Songklanakarin J. Sci. Technol.
44 (2), 323-331, Mar. - Apr. 2022

http://www.sjst.psu.ac.th

Original Article

Periods of k-step Fibonacci functions modulo m

Yanapat Tongron, and Supattra Kerdmongkon*

Mathematics and Statistics Program, Faculty of Science and Technology,
Nakhon Ratchasima Rajabhat University, Mueang, Nakhon Ratchasima, 30000 Thailand

Received: 11 April 2021; Revised: 5 August 2021; Accepted: 30 August 2021

Abstract

For an integer $k \geq 2$, a k-step Fibonacci function is a function $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n+k) = f(n+k-1) + f(n+k-2) + \cdots + f(n)$ for any integer n. We mainly show the existence of primitive period of a k-step Fibonacci function in modulo m. Moreover, the explicit primitive period of a k-step Fibonacci function, when $k = 2,3,4$, under some conditions is also established.

Keywords: Fibonacci functions, k-step Fibonacci function, primitive period modulo m

1. Introduction

The Fibonacci numbers, commonly denoted by F_n, form a sequence called the Fibonacci sequence for which each number is the sum of the two preceding ones starting from 0 and 1. That is, (Koshy, 2001)

$$F_0 = 0,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2}$$

for any natural number $n \geq 2$. The beginning of the sequence is thus:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

Similar to the Fibonacci numbers, each Lucas number, commonly denoted by L_n is defined to be the sum of the two previous terms starting from 2 and 1. That is (Koshy, 2001)

$$L_0 = 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2}$$

for any natural number $n \geq 2$. The beginning of the sequence is thus:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \ldots$$

This sequence is called the Lucas sequence.

In 1967, Elmore (1967) studied a Fibonacci function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x+2) = f(x+1) + f(x)$ for any real number x. Considering such a Fibonacci function $f: \mathbb{Z} \to \mathbb{Z}$, we have the Fibonacci sequence if $f(0) = 0$ and $f(1) = 1$. In addition, we have the Lucas sequence if $f(0) = 2$ and $f(1) = 1$.

Thongngam and Chinram (2019) recently studied periods of a Fibonacci function $f: \mathbb{Z} \to \mathbb{Z}$ modulo m as follows:

Theorem 1. (Thongngam & Chinram, 2019) Let $f: \mathbb{Z} \to \mathbb{Z}$ be a Fibonacci function.
1) If $f(0) = 0$, then $f(-n) = (-1)^{n+1} f(n)$ for any positive integer n.
2) If $f(0) = 2f(1)$, then $f(-n) = (-1)^n f(n)$ for any positive integer n.
Here is the result that we are mainly going to generalize in our work.

Theorem 2. (Thongngam & Chinram, 2019) If $f: \mathbb{Z} \to \mathbb{Z}$ is a Fibonacci function and m is a positive integer > 1, then there exists an integer $1 \leq l \leq m^2$ such that $f(n+l) \equiv f(n) \pmod{m}$ for any integer n.

Definition 1. (Thongngam & Chinram, 2019) Let $f: \mathbb{Z} \to \mathbb{Z}$ be a Fibonacci function and m be a positive integer > 1. A positive integer l such that $f(n+l) \equiv f(n) \pmod{m}$ for any integer n is called a *Period* of f modulo m. The *Primitive Period* of f modulo m, written as $l \coloneqq l(m)$ is the smallest of such positive integers l.

Theorem 3. (Thongngam & Chinram, 2019) Let l be a positive integer. l is a period of f modulo m if and only if $l(m) | l$.

Theorem 4. (Thongngam & Chinram, 2019) Let m and n be positive integers > 1. If $\gcd(m,n) = 1$, then $l(mn) = \text{lcm}[l(m), l(n)]$.

*Corresponding author
Email address: supattra.k@nrru.ac.th

These results are generalizations on periods of the Fibonacci and the Lucas sequences. See (Jameson, 2018) for more information about periods of the Fibonacci sequence.

In this work, we are interested in a k-step Fibonacci function $f:\mathbb{Z}\to\mathbb{Z}$ defined by $f(n+k)=f(n+k-1)+f(n+k-2)+\cdots+f(n)$ for all integers n and $k\geq 2$. Notice that a 2-step Fibonacci function is a regular Fibonacci function. Some elementary properties on periods modulo m of such a function are provided and the above theorems are special cases of our results. Moreover, we give the primitive period of some certain k-step Fibonacci functions under some additional conditions.

2. Main Results

Again, a k-step Fibonacci function $f:\mathbb{Z}\to\mathbb{Z}$ is defined by $f(n+k)=f(n+k-1)+f(n+k-2)+\cdots+f(n)$ for all integers n and $k\geq 2$.

Example 1. Let $f:\mathbb{Z}\to\mathbb{Z}$ be a 3-step Fibonacci function such that $f(0)=0$, $f(1)=1$ and $f(2)=2$. Then we have the following tables:

Table 1. The values of the 3-step Fibonacci function $f(n)$

n	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	9	-38	27	-2	-13	12	-3	-4	5	-2	-1	2	-1	0	1
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f(n)$	0	1	2	3	6	11	20	37	68	125	230	423	778	1431	2632

Note that $f(n)=2f(n+3)-f(n+4)$, for example, $f(-16)=2f(-13)-f(-12)$, $f(-15)=2f(-12)-f(-11)$, $f(-14)=2f(-11)-f(-10)$, ...
The following lemma shows that $f(n)$ depends on only some two consecutive functions for any integer n.

Lemma 1. Let f be a k-step Fibonacci function. Then $f(n)=2f(n+k)-f(n+k+1)$ for any integer n.

Proof. Let n be an integer. Then we have

$$
\begin{aligned}
(f(n) &= f(n+k+1)+f(n)-f(n+k+1)\\
&= f(n+k)+f(n+k-1)+\cdots+f(n+1)+f(n)-f(n+k+1)\\
&= f(n+k)+f(n+k)-f(n+k+1)\\
&= 2f(n+k)-f(n+k+1)\ )
\end{aligned}
$$

as desired. □

Lemma 1 yields another proof of Theorem 1 as follows:
Another proof of Theorem 1.

1) Let $f(0)=0$. From Lemma 1, we have

$$
\begin{aligned}
f(-1) &= 2f(1)-f(2)\\
&= 2f(1)-\big(f(1)+f(0)\big)\\
&= (-1)^{1+1}f(1),\\
f(-2) &= 2f(0)-f(1)\\
&= -\big(f(2)-f(0)\big)\\
&= (-1)^{2+1}f(2),\\
f(-3) &= 2f(-1)-f(0)\\
&= f(1)+f(1)+f(0)\\
&= (-1)^{3+1}f(3).)
\end{aligned}
$$

Assume that $f(-l)=(-1)^{l+1}f(l)$ for all $1\leq l\leq n$ and $n\geq 3$. By Lemma 1, consider

$$
\begin{aligned}
(f(-(n+1)) &= 2f(-(n-1))-f(-(n-2))\\
&= (-1)^{n}\big(f(n-1)+f(n-1)+f(n-2)\big)\\
&= (-1)^{n+1+1}f(n+1).
\end{aligned}
$$

The statement holds by the Principle of Strong Mathematical Induction.

2) Let $f(0)=2f(1)$. From Lemma 1, we have

$$
\begin{aligned}
f(-1) &= 2f(1)-f(2)\\
&= 2f(1)-\big(f(1)+f(0)\big)\\
&= (-1)^{1}f(1),\\
f(-2) &= 2f(0)-f(1)\\
&= f(1)+f(0)\\
&= (-1)^{2}f(2),
\end{aligned}
$$

$$f(-3) = 2f(-1) - f(0)$$
$$= -\big(f(1) + f(1) + f(0)\big)$$
$$= (-1)^3 f(3).$$

Assume that $f(-l) = (-1)^l f(l)$ for all $1 \le l \le n$ and $n \ge 3$. By Lemma 1, consider

$$(f\big(-(n+1)\big) = 2f(-(n-1)) - f(-(n-2))$$
$$= (-1)^{n-1}(f(n-1) + f(n-1) + f(n-2))$$
$$= (-1)^{n-1} f(n+1).)$$

The statement holds by the Principle of Strong Mathematical Induction. □

Next, we study some properties on periods modulo m of k-step Fibonacci functions.

Theorem 5. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a k-step Fibonacci function and m be a positive integer > 1. Then there exists an integer $1 \le l \le m^k$ such that $f(n+l) \equiv f(n) \pmod{m}$ for any integer n.

Proof. Consider all integers $0 \le a \le m^k$ which can be $m^k + 1$ possible values. And consider k-tuple $\big(f(a), f(a+1), \dots, f(a+k-1)\big)$ modulo m which can be m^k possible values as follows:

$$\begin{array}{cccc} (0,0,\dots,0,0), & (0,0,\dots,0,1), & \dots, & (0,0,\dots,0,m-1), \\ (0,0,\dots,1,0), & (0,0,\dots,1,1), & \dots, & (0,0,\dots,1,m-1), \\ \vdots & \vdots & & \vdots \\ (m-1,\dots,m-1,0), & (m-1,\dots,m-1,1), & \dots, & (m-1,\dots,m-1,m-1). \end{array}$$

By the Pigeonhole Principle, there exist integers $0 \le i < j \le m^k$ such that $\big(f(j), f(j+1), \dots, f(j+k-1)\big) \equiv \big(f(i), f(i+1), \dots, f(i+k-1)\big) \pmod{m}$.

Then $f(j+\alpha) \equiv f(i+\alpha) \pmod{m}$ for all $\alpha \in \{0,1,\dots,k-1\}$. Put $l := j - i$ and observe that $1 \le l \le m^k$. Note that $f(i+\alpha+l) \equiv f(i+\alpha) \pmod{m}$ for all $\alpha \in \{0,1,\dots,k-1\}$. We divide our proof into two cases: $n \ge i$ and $n \le i$.

Case 1: Assume that $f(r+l) \equiv f(r) \pmod{m}$ for all $i \le r \le n$ and $n \ge i + k - 1$. Then

$$f(n+1) = f(n) + f(n-1) + \cdots + f(n-k+1)$$
$$\equiv f(n+l) + f(n-1+l) + \cdots + f(n-k+1+l) \pmod{m}$$
$$\equiv f(n+1+l) \pmod{m}.$$

By the Principle of Strong Mathematical Induction, $f(n+l) \equiv f(n) \pmod{m}$ for any integer $n \ge i$.

Case 2: Assume that $f(r+l) \equiv f(r) \pmod{m}$ for all $n \le r \le i + k - 1$ and $n \le i$. Then

$$f(n-1) = f(n+k-1) - f(n+k-2) - \cdots - f(n)$$
$$\equiv f(n+k-1+l) - f(n+k-2+l) - \cdots - f(n+l) \pmod{m}$$
$$\equiv f(n-1+l) \pmod{m}.$$

By the Principle of Strong Mathematical Induction, $f(n+l) \equiv f(n) \pmod{m}$ for any integer $n \le i$. These complete the proof.
Observe that a period of any k-step Fibonacci function always exists.

Definition 2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a k-step Fibonacci function and m be a positive integer > 1. A positive integer l such that $f(n+l) \equiv f(n) \pmod{m}$ for any integer n is called a *Period* of f modulo m. The smallest positive integer l such that $f(n+l) \equiv f(n) \pmod{m}$ for any integer n is called the *Primitive Period* of f modulo m and written $l := l_f(m)$.

By The Well Ordering Principle, the unique primitive period of any k-step Fibonacci function modulo m always exists. The following corollary shows bounds on the primitive period. The proof of the corollary immediately follows from Theorem 6.

Corollary 1. If $l_f(m)$ is the primitive period of a k-step Fibonacci function f modulo m, then $1 \le l_f(m) \le m^k$.

The relation between a period and the primitive period of a k-step Fibonacci function modulo m is described in the following theorem.

Theorem 6. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a k-step Fibonacci function and l, m be positive integers > 1. l is a period of f modulo m if and only if $l_f(m) \mid l$.

Proof. Assume that l is a period of f modulo m. Then $l \ge l_f(m)$. By the Division Algorithm, there exist two positive integers q and r with $0 \le r < l_f(m)$ for which $l = q l_f(m) + r$.
Consider

$$\begin{aligned} f(n) &\equiv f(n+l) \pmod m \\ &\equiv f\left(n+r+ql_f(m)\right) \pmod m \\ &\equiv f\left(n+r+(q-1)l_f(m)+l_f(m)\right) \pmod m \\ &\equiv f\left(n+r+(q-1)l_f(m)\right) \pmod m \\ &\equiv f\left(n+r+(q-2)l_f(m)+l_f(m)\right) \pmod m \\ &\equiv f\left(n+r+(q-2)l_f(m)\right) \pmod m \\ &\vdots \\ &\equiv f\left(n+r+l_f(m)\right) \pmod m \\ &\equiv f(n+r) \pmod m \end{aligned}$$

for any integer n. If $r > 0$, then r is a period of f modulo m with $r < l_f(m)$: a contradiction. Thus, $r = 0$ and so $l = ql_f(m)$. It follows that $l_f(m) \mid l$.

Conversely, assume that $l_f(m) \mid l$. Then $l = ql_f(m)$ for some positive integer q. Consider

$$\begin{aligned} f(n) &\equiv f\left(n+l_f(m)\right) \pmod m \\ &\equiv f\left(n+2l_f(m)\right) \pmod m \\ &\equiv f\left(n+3l_f(m)\right) \pmod m \\ &\vdots \\ &\equiv f\left(n+ql_f(m)\right) \pmod m \\ &\equiv f(n+l) \pmod m \end{aligned}$$

for any integer n. Hence, l is a period of f modulo m. □

Example 2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 3-step Fibonacci function such that $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$. From Example 1, we have the following tables:

Table 2. The values of the 3-step Fibonacci function $f(n)$ in modulo 2 and 3

n	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	27	-2	-13	12	-3	-4	5	-2	-1	2	-1	0	1
$f(n)$ (mod 2)	1	0	1	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	0	1	2	0	0	2	2	1	2	2	2	0	1

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	0	1	2	3	6	11	20	37	68	125	230	423	778
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1	0	1	0
$f(n)$ (mod 3)	0	1	2	0	0	2	2	1	2	2	2	0	1

Then $1 \le l_f(2) = 2 \le 2^3$ and $1 \le l_f(3) = 13 \le 3^3$. Observe that 4 is also a period of f modulo 2 and $l_f(2) \mid 4$. Similarly, we can calculate that 26 is also a period of f modulo 3 and $l_f(3) \mid 26$.

The final theorem of this section explains a relation among $l_f(m)$, $l_f(n)$ and $l_f(mn)$ when $\gcd(m,n) = 1$.

Theorem 7. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a k-step Fibonacci function and m, n be positive integers > 1. If $\gcd(m,n) = 1$, then $l_f(mn) = \operatorname{lcm}\left[l_f(m), l_f(n)\right]$.

Proof. Assume that $\gcd(m,n) = 1$. Since $l_f(mn)$ is a period of f modulo mn, $f\left(N + l_f(mn)\right) \equiv f(N) \pmod{mn}$ for any integer N.

Then $f\left(N + l_f(mn)\right) \equiv f(N) \pmod m$ and $f\left(N + l_f(mn)\right) \equiv f(N) \pmod n$

for any integer N. Consequently, $l_f(mn)$ is a period of f modulo m and n respectively. By Theorem 6, we have $l_f(m) \mid l_f(mn)$ and $l_f(n) \mid l_f(mn)$.

Therefore, $\operatorname{lcm}\left[l_f(m), l_f(n)\right] \mid l_f(mn)$.

Conversely, since $l_f(m) \mid \operatorname{lcm}\left[l_f(m), l_f(n)\right]$ and $l_f(n) \mid \operatorname{lcm}\left[l_f(m), l_f(n)\right]$, we obtain from Theorem 6 that $\operatorname{lcm}\left[l_f(m), l_f(n)\right]$ is a period of f modulo m and n respectively. Hence,

$$f\big(N+\mathrm{lcm}[l_f(m),l_f(n)]\big)\equiv f(N) \pmod m \text{ and } f\big(N+\mathrm{lcm}[l_f(m),l_f(n)]\big)\equiv f(N) \pmod n$$

for any integer N. Since $\gcd(m,n)=1$,

$$f\big(N+\mathrm{lcm}[l_f(m),l_f(n)]\big)\equiv f(N) \pmod{mn}$$

for any integer N and so $\mathrm{lcm}[l_f(m),l_f(n)]$ is a period of f modulo mn. We conclude from Theorem 6 that

$$l_f(mn)\mid \mathrm{lcm}[l_f(m),l_f(n)].$$

The proof is complete.

Example 3. Let $f:\mathbb{Z}\to\mathbb{Z}$ be a 3-step Fibonacci function such that $f(0)=0$, $f(1)=1$ and $f(2)=2$. Consider the following tables:

Table 3. The values of the 3-step Fibonacci function $f(n)$ in modulo 2, 3, and 6

n	−32	−31	−30	−29	−28	−27	−26	−25	−24	−23
$f(n)$	−5264	−3055	5426	−2893	−522	2011	−1404	85	692	−627
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	1	2	2	2	0	1	0	1	2	0
$f(n)$ (mod 6)	4	5	2	5	0	1	0	1	2	3

n	−22	−21	−20	−19	−18	−17	−16	−15	−14	−13
$f(n)$	150	215	−262	103	56	−103	56	9	−38	27
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	0	2	2	1	2	2	2	0	1	0
$f(n)$ (mod 6)	0	5	2	1	2	5	2	3	4	3

n	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3
$f(n)$	−2	−13	12	−3	−4	5	−2	−1	2	−1
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	1	2	0	0	2	2	1	2	2	2
$f(n)$ (mod 6)	4	5	0	3	2	5	4	5	2	5

n	−2	−1	0	1	2	3	4	5	6	7
$f(n)$	0	1	0	1	2	3	6	11	20	37
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	0	1	0	1	2	0	0	2	2	1
$f(n)$ (mod 6)	0	1	0	1	2	3	0	5	2	1

n	8	9	10	11	12	13	14	15	16	17
$f(n)$	68	125	230	423	778	1431	2632	4841	8904	16377
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	2	2	2	0	1	0	1	2	0	0
$f(n)$ (mod 6)	2	5	2	3	4	3	4	5	0	3

n	18	19	20	21	22	23	24	25	26	27
$f(n)$	30122	55403	101902	187427	344732	634061	1166220	2145013	3945294	7256527
$f(n)$ (mod 2)	0	1	0	1	0	1	0	1	0	1
$f(n)$ (mod 3)	2	2	1	2	2	2	0	1	0	1
$f(n)$ (mod 6)	2	5	4	5	2	5	0	1	0	1

We see that $l_f(6)=26=\mathrm{lcm}[2,13]=\mathrm{lcm}[l_f(2),l_f(3)]$.

4. Explicit Primitive Periods

In this section, we find the explicit period of a k-step Fibonacci function f under some conditions for $k = 2,3,4$. Let $f(0), f(1), \dots, f(k-1)$ be the starting values. We observe that l is a period of f modulo m if and only if l is the first positive integer so that $f(n+l) \equiv f(n) \pmod{m}$ for all $n \in \{0,1,\dots,k-1\}$. First, we provide necessary and sufficient conditions for a k-step Fibonacci function f such that $l_f(m) = 1$.

Lemma 2. Let m be a positive integer and $f: \mathbb{Z} \to \mathbb{Z}$ be a k-step Fibonacci function with the starting values $f(0) = a_0, f(1) = a_1, \dots, f(k-1) = a_{k-1}$ and $\gcd(m, k-1) = 1$. Then $m|a_i$ for all $i \in \{0,1,\dots,k-1\}$ if and only if $l_f(m) = 1$.

Proof. Assume that $a_0 \equiv a_1 \equiv \cdots \equiv a_{k-1} \equiv 0 \pmod{m}$. We can prove by induction on n with the definition of a k-step Fibonacci function that $f(n+1) \equiv f(n) \equiv 0 \pmod{m}$ for all integers n and so $l_f(m) = 1$. On the other hand, assume that $l_f(m) = 1$. Now, we have that $a_0 \equiv a_1 \equiv \cdots \equiv a_{k-1} \equiv a_0 + a_1 + \cdots + a_{k-2} + a_{k-1} \pmod{m}$.
Then

$$(k-1)a_i = \overbrace{a_i + a_i + \cdots + a_i}^{k-1 \text{ terms}} \equiv a_0 + a_1 + \cdots + a_{k-2} \equiv 0 \pmod{m}.$$

for all $i \in \{0,1,\dots,k-1\}$. The result follows from $\gcd(m, k-1) = 1$.

Now, we show the desired results for a k-step Fibonacci function with a period greater than 1. We start with the result for $k = 2$ as follows:

Theorem 8. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 2-step Fibonacci function with the starting values $f(0) = a$ and $f(1) = b$. Assume that $2m \nmid a$ or $2m \nmid b$. For a positive integer m, $m|a$ and $m|b$ if and only if $l_f(2m) = 3$.

Proof. Let m be a positive integer. Then

$$f(3) = a + 2b$$
$$f(4) = 2a + 3b.$$

Assume that $m|a$ and $m|b$. Then there exist integers i and j such that $a = mi$ and $b = mj$. It follows that

$$f(3) = mi + 2mj \equiv mi = f(0) \pmod{2m},$$
$$f(4) = 2mi + 3mj \equiv mj = f(1) \pmod{2m}.$$

Thus, 3 is a period of f modulo $2m$. We conclude by Theorem 6 and Lemma 2 that $l_f(2m) = 3$. Conversely, assume that $l_f(2m) = 3$. Then

$$a + 2b = f(3) \equiv f(0) = a \pmod{2m},$$
$$2a + 3b = f(4) \equiv f(1) = b \pmod{2m}.$$

These implies that $m|b$ and $m|a$.

Example 4. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 2-step Fibonacci function with the starting values $f(0) = 4$ and $f(1) = 6$. Observe that $4 \nmid f(1)$, $2|f(0)$, and $2|f(1)$. Consider

$$f(0) = 4 \equiv 0 \pmod{4},$$
$$f(1) = 6 \equiv 2 \pmod{4},$$
$$f(2) = 10 \equiv 2 \pmod{4},$$
$$f(3) = 16 \equiv 0 \pmod{4},$$
$$f(4) = 26 \equiv 2 \pmod{4},$$
$$f(5) = 42 \equiv 2 \pmod{4},$$
$$\vdots$$

We see that $l_f(4) = 3$.

Example 5. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 2-step Fibonacci function with the starting values $f(0) = 1$ and $f(1) = 2$. Observe that $f(0)$ and $f(1)$ are not divisible by 3. Consider

$$f(0) = 1 \equiv 1 \pmod{6},$$
$$f(1) = 2 \equiv 2 \pmod{6},$$
$$f(2) = 3 \equiv 3 \pmod{6},$$
$$f(3) = 5 \equiv 5 \pmod{6},$$
$$\vdots$$

We see that $l_f(4) \neq 3$.
Next, the result for a 3-step Fibonacci function is as follows:

Theorem 9. Let m be a positive odd integer and $f: \mathbb{Z} \to \mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0) = a, f(1) = b$, and $f(2) = c$. Assume that $3m \nmid a, 3m \nmid b$, or $3m \nmid c$. Then the following statements hold.

1. If $m|a, m|b$, and $m|c$, then $l_f(3m) = 13$.
2. If $l_f(3m) = 13$, then

$$\begin{aligned} 91a + 141b + 168c &\equiv 0 \pmod{m} \\ 168a + 259b + 309c &\equiv 0 \pmod{m} \\ 309a + 477b + 568c &\equiv 0 \pmod{m}. \end{aligned}$$

Proof. We have from the assumption that

$$\begin{aligned} f(13) &= 274a + 423b + 504c \\ f(14) &= 504a + 778b + 927c \\ f(15) &= 927a + 1431b + 1705c. \end{aligned}$$

1. Assume that $m|a, m|b$, and $m|c$. Then there exist integers r, s, and t such that $a = mr, b = ms$, and $c = mt$. We obtain from the above that

$$\begin{aligned} f(13) &= 274mr + 423ms + 504mt \equiv mr = f(0) \pmod{3m}, \\ f(14) &= 504mr + 778ms + 927mt \equiv ms = f(1) \pmod{3m}, \\ f(15) &= 927mr + 1431ms + 1705mt \equiv mt = f(2) \pmod{3m}. \end{aligned}$$

Therefore, 13 is a period of f modulo $3m$. It follows by Theorem 6 and Lemma 2 that $l_f(3m) = 13$.

2.Assume that $l_f(3m) = 13$. The result immediately follows from the fact that

$$\begin{aligned} 274a + 423b + 504c &= f(13) \equiv f(0) = a \pmod{3m}, \\ 504a + 778b + 927c &= f(14) \equiv f(1) = b \pmod{3m}, \\ 927a + 1431b + 1705c &= f(15) \equiv f(2) = c \pmod{3m}. \end{aligned}$$

Example 6. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0) = 6$, $f(1) = 3$ and $f(2) = 0$. Observe that $f(0)$ and $f(1)$ are not divisible by 9. We also note that $3|f(0)$, $3|f(1)$, and $3|f(2)$. Consider

$$\begin{aligned} f(0) &= 6 \equiv 6 \pmod{9}, \\ f(1) &= 3 \equiv 3 \pmod{9}, \\ f(2) &= 0 \equiv 0 \pmod{9}, \\ &\vdots \\ f(13) &= 2913 \equiv 6 \pmod{9}, \\ f(14) &= 5358 \equiv 3 \pmod{9}, \\ f(15) &= 9855 \equiv 0 \pmod{9}, \\ &\vdots \end{aligned}$$

We see that $l_f(9) = 13$.

It is not hard to prove that if $m = 3$ or $m = 7$ in Theorem 9, then $l_f(3m) = 13$ implies $m|a, m|b$, and $m|c$.

Corollary 2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0) = a, f(1) = b$, and $f(2) = c$. Then the following statements hold.

1. If $l_f(9) = 13$ and a, b, or c is not divisible by 9, then $3|a, 3|b$, and $3|c$.
2. If $l_f(21) = 13$ and a, b, or c is not divisible by 21, then $7|a, 7|b$, and $7|c$.

Finally, the next theorem yields the result for a 4-step Fibonacci function is as follows:

Theorem 10. Let m be a positive integer and $f: \mathbb{Z} \to \mathbb{Z}$ be a 4-step Fibonacci function with the starting values $f(0) = a, f(1) = b, f(2) = c$, and $f(3) = d$. Assume that $\gcd(4m, 3) = 1$ and a, b, c, or d is not divisible by $4m$. Then the following statements hold.

1. If $m|a, m|b, m|c$, and $m|d$, then

$$l_f(4m) = \begin{cases} 5 & \text{if } b + c + d, a + b + 2c + 2d, 2a + 3b + 3c + 4d, \\ & \text{and } 4a + 6b + 7c + 7d \text{ are divisible by } 2m, \\ 10 & \text{otherwise.} \end{cases}$$

2. If $l_f(4m) = 10$, then

$$\begin{aligned} 7a + 11b + 13c + 14d &\equiv 0 \pmod{m} \\ 14a + 21b + 25c + 27d &\equiv 0 \pmod{m} \\ 27a + 41b + 48c + 52d &\equiv 0 \pmod{m} \\ 52a + 79b + 93c + 100d &\equiv 0 \pmod{m}. \end{aligned}$$

Proof. Consider

$$\begin{aligned} f(0) &= a \\ f(1) &= b \\ f(2) &= c \\ f(3) &= d \\ f(4) &= a + b + c + d \\ f(5) &= a + 2b + 2c + 2d \\ f(6) &= 2a + 3b + 4c + 4d \\ f(7) &= 4a + 6b + 7c + 8d \end{aligned}$$

$$
\begin{aligned}
f(8) &= 8a + 12b + 14c + 15d \\
f(9) &= 15a + 23b + 27c + 29d \\
f(10) &= 29a + 44b + 52c + 56d \\
f(11) &= 56a + 85b + 100c + 108d \\
f(12) &= 108a + 164b + 193c + 208d \\
f(13) &= 208a + 316b + 372c + 401d.
\end{aligned}
$$

1. Assume that $m|a, m|b, m|c$, and $m|d$. Then there exist integers r, s, t and u such that $a = mr, b = ms, c = mt$, and $d = mu$. We have from the above that

$f(10) = 29mr + 44ms + 52mt + 56mu \equiv mr = f(0) \pmod{4m}$,
$f(11) = 56mr + 85ms + 100mt + 108mu \equiv ms = f(1) \pmod{4m}$,
$f(12) = 108mr + 164ms + 193mt + 208mu \equiv mt = f(2) \pmod{4m}$,
$f(13) = 208mr + 316ms + 372mt + 401mu \equiv mu = f(3) \pmod{4m}$.

Hence, 10 is a period of f modulo $4m$. It follows by Theorem 6 and Lemma 2 that $l_f(4m) = 2,5$, or 10. If $l_f(4m) = 2$, then

$a \equiv c \equiv a + b + c + d \pmod{4m}$,
$b \equiv d \equiv a + 2b + 2c + 2d \pmod{4m}$.

Thus,

$a \equiv 2a + 2b \pmod{4m}$,
$b \equiv 3a + 4b \equiv a \pmod{4m}$.

These mean that $a \equiv b \equiv c \equiv d \pmod{4m}$. Since a, b, c, or d is not divisible by $4m$, all a, b, c, and d are not divisible by $4m$. Now, we also have that

$a + b + c + d \equiv 2a + 3b + 4c + 4d \pmod{4m}$.

Consequently, $9a \equiv 0 \pmod{4m}$ which is a contradiction. Therefore, it is impossible that $l_f(4m) = 2$. We note that $l_f(4m) = 5$ if and only if

$a + 2b + 2c + 2d = f(5) \equiv f(0) = a \pmod{4m}$,
$2a + 3b + 4c + 4d = f(6) \equiv f(1) = b \pmod{4m}$,
$4a + 6b + 7c + 8d = f(7) \equiv f(2) = c \pmod{4m}$,
$8a + 12b + 14c + 15d = f(8) \equiv f(3) = d \pmod{4m}$

if and only if $b + c + d, a + b + 2c + 2d, 2a + 3b + 3c + 4d$, and $4a + 6b + 7c + 7d$ are divisible by $2m$.

2. Assume that $l_f(4m) = 10$. The result immediately follows from the fact that

$29a + 44b + 52c + 56d = f(10) \equiv f(0) = a \pmod{4m}$,
$56a + 85b + 100c + 108d = f(11) \equiv f(1) = b \pmod{4m}$,
$108a + 164b + 193c + 208d = f(12) \equiv f(2) = c \pmod{4m}$,
$208a + 316b + 372c + 401d = f(13) \equiv f(3) = d \pmod{4m}$.

Example 7. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 4-step Fibonacci function with the starting values $f(0) = -4$, $f(1) = 0$, $f(2) = 0$ and $f(3) = 4$. Observe that all $f(i)$ are not divisible by 8 for $i = 0,1,2,3$. We also note that $2|f(0)$, $2|f(1)$, $2|f(2)$ and $2|f(3)$. Furthermore, we see that 4 divides $f(1) + f(2) + f(3)$, $f(0) + f(1) + 2f(2) + 2f(3)$, $2f(0) + 3f(1) + 3f(2) + 4f(3)$, and $4f(0) + 6f(1) + 7f(2) + 7f(3)$. Consider

$f(0) = -4 \equiv 4 \pmod 8$,
$f(1) = 0 \equiv 0 \pmod 8$,
$f(2) = 0 \equiv 0 \pmod 8$,
$f(3) = 4 \equiv 4 \pmod 8$,
$f(4) = 0 \equiv 0 \pmod 8$,
$f(5) = 4 \equiv 4 \pmod 8$,
$f(6) = 8 \equiv 0 \pmod 8$,
$f(7) = 16 \equiv 0 \pmod 8$,
$f(8) = 28 \equiv 4 \pmod 8$,
$\vdots$

We see that $l_f(8) = 5$.

Example 8. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a 4-step Fibonacci function with the starting values $f(0) = 0$, $f(1) = 10$, $f(2) = 5$ and $f(3) = 0$. Observe that $f(1)$ and $f(2)$ are not divisible by 20. We also note that $5|f(0)$, $5|f(1)$, $5|f(2)$, and $5|f(3)$. Furthermore, we see that 10 does not divide $f(1) + f(2) + f(3)$. Consider

$f(0) = 0 \equiv 0 \pmod{20}$,
$f(1) = 10 \equiv 10 \pmod{20}$,
$f(2) = 5 \equiv 5 \pmod{20}$,
$f(3) = 0 \equiv 0 \pmod{20}$,
$\vdots$
$f(10) = 700 \equiv 0 \pmod{20}$,

$$f(11) = 1350 \equiv 10 \pmod{20},$$
$$f(12) = 2605 \equiv 5 \pmod{20},$$
$$f(13) = 5020 \equiv 0 \pmod{20},$$
$$\vdots$$

We see that $l_f(20) = 10$.

5. Conclusions

In this paper, we present the existence of primitive period modulo m of a k-step Fibonacci function which is a generalization of a regular Fibonacci function that appeared in Thongngam and Chinram (2019). A regular Fibonacci function is also a generalization of the Fibonacci and Lucas sequence. Furthermore, the primitive period of some certain k-step Fibonacci functions under some additional conditions is established. An interesting direction for our future work is to find the explicit primitive period of the other k-step Fibonacci functions and to find the sharp bound of the primitive period of any k-step Fibonacci function.

References

Elmore, M. (1967). Fibonacci functions. *Fibonacci Quarterly*, *5*, 371-382.

Jameson, G. J. O. (2018). Fibonacci periods and multiples. *Mathematical Gazette*, *102*(553), 63-76.

Koshy, T. (2001). *Fibonacci and Lucas number with applications.* New Jersey, NJ: John Wiley and Sons.

Thongngam, N., & Chinram, R. (2019). Periods of fibonacci functions Modulo m. *Thaksin University Journal*, *22*(2), 53-58.