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Original Article

Log Akash regression model with application

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Abstract

The current study proposes and presents a new regression model for the response variable following the Akash distribution. The unknown parameters of the regression model are estimated using the maximum likelihood method. A simulation study is conducted to evaluate the performance of the maximum likelihood estimates (MLEs). Additionally, a residual analysis is performed for the proposed regression model. The log-Akash model is compared to several other models, including Weibull regression and gamma regression, using various statistical criteria. The results show that the suggested model fits the data better than these other models. It is anticipated that the model will have applications in fields such as economics, biological studies, mortality and recovery rates, health, hazards, measuring sciences, medicine, and engineering.

Keywords: definition of Akash distribution, log Akash regression model, maximum likelihood, residual analysis, deviance and martingale residual

1. Introduction

Several distributions have been used to model data in various fields, including economics, biological studies, mortality, recovery rates, health, risks, measurement sciences, medicine, engineering, insurance, and finance. In recent years, there have been studies that have attempted to provide modeling of data based on its distributions. For example, Cordeiro and Altun (2020) suggested the unit-improved second-degree Lindley distribution for inference and regression modeling. Gauss M. (Cordeiro & Altun, 2015) proposed the log-generalized modified Weibull regression model. Mazucheli, Korkmaz, Menezes, and Leiva (2023) introduced a new quantile regression for modeling bounded data using the Birnbaum-Saunders distribution. Silva, Ortega, Cancho, and Barreto (2008) introduced Log-Burr XII regression models. Ortega, Cordeiro, and Kattan (2013) introduced the Log Beta Generalized Weibull Regression Model for lifetime data.

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Mazucheli, Leiva, Alves, and Menezes (2021) suggested the quantile regression modeling on the unit Burr-XII, Shahedul and Khan (2021) suggested the Exponentiated Weibull regression, Sılvio and Junior (2021) suggested the Log-generalized inverse Weibull Regression Model, Daniele and Granzotto (2018) introduced the Transmuted Weibull Regression Model, Selasi, Kwaku, and Ocloo (2023) proposed an extension of the Burr XII distribution with applications and regression, Raid Al-Aqtash, and Selasi (2021) suggested the Gumbel-Burr XII regression model, Altun, Yousof, Chakraborty, and Handique (2023) proposed Zografos–Balakrishnan Burr XII regression model, Mazucheli, Korkmaz, Menezes, and Leiva (2023) suggested Unit-Chen quantile regression model, Josmar and Mazucheli (2023) suggested the unit generalized half-normal quantile regression model, Silva, Ortega, Cancho, and Barreto (2008) had Log-Burr XII regression models with censored data.

Akash distribution can be characterized as over-dispersed when (mu > sigma^2), equi-dispersed when (mu = sigma^2), and under-dispersed when (mu < sigma^2). Its hazard rate function increases with (x) and when (beta) is involved. It is considered superior to Lindley and exponential distributions for modeling lifetime data in medical science and engineering. Shanker introduced a quasi-Akash distribution to model lifetime data, discussing its statistical properties and potential applications. Additionally, a comparative study of one-parameter Akash, Lindley, and exponential distributions showed that the Akash distribution sometimes provided a better fit for certain datasets.

This article is organized as follows: Section 2 introduces the definition of Akash distribution, while Section 3 suggests a log Akash regression model of location-scale. Section 4 employs the maximum likelihood method to estimate the parameters, and Section 5 presents different types of residual analysis. Section 6 is simulation study**,** Section 7 has real data, and Section 8 presents conclusions.

2. Definition of Akash Distribution

The importance of modeling and lifetime data analysis is emphasized in various fields, with several continuous distributions being utilized to describe lifetime data. The exponential, Lindley, gamma, lognormal, and Weibull distributions are among the commonly used distributions for modeling lifetime data. However, the gamma and lognormal distributions' survival functions cannot be expressed in closed form and require numerical integration, making the exponential, Lindley, and Weibull distributions more popular choices. One advantage of the Lindley distribution over the exponential distribution is that the former has monotonically decreasing danger rate, whereas the latter has a constant hazard rate. This property makes the Lindley distribution more flexible and realistic in modeling certain types of lifetime data. The cumulative distribution function (c.d.f.) and probability density function (p.d.f.) of the Lindley distribution, as introduced by Lindley (1958), are given by:

$$f(x,\beta)=\frac{\beta^2}{\beta+1}(x+1)\exp(-\beta x)\quad x\geq 0,\beta\geq 0 \tag{1}$$

$$F(x,\beta)=1-(1+\frac{\beta x}{\beta+1})\exp(-\beta x)\quad x\geq 0,\beta\geq 0 \tag{2}$$

Although the Lindley distribution has been widely used in modeling lifetime data and has been shown to be useful in stress-strength reliability modeling by Hussain (2006), there are still some limitations and restrictions when applying it to real-world data. To address these issues, (Shanker *et al*., 2018) proposed a new distribution that is a mixture of an exponential distribution and a gamma distribution. This new distribution has the advantage of being more flexible and can better fit various types of lifetime data. The probability density function (p.d.f.) of the new distribution is given by:

$$f(x,\beta)=\frac{\beta^3}{\beta^2+1}\,[x^2+1].\exp(-\beta x)\quad x\geq 0,\beta\geq 0. \tag{3}$$

Figure 1 illustrates the probability density function (PDF) of the Akash distribution.

This distribution is known as the Akash distribution. The cumulative distribution function (c.d.f.) for (3) is given by

$$F(x,\beta)=1+\left[1+\frac{\beta x(\beta x+1)}{\beta^2+2}\right]\exp(-\beta x)\;x\geq 0,\beta\geq 0 \tag{4}$$

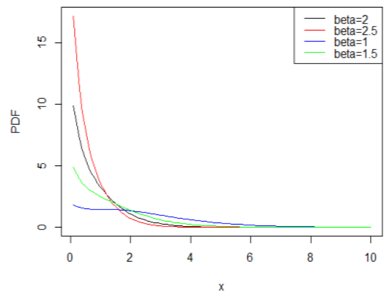

Figure 1. The probability density function (PDF) of the Akash distribution

Figure 2 shows the cumulative distribution function (CDF) of the Akash distribution at different values of the parameters.

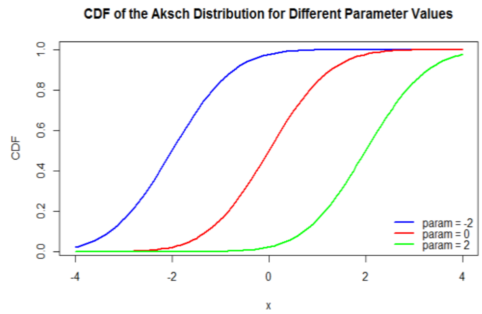

Figure 2. The cumulative distribution function (CDF) of the Akash distribution at different values of the parameters

3. The Log-Akash Regression Model

The main objective of this paper is to introduce a novel application of the Akash distribution in regression modeling. The proposed model utilizes the log-Akash distribution, which is derived from the positive Akash random quantity through a log transformation. This approach is commonly used in survival analysis and allows for the handling of both censored and uncensored data. The model assumptions of the log Akash regression mode are as follows. The model assumes constant variances for all observations, which is a standard assumption in regression models with censoring in survival analysis and reliability studies

These assumptions vary slightly depending on the model type, but they often include:

1. Linearity: There is a linear relationship between the outcome and the variables that predicted it. This indicates that a linear combination of the predictor variables (X) can be used to describe the expected value of the dependent variable (Y).

2. Independence: Observations do not depend on one another. This indicates that there is no correlation between the residuals (errors), which is especially important for time series or hierarchical data where observations may be grouped.

3. Homoscedasticity: At every level of the independent variables, the variance of errors remains constant. Stated otherwise, the “scatter” or spread of residuals should be roughly constant across all predictor values.

4. No Perfect Multicollinearity: The predictors in multiple regression shouldn't have a perfect correlation with one another. It may be challenging to discern each predictor's unique impact on the result when there is substantial multicollinearity, which occurs when predictors are highly correlated.

5. Normality of Errors: The errors, or residuals, follow a normal distribution. Although it is less important for prediction accuracy in big samples, this assumption is especially pertinent for hypothesis testing and creating confidence ranges.

6. No Autocorrelation: In time series data, where autocorrelation (correlation of residuals across time) should be minimized, this assumption is most applicable. The residuals' autocorrelation indicates a pattern or trend that the model may have missed, suggesting the necessity for extra terms or transformations.

Let X be a random variable having the Akash density function and let random variable $y = \sigma log x$, $\beta = exp(-\frac{\mu}{\sigma})$, for $y \in R$ Differentiating the hypothesis we get the following $dy = \frac{\sigma dx}{x}$ implies $\frac{dy}{dx} = \frac{\sigma}{x}$ but the Jacob equal $\frac{dx}{dy} = \frac{x}{\sigma}$, the density function of Y can be written as

$$f(y,\mu,\sigma) = f^{-1}(x).\,|J|\,. \quad (5)$$

$$f(y,\mu,\sigma) = \frac{e^{-\left(\frac{3\mu}{\sigma}\right)}}{2+e^{-\left(\frac{2\mu}{\sigma}\right)}}\left[1+e^{\left(\frac{2y}{\sigma}\right)}\right].\; exp\left(-e^{\frac{y-\mu}{\sigma}}\right).\frac{e^{\left(\frac{y}{\sigma}\right)}}{\sigma}. \quad (6)$$

$$f(y,\mu,\sigma) = \frac{e^{\left(\frac{y-\mu}{\sigma}\right)}}{\left[2+e^{-\left(\frac{2\mu}{\sigma}\right)}\right]\sigma}.\left[e^{-\frac{2\mu}{\sigma}}+e^{\left(\frac{2(y-\mu)}{\sigma}\right)}\right].\; exp\left(-e^{\frac{y-\mu}{\sigma}}\right). \quad (7)$$

We define the standardized $z = \frac{y-\mu}{\sigma}$ with pdf (for $z \in R$) given by

$$f(z,\mu,\sigma) = \frac{e^{z}}{\left[2+e^{-\left(\frac{2\mu}{\sigma}\right)}\right]\sigma}.\left[e^{-\frac{2\mu}{\sigma}}+e^{2(z)}\right].\; exp(-e^{z}). \quad (8)$$

The survival function is given by

$$s(z,\mu,\sigma) = \frac{e^{\left(\frac{y}{\sigma}\right)}}{\sigma}.\left[1+\frac{e^{(z)}\left(1+e^{(z)}\right)}{\left(e^{-\frac{2\mu}{\sigma}}+2\right)}\right].\; exp(-e^{z}) \quad (9)$$

We suggest a new log-location-scale regression model based on the Akash density function. Let Y be the response variable following the Akash distribution, and $x = (x_1, x_2, \ldots\ldots x_n)$ be the independent variable. The regression model is defined as:

$$y = x^T\beta + \sigma z \qquad i = 1, 2 \ldots\ldots\ldots n \quad (10)$$

The variable y conforms to the Akash distribution with unspecified parameters, where μ_i is a real number and σ_i is also a real number, using the identity link function $\mu_i = x^T\beta$. The vector μ_i which consists of $(\mu_1, \mu_2, \ldots, \mu_n)$, is a known design matrix

4. Estimation of The Model Parameters

For right-censored lifetime data, we have ti = min(f_i, c_i), where f_i is the lifetime and ci is the censoring time, then, we have $y_i = \log(t_i)$ for the i[th] individual $i = 1, \ldots, n$. If we have random sample with n observations $(y_1,, x_1)$,...,$(y_n,, x_n)$, where =

$$\delta_i = \begin{cases} 1 & for y_i = \log(t_i) \\ 0 & for y_i = log(c_i) \end{cases}$$

the function of log-likelihood is given by

$$L(\theta) = \sum_{i\in F}^{n} \delta_i \,.\, log f(y_i) + \sum_{i\in s}^{n}(1-\delta_i)\log(s(y_i)) \quad (11)$$

where $f(y_i)$ is the density function of Akash distribution and $s(y_i)$ is survival function

$$L(\theta) = K_1 + K_2. \quad (12)$$

$$K_1 = \sum_{i\in F}^{n} \delta_i \,.\, log f(y_i), K_2 = \sum_{i\in s}^{n}(1-\delta_i)\log(s(y_i)) \quad (13)$$

$$K_1 = \sum_{i\in F}^{n} \delta_i \frac{y-\mu}{\sigma}. - \sum_{i\in F}^{n} \delta_i\, log\left(\left(e^{-2\mu/\sigma}\right)+2)\sigma\right) - \sum_{i\in F}^{n} \delta_i e^{\frac{y-\mu}{\sigma}} + \sum_{i\in F}^{n} \delta_i\, log(e^{-2\mu/\sigma} + e^{2(\frac{y-\mu}{\sigma})}) \quad (14)$$

$$k_2 = \sum_{i\in c}^{n}(1-\delta_i) log\left[\left(e^{-\frac{2\mu}{\sigma}}+2\right)+e^{(z)}(1+e^{(z)})\right]. - \sum_{i\in c}^{n}(1-\delta_i) log\left[\left(e^{-\frac{2\mu}{\sigma}}+2\right)\right] - \sum_{i\in c}^{n}(1-\delta_i)e^{z} \quad (15)$$

Substituting in the value $\mu = \beta_0 + x\beta_1$, into the previous equation, we get the following

$$K_1 = \sum_{i\in F}^{n} \delta_i \frac{y-\beta_0-x\beta_1}{\sigma}. - \sum_{i\in F}^{n} \delta_i\, log\left(\left(e^{-2(\beta_0+x\beta_1)/\sigma}\right)+2)\sigma\right) - \sum_{i\in F}^{n} \delta_i e^{\frac{y-\beta_0-x\beta_1}{\sigma}} + \sum_{i\in F}^{n} \delta_i\, log(e^{-2(\beta_0+x\beta_1)/\sigma} + e^{2(\frac{y-\beta_0-x\beta_1}{\sigma})}) \quad (16)$$

To get the value of k_2

$$k_2 = \sum_{i\in c}^{n}(1-\delta_i) log\left[\left(e^{-\frac{2(\beta_0+x\beta_1)}{\sigma}}+2\right)+e^{(z)}(1+e^{(z)})\right]. - \sum_{i\in c}^{n}(1-\delta_i) log\left[\left(e^{-\frac{2(\beta_0+x\beta_1)}{\sigma}}+2\right)\right] - \sum_{i\in c}^{n}(1-\delta_i)e^{z} \quad (17)$$

Estimate the coefficients of regression by minimizing the log-likelihood function

$$\frac{\partial l(\theta)}{\partial \beta_0} = \frac{\partial K_1}{\partial \beta_0} + \frac{\partial k_2}{\partial \beta_0}. \quad (18)$$

$$\frac{\partial l(\theta)}{\partial \beta_1} = \frac{\partial K_1}{\partial \beta_1} + \frac{\partial k_2}{\partial \beta_1}. \quad (19)$$

From the previous three equations, we obtain non-linear equations by solving them using the software R, to obtain the value of the regression coefficients.

5. Residual Analysis

After fitting a model, it is essential to evaluate its suitability and ensure that it meets certain assumptions. One way to do this is by analyzing residuals, which can help identify any issues with the model's fit. In survival analysis, which involves right-censored data, martingale residuals can be used to assess the quality of fit and leverage of the model.

5.1 Martingale residuals

Martingale residuals are defined as the difference between the counting process and the integrated density function (also known as the hazard rate function) in parametric lifetime models. This method was introduced by Barlow and Prentice (2014) and has been used by researchers such as Therneau (2020), Commenges and Rondeau (2000), and Elgmati (2015).

$$r_M = \delta_i + \int_0^y k(u)du. \qquad i = 1,2,3 \quad (20)$$

where δ_i=1 or 0, one when observation is censored and 0 when observation is uncensored, the r_{Mi} reduce to

$$r_M = \delta_i + log(S(y)). \quad (21)$$

$$r_M = \begin{cases} \delta_i + log(s(y)) & \delta_i = 1 \\ log(s(y)) & \delta_i = 0 \end{cases}. \quad (22)$$

$$r_M = \begin{cases} \delta_i + log\left(\frac{e^{(\frac{y}{\sigma})}}{\sigma}.\left[1 + \frac{e^{(z)}(1+e^{(z)})}{(e^{-\frac{2\mu}{\sigma}}+2)}\right].exp(-e^z)\right) & \delta_i = 1 \\ log\left(\frac{e^{(\frac{y}{\sigma})}}{\sigma}.\left[1 + \frac{e^{(z)}(1+e^{(z)})}{(e^{-\frac{2\mu}{\sigma}}+2)}\right].exp(-e^z)\right) & \delta_i = 0 \end{cases} \quad (23)$$

5.2 Deviance residual

In statistics and machine learning, the deviance residual measures the discrepancy between a model's predictions and the actual values of the response variable. It assesses how well a model fits the data. In regression analysis, the deviance residual is calculated as the difference between the observed response variable and the predicted response variable, raised to a power, known as the deviance exponent. The deviance residual is used to evaluate a model's fit, with lower values indicating a better fit. The formula for deviance residual is:

Deviance residual = (observed response - predicted response)^(deviance exponent)

The deviance exponent is usually set to 2, which means that the deviance residual is calculated as the squared difference between the observed and predicted responses. This makes the residuals have a mean of 0 and a variance of 1, facilitating result interpretation. Deviance residuals have various applications, such as in model selection. Deviance residuals can be used to compare the fit of different models. In summary the deviance residual is a measure of the difference between a model's predictions and the actual values, used to compare different models. Deviance residuals for the Cox model without time-dependent explanatory variables were proposed by Therneau *et al*. (1997) as follows

$$r_D. = sign(r_M)[-2(r_M + \delta_i + \log(\delta_i - r_M)]^{1/2} \quad (24)$$

From the martingale residual, r_{Di} is more symmetrically about zero in this case. Consequently, the residual deviation for Akash is defined as follows

$$r_D = \begin{cases} sign(r_M)[-2(r_M + \delta_i + \log(\delta_i - r_M)]^{\frac{1}{2}} & \delta_i = 1 \\ sign(r_M)[-2(r_M + \delta_i + \log(\delta_i - r_M)]^{\frac{1}{2}} & \delta = 0 \end{cases} \quad (25)$$

5.3 Modified martingale-type residual

We have proposed a change in the martingale-type residual, and it can be written as

$$r_{MD} = (1 - \delta_i) + r_{D_i}. \quad (26)$$

where δ_i = 0 denotes censored observation and δ_i = 1 uncensored and r_{Di} is the martingale type residual that is defined in Section 5.2. In the log-Akash regression models, the modified martingale-type residual is defined by

$$r_{MD} = \begin{cases} sign(r_M)[-2(r_M + \delta_i + log(\delta_i - r_M)]^{\frac{1}{2}} & \delta_i = 1 \\ 1 + sign(r_M)[-2(r_M + \delta_i + log(\delta_i - r_M)]^{\frac{1}{2}} & \delta = 0 \end{cases} \quad (27)$$

5.4 Pearson residuals

The Pearson residual is widely used for detecting outliers in data. It is based on the idea of subtracting the mean and dividing by the standard deviation, which helps to identify potential outliers by comparing the relative distances of each data point from the mean. This method is particularly useful in linear regression, where it can help assess the fit of the model and detect any observations that do not conform to the overall pattern.

$$r_i = \frac{y_i - mean(y_i)}{\sqrt{var(y_i)}}. \quad (28)$$

where x_i is following Akash distribution, and $mean(y_i) = \widehat{\mu_i}$. This approach enables researchers to readily identify the extreme values that arise in the data due to measurement errors or data collection, as well as the values that do not conform to the overall pattern of the data. The Pearson residual is considered a vital tool in evaluating the precision of the model and its stability.

6. Simulation Study

In this section, a simulation study is given to evaluate the MLEs of the parameters of Akash regression model. Three censoring rates τ =(10%, 20%, 30%) and sample sizes (n = 20; 50; 100) are used. The simulation replication is N = 1,000. The lifetimes are generated by using the function of the Akash distribution The following parameter vector is used (β_0=0.6, β_1=1.6, σ=1). For each generated sample size, the biases, average of estimates (AEs) and MSEs are calculated. The simulation results are reported in Table 1.

$$bais = \sum_{i=1}^{n}(\widehat{\beta} - \beta) \qquad MSE = \frac{1}{n}\sum_{i=1}^{n}(\widehat{\beta} - \beta)^2$$

Table 1. Bias for log Akash regression model

Model	Log Akash regression model			
τ	n	β_1	β_0	σ
0.20	20	0.013	0.012	0.014
	50	0.214	0.231	0.112
	100	0.430	0.341	0.352
0.30	20	0.012	0.011	0.013
	50	0.118	0.113	0.121
	100	0.214	0.301	0.254
0.50	20	0.001	0.002	0.001
	50	0.191	0.201	0.203
	100	0.291	0.301	0.342

The simulation results presented in Table 1 indicate that the biases and MSEs approach zero as the sample size decreases. This suggests that the lack of bias is minimal at small sample sizes, with the smallest value of τ=(20%, 30%, 50%) corresponding to the smallest sample size.

Table 2. MSE for log Akash regression model

Model	Log Akash regression model			
τ	n	β_0	β_1	σ
0.20	20	0.0215	0.0057	0.0020
	50	0.4412	0.2611	0.1184
	100	0.7836	0.3029	0.1866
0.30	20	0.0136	0.0020	0.0001
	50	0.2186	0.0008	0.0080
	100	0.4214	0.2504	0.0722
0.50	20	0.0030	0.0061	0.0027
	50	0.0236	0.1559	0.3145
	100	0.2911	0.2145	0.0262

By extrapolating the previous figure, we can observe that the MSE is as small as possible at the small sample size, as well as at the smallest value.

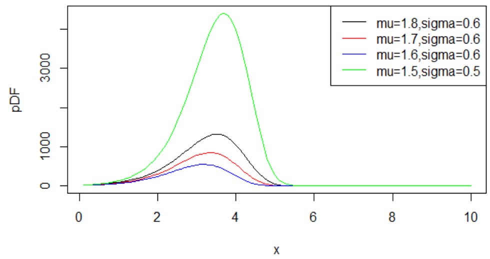

Figure 3. Probability density function of the Akash distribution at different values of the parameters

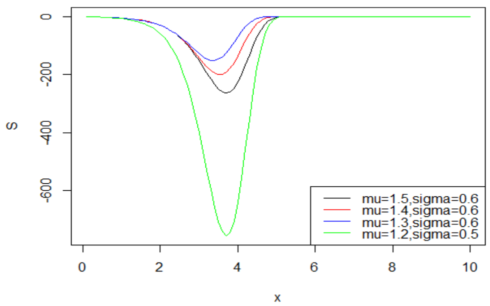

Figure 4. The survival function of the Akash distribution at different values of the parameters

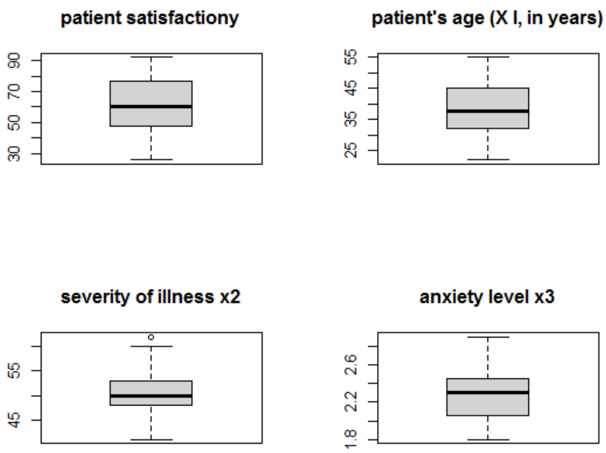

Figure 5. Box plots for the independent variables and the dependent variable

7. Real Data

The data in the following experiment consists of four variables, which are categorized into one dependent variable and three independent variables. It is noted that the data scale used in the current study is continuous data. The dependent variable is the patient's satisfaction, while the independent variables are the patient's age (x_1), anxiety level (x_3), and disease severity (x_2, an index). The study focuses on the relationship between the dependent variable, patient satisfaction, and the independent variable, patient age. There are 200 observation records. The descriptive data for the two variables are as follows.

Table 3. Descriptive statistics for the dependent and independent variables

Variable	Min.	Max	Median	Mean	Q1
y	26	92	60	37	49
X1	22	55	62	38	31

Table 4. Goodness-of –fit (test and criteria)

Goodness-of –fit (criteria)	Akash	Weibull	Gamma
Akaike Information Criterion (AIC)	52.24356	56.23672	58.23147
Bayesian Information Criterion (BIC)	51.23456	54.32466	56.23146
Quinn Information Criterion (HQIC)	55.21432	56.23142	57.32451

Table 5. Goodness-of –fit (test)

Goodness-of –fit (criteria)	Akash	Weibull	Gamma
Kolmogorov-Smirnov Statistic	0.0871768	0.0988232	0.164698
Cramer-Von-Mises Statistic	0.0171432	0.0976642	0.056378
Anderson–Darling Statistic	0.1463242	0.6248354	0.309834

8. Results and Discussion

From Table 4, we notice that the statistical criteria Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Quinn Information Criterion (HQIC) for the Akash distribution have values less than those for the Gamma distribution and the Weibull distribution. Therefore, these results suggests that the Akash distribution is a good fit for the data to a large degree.

From Table 5, we notice that the values of the Cramer-Von Mises statistic, the Kolmogorov-Smirnov statistic, and the Anderson-Darling statistic for the Akash distribution are smaller than those for the other distributions, indicating that the data are more consistent with the proposed distribution.

By extrapolating Table 6, we can verify whether the data follow a normal distribution or not, by using two tests: the Kolmogorov-Smirnov test and the Shapiro-Wilk test. The results of both tests indicate that the p-value is less than 0.05, which suggests that the data do not follow a normal distribution.

Table 6. Normality test for data

Goodness-of –fit (criteria)	Value test	p-value
Kolmogorov-Smirnov test	1	0.0271
shapiro.test	0.97302	2.2e-16

9. Fitted Regression Model

In this section, after determining the appropriate model for the data, it is necessary to compare the proposed model with other models in light of some evaluation criteria from the model selection process. The next table shows the BIC and AIC criteria for the proposed model

Extrapolating from Table 7 we notice that the AIC and BIC for the Akash regression are less than those for the Weibull regression and the Gamma regression – therefore, the Akash regression is better than the Weibull regression and the Gamma regression.

10. Conclusions

The current study proposed a new regression model called the Akash regression model. The maximum log-likelihood estimation method was employed to estimate the unknown parameters. A simulation study demonstrated that the maximum log-likelihood method outperformed other methods in the case of small samples. The study relied on some tools to test the suitability of the data used in the research under study, including Kolmogorov-Smirnov, Cramer-Von-Mises, and Anderson-Darling statistics, as all the previous measures were smaller in the case of the distribution under study. The suggested regression model was compared with sub-models, specifically the Weibull regression model and the gamma regression model, using the AIC and BIC criteria. According to the results, the proposed regression model had a better performance than the other models.

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Table 7. AIC, BIC, and R^2 for different regression models

Model	β_0	β_1	AIC	BIC	R^2
Log Akash Regression Model	1.423	0.632	142.323	139.212	0.856
Log Weibull regression model	1.532	0.532	154.234	151.542	0.653
Log gamma regression model	1.623	0.623	156.324	153.274	0.527

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