SS(a \cup SSa]\big]$
$\subseteq (SSa \cup SSSSa]$
$= (SSa] \cup (SSSSa]$
$\subseteq (SSa]$.
(xix): We can similarly prove as (xviii).

4. Connections among Regularities of Ordered Ternary Semigroups

Now, we present connections among the regularities as mentioned in the preceding sections. The proof of the following proposition is not difficult.

Proposition 4.1. Let S be an ordered ternary semigroup and a be an element of S.

(1) If $a \in (aaSaa]$, then $a \in (aSaaa]$, $a \in (aaaSa]$ and $a \in (aSaSa]$.

(2) If $a \in (aSaaa]$ (resp. $a \in (aaaSa]$), then $a \in (Saaaa]$ (resp. $a \in (aaaaS]$).

(3) If $a \in (Saaaa]$ (resp. $a \in (aaaaS]$), then $a \in (SaaaS]$ and $a \in (SaaSa]$ (resp. $a \in (SaaaS]$ and $a \in (aSaaS]$.

(4) If $a \in (SaaSa]$ (resp. $a \in (aSaaS]$), then $a \in (SSaaS]$ and $a \in (SaSSa]$ (resp. $a \in (SaaSS]$ and $a \in (aSSaS]$).

(5) If $a \in (SaSSa]$ (resp. $a \in (aSSaS]$), then $a \in (SaSaS]$ and $a \in (SSaSa]$ (resp. $a \in (SaSaS]$ and $a \in (aSaSS]$).

(6) If $a \in (SSaSa]$ (resp. $a \in (aSaSS]$), then $a \in (SSSSa]$ and $a \in (SSaSS]$ (resp. $a \in (aSSSS]$ and $a \in (SSaSS]$).

(7) If $a \in (aSaSa]$, then $a \in (SSaSa]$ and $a \in (aSaSS]$.

(8) If $a \in (SaaaS]$, then $a \in (SSaaS]$.

(9) If $a \in (SSaaS]$, then $a \in (SaSaS]$.

(10) If $a \in (SaSaS]$, then $a \in (SaSSS]$.

(11) If $a \in (SaSSS]$, then $a \in (SSaSS]$.

Proof. It is clear (3), (6), (7), (8) and (10) are true.

(1): Let $a \in (aaSaa]$. We obtain $a \in (aaSaa] \subseteq (a(aaSaa]Saa] \subseteq (aSaSa]$ and $a \in (aaSaa] \subseteq (aaS(aaSaa]a] \subseteq (aaSaaSaaa] \subseteq (aSaaa]$. Similarly, we can show that if $a \in (aaSaa]$, then $a \in (aaaSa]$.

(2): Let $a \in (aSaaa]$. We obtain $a \in (aSaaa] \subseteq (aS(aSaaa]aa] \subseteq (Saaaa]$. Similarly, we can show that if $a \in (aaaSa]$, then $a \in (aaaaS]$.

(4): Let $a \in (SaaSa]$. We obtain $a \in (SaaSa] \subseteq (S(SaaSa]aSa] \subseteq (SSaaS]$. It is clear that if $a \in (SaaSa]$, then $a \in (SaaSS]$. Similarly, we can show that if $a \in (aSaaS]$, then $a \in (SaaSS]$ and $a \in (aSSaS]$.

(5): Let $a \in (SaSSa]$. We obtain $a \in (SaSSa] \subseteq (SaSS(SaSSa] \subseteq (SaSaS]$ and $a \in (SaSSa] \subseteq (S(SaSSa]SSa] \subseteq (SSaSa]$. Similarly, we can show that if $a \in (aSSaS]$, then $a \in (SaSaS]$ and $a \in (aSaSS]$.

(9): Let $a \in S$. Then $a \in (SSaaS] \subseteq (SS(SSaaS]aS] \subseteq (SaSaS]$.

(11): Let $a \in (SaSSS]$. We obtain $a \in (SaSSS] \subseteq (S(SaSSS]SSS] \subseteq (SSaSS]$.

For an ordered ternary semigroup S and $a \in S$, we conclude the connections among nineteen types of regular element a as the following figure.

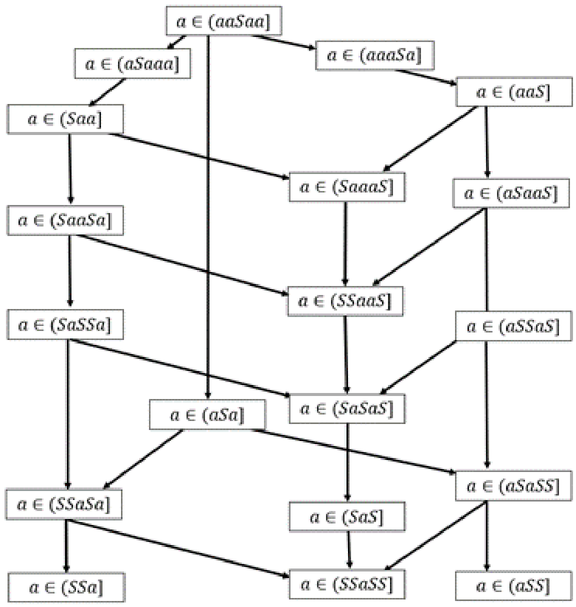

Figure 1. The connections of the regularities.

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