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Original Article

Improving efficiency within the manufacturing sector through the implementation of queue-length dependent bulk service strategies with vacation scheduling

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Abstract

The study examines the batch arrival queue-length-dependent bulk service with multiple vacations. Service provision is that the batch depends upon the queue length. The study incorporates supplementary variables on account of variations in service time (both high and low batch) and vacation periods. It derives the probability generating functions of queue length distributions and explicit expressions of key performance indices. Numerical results are presented to validate the analytical findings.

Keywords: general bulk service, non-Markovian queue, multiple vacation

1. Introduction

In the manufacturing world, the raw materials play a crucial role. When materials are insufficient it leads to production halts and delay in delivering products to consumers. Let us consider a car manufacturing factory that requires steel. Without enough steel, car production will stop. This shortage affects the entire supply chain, from production to the dealership. These delays are significant. It creates problems for both manufacturers and consumers. Manufacturers struggle to do their jobs, while consumers grow frustrated with the extended waiting times.

General reasons for raw material shortages: the following are a few reasons for factories not having enough raw materials.

Supplier Issues: Factories depend on suppliers for raw materials. If supplier encounters any problems such as production delays or shortages, then factories cannot run manufacturing.

Transportation Challenges: Transporting materials from suppliers to factory is similar to bringing groceries home. Again issues like delayed shipments or high transportation costs can slow down the arrival of raw materials. If there is a sudden surge in demand for a popular toy, factories might face unexpectedly high demand for their products, leading to shortages if they haven't been adequately prepared. Production with low raw materials: a solution that decreases waiting times.

When factories start making products again, even with a low quantity of raw materials, it means the waiting time for customers decreases. The customers get the products earlier. Optimizing the skills that workers need can help ensure that their skills are utilized effectively.

Restarting production with a low raw material stock helps to minimize the financial strain on factories. While they may not be making as much as usual, they are still generating some income. It's a bit like keeping the business engine running, even if at a slower pace.

Meeting consumer demand: consumers are like hungry guests at a dinner table - they want their products! Restarting production, even with limited materials, allows factories to meet consumer demand and keep them satisfied.

The model deals with having enough materials, while stopping production and making people wait longer is common in manufacturing industries. It shows how important for a company it is to manage their materials well and make sure they have enough materials to keep the work running smoothly. It leads to customer satisfaction. Suppose, a manufacturing

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industry starts production with a low quantity: the proposed model is described in this work.

The idea of starting production, even with a low quantity of raw materials, emerges as a solution that can benefit everyone involved. By this process, waiting times are reduced, workers stay engaged, operational flow is maintained, available resources are utilized wisely, financial strain is minimized, and consumer demand is met. It is a practical approach that keeps the wheels of production turning, ensuring that the products will reach the customer, finding the way into their hands sooner rather than later.

The versatile bulk service queuing system finds applications across various domains. General Bulk Service Rule (GBSR) with queue length is 'r ', where

- When $0 \le r \le a-1$, the server remains idle.
- For $a \le r \le b-1$, all customers are taken for service.
- If $r \ge b$, then the first 'b' customers are taken for service.

Two different service patterns with a single vacation were discussed by Thangaraj and Rajendran (2017). The bulk service was offered if the queue length (Q) is less than or equal to b; otherwise, the single service or single vacation is provided. Under (a, c, and b) policy, Jeyakumar and Arumuganathan (2008) investigated the case of two service nodes. When the number of requests in the queue reaches 'a', the server provides a single service. When the number of requests in the queue reaches 'c' (which is greater than 'a'), the server provides a batch service with a variable number of requests, up to a maximum of 'b' (which is greater than 'c'). Only when there are at least 'a' customers in the queue does the server start the service. The server transitions from single service to batch service only during service initiation epochs, based on the current queue length.

Bulk service queueing models are studied by various researchers. Recently, Achyutha Krishnamoorthy, Anu Nuthan Joshua and Vladimir Vishnevsky (2021), Gopal K Gupta, and Banerje (2021), Shanthi, Muthu Ganapathi Subramanian, and Gopal Sekar (2022) have investigated a single server bulk service queueing model with different contexts. Mohan Chaudhry and Jing Gai (2022) studied bulk service queueing system with multi-server. Shakila Devi and Vijiyalakshmi (2023) examined bulk queueing models with various service interruptions. Meanwhile, Chaudhry, Datta Banik, Barik, and Goswami (2023) developed an innovative computational procedure for analyzing the waiting-time distribution within bulk-service finite-buffer queues with Poisson input. Keerthiga and Indhira (2024) review bulk arrival and batch service queueing models, focusing on vacations, breakdowns, and performance measures to reduce congestion.

The batch-size-dependent service queueing system was studied by Pradhan and Gupta (2019, 2022), Gupta, Banerjee and Gupta (2020), Sourav Pradhan (2020), Gupta and Banerjee (2019). Umesh Chandra Gupta, Nitin Kumar, Sourav Pradhan, Farida Parvez Barbhuiya and Chaudhry (2021), Tamrakar and Banerjee (2021), Tamrakar, Banerjee and Gupta (2022), and Nandy and Pradhan (2021, 2021a), using various parameters. In a study conducted by Pradhan Sourav and Karan Prasenjit (2023), a bulk service queue with server breakdown and multiple vacations was examined, incorporating batch-size dependence and an infinite buffer. Additionally, Panda and Goswami (2023) conducted a separate analysis of a discrete-time queue, where a modified batch service policy and service dependency on batch size were considered. Pradhan, Nandy and Gupta (2024) studied bulk-service queueing systems with vacation policies, analyzing steady-state distributions and giving numerical examples for diverse applications. Niranjan and devi Latha (2024) investigated two-phase heterogeneous and batch service queuing system with breakdown in two phases, feedback, and vacation

The following ways are employed in this work: As a batch arrival queuing model with more practical significance, consider the bulk service concept. But in this modal the bulk service depends on queue length. If queue length (Q) is

1. $Q \ge a$ then the general bulk service rule is followed to serve the batch.
2. $1 \le Q \le a-1$ then the whole batch is taken for service with different service rates.
3. Q=0 then the server starts its multiple vacation.

The lack of existing research on batch arrival multiple vacations with the specified queue length dependent service motivated the creation of this work.

The structure of the article is as follows: In Section 2, a real-world example of the queuing problem is presented and the system equations are formulated. Section 3 focuses on the distribution of queue sizes. In Section 4, the Probability Generating Function (PGF) for the equilibrium queue length distribution is established. Section 5 involves the computation of various performance metrics for the queuing system. Section 6 introduces an analytical cost model. A numerical example is presented in Section 7. Finally, Section 8 provides the conclusion of the paper.

2. Mathematical Description of the Model

The analysis of a queuing system characterized by batch arrivals of customers is examined, where the server delivers bulk service influenced by the queue length and takes multiple vacations. The initiation of service to a batch by the server depends upon the queue length (r), as detailed below:

$1 \le r \le a-1$: all the customers are taken for service with Low-capacity service.

1. $a \le r \le b-1$: all the customers are taken for service with High-capacity service.
2. $r \ge b$: then first 'b' customers are taken for service with High-capacity service.
3. $r = 0$: then the server goes for multiple vacation.

At the completion of multiple vacations, if the queue length is either equal to or exceeds 'a', the server will provide batch service. The batch service offered is categorized as either High-capacity or Low-capacity service, depending on the queue length. If the queue length is less than 'a', the server enters another vacation period. This utilizes the supplementary variable technique to derive the system state probabilities and formally define this model. Figure 1 (Pictorial representation of the model) depicts a diagram of the proposed model.

2.1 Notation and probabilities

The assumptions and notations and their descriptions used in this article are given in Table 1 Notations.

The symbols for Cumulative Distribution Functions (CDF), Probability Density Function (PDF), and their Laplace-Stieltjes Transform (LST) are given in Table 2 Symbols.

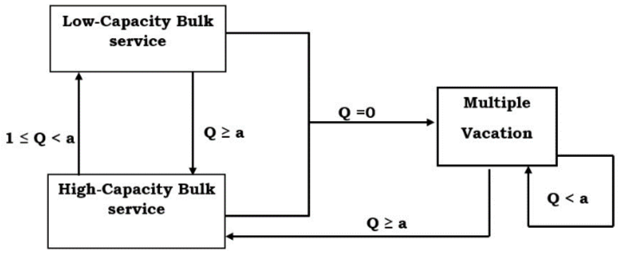

Figure 1. Pictorial representation of the model

Table 1. Notation

Notation	Description
λ	Group arrival rate
$g_k = \Pr(X = k)$, $k = 1,2,3,\ldots$	
$X(z)$	Probability Generating Function (PGF) of X
$N_1(t)$	The number of customers who are being serviced at the station
$N_2(t)$	The count of customers in the queue
$H^0(t)$	Remaining High-capacity service time
$L^0(t)$	Remaining Low-capacity service time
$V^0(t)$	Remaining vacation time

Table 2. Symbols

Parameter	CDF	PDF	LST
High-capacity Service time	H	$h(w)$	$\tilde{H}(\tau)$
Low-capacity Service time	L	$l(w)$	$\tilde{L}(\tau)$
Vacation	V	$v(w)$	$\tilde{V}(\tau)$

$\psi(t)$= [1] means the server is busy in High-capacity batch service, $\psi(t)$= [2] means the server is busy in Low-capacity batch service, $\psi(t)$= [3] means the server is on vacation.

$$H_{m,n}(w,t)dt = \Pr\left\{\begin{matrix} N_1(t) = m, N_2(t) = n, w \le H^0(t) < w + dt, \\ \psi(t) = 1 \end{matrix}\right\} \quad a \le m \le b, n \ge 1,$$

$$L_{m,n}(w,t)dt = \Pr\left\{\begin{matrix} N_1(t) = m, N_2(t) = n, w \le L^0(t) < w + dt, \\ \psi(t) = 2 \end{matrix}\right\} \quad 1 \le m \le a-1, n \ge 1,$$

$$V_{k,j}(w,t)dt = \Pr\{Z(t) = k, N_2(t) = j, w \le V^0(t) < w + dt, \psi(t) = 3\} \quad k \ge 1, j \ge 0.$$

3. Queue Size Distribution

The Kolmogorov backward equation which describes the system is as follows:

$$-H'_{m,0}(w) = -\lambda H_{m,0}(w) + \sum_{r=1}^{a-1} L_{r,m}(0)h(w) + \sum_{r=a}^{b} H_{r,m}(0)h(w) + \sum_{k=1}^{\infty} V_{k,m}(0)h(w), a \le m \le b, \quad (1)$$

$$-H'_{m,n}(w) = -\lambda H_{m,n}(w) + \sum_{d=1}^{n} H_{m,n-d}(w)\lambda g_d, \quad a \le m \le b-1, n \ge 1, \quad (2)$$

$$-H'_{b,n}(w) = -\lambda H_{b,n}(w) + \sum_{r=1}^{a-1} L_{r,b+n}(0)h(w) + \sum_{r=a}^{b} H_{r,b+n}(0)h(w) + \sum_{k=1}^{\infty} V_{k,b+n}(0)h(w) + \sum_{d=1}^{n} H_{b,n-d}(w)\lambda g_d, n \ge 1, \quad (3)$$

$$-L'_{m,0}(w) = -\lambda L_{m,0}(w) + \sum_{r=1}^{a-1} L_{r,m}(0)l(w) + \sum_{r=a}^{b} H_{r,m}(0)l(w), 1 \le m \le a-1, \quad (4)$$

$$-L'_{m,n}(w) = -\lambda L_{m,n}(w) + \sum_{d=1}^{n} L_{m,n-d}(w)\lambda g_d, 1 \le m \le a-1, n \ge 1, \quad (5)$$

$$-V'_{1,0}(w) = -\lambda V_{1,0}(w) + \sum_{r=1}^{a-1} L_{r,0}(0)v(w) + \sum_{r=a}^{b} H_{r,0}(0)v(w), \quad (6)$$

$$-V'_{1,n}(w) = -\lambda V_{1,n}(w) + \sum_{d=1}^{n} V_{1,n-d}(w)\lambda g_d, n \ge 1, \quad (7)$$

$$-V_{k,0}^{'}(w) = -\lambda V_{k,0}(w) + V_{k-1,0}(0)v(w), k \geq 2, \quad (8)$$

$$-V_{k,n}^{'}(w) = -\lambda V_{k,n}(w) + V_{k-1,n}(0)v(w) + \sum_{d=1}^{n} V_{k,n-d}(w)\lambda g_d, k \geq 2, 1 \leq n \leq a-1, \quad (9)$$

$$-V_{k,n}^{'}(w) = -\lambda V_{k,n}(w) + \sum_{d=1}^{n} V_{k,n-d}(w)\lambda g_d, k \geq 2, n \geq a. \quad (10)$$

While applying LST to the above equations (1) to (10), we get,

$$\tau\widetilde{H}_{m,0}(\tau) - H_{m,0}(0) = \lambda\widetilde{H}_{m,0}(\tau) - \sum_{r=1}^{a-1} L_{r,m}(0)\widetilde{H}(\tau) - \sum_{r=a}^{b} H_{r,m}(0)\widetilde{H}(\tau) - \sum_{k=1}^{\infty} V_{k,m}(0)\widetilde{H}(\tau), a \leq m \leq b, \quad (11)$$

$$\tau\widetilde{H}_{m,n}(\tau) - H_{m,n}(0) = \lambda\widetilde{H}_{m,n}(\tau) - \sum_{d=1}^{n} \widetilde{H}_{m,n-d}(\tau)\lambda g_d, a \leq m \leq b-1, n \geq 1, \quad (12)$$

$$\tau\widetilde{H}_{b,n}(\tau) - H_{b,n}(0) = \lambda\widetilde{H}_{b,n}(\tau) - \sum_{r=1}^{a-1} L_{r,b+n}(0)\widetilde{H}(\tau) - \sum_{r=a}^{b} H_{r,b+n}(0)\widetilde{H}(\tau) - \sum_{k=1}^{\infty} V_{k,b+n}(0)\widetilde{H}(\tau) \sum_{d=1}^{n} \widetilde{H}_{b,n-d}(\tau)\lambda g_d, n \geq 1, \quad (13)$$

$$\tau\tilde{L}_{m,0}(\tau) - L_{m,0}(0) = \lambda\tilde{L}_{m,0}(\tau) - \sum_{r=1}^{a-1} L_{r,m}(0)\tilde{L}(\tau) - \sum_{r=a}^{b} H_{r,m}(0)\tilde{L}(\tau), \ 1 \leq m \leq a-1, \quad (14)$$

$$\tau\tilde{L}_{m,n}(\tau) - L_{m,n}(0) = \lambda\tilde{L}_{m,n}(\tau) - \sum_{d=1}^{n} L_{m,n-d}(\tau)\lambda g_d, 1 \leq m \leq a-1, n \geq 1, \quad (15)$$

$$\tau\tilde{V}_{1,0}(\tau) - V_{1,0}(0) = \lambda\tilde{V}_{1,0}(\tau) - \sum_{r=1}^{a-1} L_{r,0}(0)\tilde{V}(\tau) - \sum_{r=a}^{b} H_{r,0}(0)\tilde{V}(\tau), \quad (16)$$

$$\tau\tilde{V}_{1,n}(\tau) - V_{1,n}(0) = \lambda\tilde{V}_{1,n}(\tau) - \sum_{d=1}^{n} \tilde{V}_{1,n-d}(\tau)\lambda g_d, n \geq 1, \quad (17)$$

$$\tau\tilde{V}_{k,0}(\tau) - V_{k,0}(0) = \lambda\tilde{V}_{k,0}(\tau) - V_{k-1,0}(0)\tilde{V}(\tau), k \geq 2, \quad (18)$$

$$\tau\tilde{V}_{k,n}(\tau) - V_{k,n}(0) = \lambda\tilde{V}_{k,n}(\tau) - V_{k-1,n}(0)\tilde{V}(\tau) - \sum_{d=1}^{n} \tilde{V}_{k,n-d}(\tau)\lambda g_d, k \geq 2, 1 \leq n \leq a-1, \quad (19)$$

$$\tau\tilde{V}_{k,n}(\tau) - V_{k,n}(0) = \lambda\tilde{V}_{k,n}(\tau) - \sum_{d=1}^{n} \tilde{V}_{k,n-d}(\tau)\lambda g_d, k \geq 2, n \geq a. \quad (20)$$

Let us define the following PGFs:

$$\widetilde{H}_m(\mathrm{z},\tau) = \sum_{n=0}^{\infty} \widetilde{H}_{m,n}(\tau)\mathrm{z}^n, \ \ H_m(\mathrm{z},0) = \sum_{n=0}^{\infty} H_{m,n}(0)\mathrm{z}^n, a \leq m \leq b,$$

$$\tilde{L}_m(\mathrm{z},\tau) = \sum_{n=0}^{\infty} \tilde{L}_{m,n}(\tau)\mathrm{z}^n, \ \ L_m(\mathrm{z},0) = \sum_{n=0}^{\infty} L_{m,n}(0)\mathrm{z}^n, 1 \leq m \leq a-1, \quad (21)$$

$$\tilde{V}_k(\mathrm{z},\tau) = \sum_{n=0}^{\infty} \tilde{V}_{k,n}(\tau)\mathrm{z}^n, \ \ V_k(\mathrm{z},0) = \sum_{n=0}^{\infty} V_{k,n}(0)\mathrm{z}^n, k \geq 1.$$

After some mathematical manipulations, we get,

$$(\tau - \phi(\mathrm{z}))\widetilde{H}_n(\mathrm{z},\tau) = H_n(\mathrm{z},0) - \widetilde{H}(\tau)[\sum_{r=1}^{a-1} L_{r,n}(0) + \sum_{r=a}^{b} H_{r,n}(0) + \sum_{k=1}^{\infty} V_{k,n}(0)], a \leq n \leq b-1, \quad (22)$$

$$z^b(\tau - \phi(\mathrm{z}))\widetilde{H}_b(\mathrm{z},\tau) = (z^b - \widetilde{H}(\tau))H_b(\mathrm{z},0) - \widetilde{H}(\tau)\{\sum_{r=1}^{a-1} L_r(z,0) + \sum_{r=a}^{b-1} H_r(z,0) + \sum_{k=1}^{\infty} V_k(z,0) - \sum_{n=0}^{b-1} [\sum_{r=1}^{a-1} L_{r,n}(0)z^n + \sum_{r=a}^{b} H_{r,n}(0)z^n + \sum_{k=1}^{\infty} V_{k,n}(0)z^n]\}, \quad (23)$$

$$\left(\tau - \phi(\mathrm{z})\right)\tilde{L}_n(\mathrm{z},\tau) = L_n(\mathrm{z},0) - \tilde{L}(\tau)\left[\sum_{r=1}^{a-1} L_{r,n}(0) + \sum_{r=a}^{b} H_{r,n}(0)\right], 1 \leq n \leq a-1, \quad (24)$$

$$(\tau - \phi(\mathrm{z}))\tilde{V}_1(\mathrm{z},\tau) = V_1(\mathrm{z},0) - \tilde{V}(\tau)[\sum_{r=1}^{a-1} L_{r,0}(0) + \sum_{r=a}^{b} H_{r,0}(0)], \quad (25)$$

$$(\tau - \phi(\mathrm{z}))\tilde{V}_k(\mathrm{z},\tau) = V_k(\mathrm{z},0) - \tilde{V}(\tau)[\sum_{n=0}^{a-1} V_{k-1,0}(0)z^n], k \geq 2. \quad (26)$$

where $\phi(\mathrm{z}) = \lambda - \lambda X(\mathrm{z})$.

Substitute $\tau = \phi(\mathrm{z})$ in (22) to (26), to get,

$$H_n(\mathrm{z},0) = \widetilde{H}\left(\phi(\mathrm{z})\right)\left[\sum_{r=1}^{a-1} L_{r,n}(0) + \sum_{r=a}^{b} H_{r,n}(0) + \sum_{l=1}^{\infty} V_{k,n}(0)\right], a \leq n \leq b-1, \quad (27)$$

$$H_b(\mathrm{z},0) = \frac{\widetilde{H}(\phi(\mathrm{z}))}{(z^b - \widetilde{H}(\phi(\mathrm{z})))}\{\sum_{r=1}^{a-1} L_r(z,0) + \sum_{r=a}^{b-1} H_r(z,0) + \sum_{l=1}^{\infty} V_k(z,0) - \sum_{n=0}^{b-1} [\sum_{r=1}^{a-1} L_{r,n}(0)z^n + \sum_{r=a}^{b} H_{r,n}(0)z^n + \sum_{l=1}^{\infty} V_{k,n}(0)z^n]\}, \quad (28)$$

$L_n(\text{z},0)=\tilde{L}(\phi(\text{z}))[\sum_{r=1}^{a-1} L_{r,n}(0)+\sum_{r=a}^{b} H_{r,n}(0)], 1\le n\le a-1,$ (29)

$V_1(\text{z},0)=\tilde{V}(\phi(\text{z}))[\sum_{r=1}^{a-1} L_{r,0}(0)+\sum_{r=a}^{b} H_{r,0}(0)],$ (30)

$V_k(\text{z},0)=\tilde{V}(\phi(\text{z}))[\sum_{n=0}^{a-1} V_{k-1,0}(0)z^n], k\ge 2.$ (31)

After the substitution of (27) to (31) in (22) to (26) respectively then substitute $\tau=0$, to get,

$\widetilde{H}_n(\text{z},0)=\frac{1-\widetilde{H}(\phi(\text{z}))}{\phi(\text{z})}(d_n+q_n), a\le n\le b-1,$ (32)

$\widetilde{H}_b(\text{z},0)=\frac{\left(1-\widetilde{H}(\phi(\text{z}))\right)}{(\phi(\text{z}))\left(z^b-\widetilde{H}(\phi(\text{z}))\right)}\{\tilde{L}(\phi(\text{z}))\sum_{n=1}^{a-1} d_n+\widetilde{H}(\phi(\text{z}))\sum_{n=a}^{b-1}(d_n+q_n)+\tilde{V}(\phi(\text{z}))(d_0+\sum_{n=0}^{a-1} q_n z^n)$
$+\sum_{n=0}^{b-1}(d_n z^n+q_n z^n)$ (33)

$\tilde{L}_n(\text{z},0)=\frac{1-\tilde{L}(\phi(\text{z}))}{\phi(\text{z})}d_n, 1\le n\le a-1,$ (34)

$\tilde{V}_1(\text{z},0)=\frac{1-\tilde{V}(\phi(\text{z}))}{\phi(\text{z})}d_0,$ (35)

$\tilde{V}_k(\text{z},0)=\frac{1-\tilde{V}(\phi(\text{z}))}{\phi(\text{z})}\sum_{n=0}^{a-1} V_{k-1,0}(0)z^n, k\ge 2.$ (36)

where $\sum_{r=1}^{a-1} L_{r,n}(0)+\sum_{r=a}^{b} H_{r,n}(0)=d_n$ and $\sum_{l=1}^{\infty} V_{k,n}(0)=q_n$.

4. Probability Generating Function of the Queue Size

Consider P(z) as the function representing the probability of having a specific number of customers in the queue at any random time in the proposed model. Then,

$P(z)=\sum_{r=1}^{a-1}\tilde{L}_r(z,0)+\sum_{r=a}^{b}\widetilde{H}_r(z,0)+\sum_{k=1}^{\infty}\tilde{V}_k(z,0).$ (37)

The probability generating function is obtained as:

$$P(z)=\frac{\begin{array}{c}[((z^b-\widetilde{H}(\phi(\text{z})))+\tilde{L}(\phi(\text{z}))(1-z^b))\sum_{n=1}^{a-1}d_n+(1-\widetilde{H}(\phi(\text{z})))\sum_{n=a}^{b-1}(z^b-z^n)p_n\\+(z^b-1)(1-\tilde{V}(\phi(\text{z})))[d_0+\sum_{n=0}^{a-1}q_n z^n]-(1-\widetilde{H}(\phi(\text{z})))\sum_{n=1}^{a-1}d_n z^n]\end{array}}{\phi(\text{z})(z^b-\widetilde{H}(\phi(\text{z})))}. \quad (38)$$

A particular case: when $a = b = 1$ and no Low-batch service, then probability generating function of the proposed queueing model becomes

$$P(z)=\frac{(\text{z}-1)(1-\tilde{V}(\phi(\text{z})))(\text{d}_0+q_0)}{\phi(\text{z})(\text{z}-\widetilde{H}(\phi(\text{z})))}.$$

This coincides with $M^{[X]}/G/1$ queueing system with multiple vacations. The result coincides with the queue size distribution of Lee *et al*. (1994) with N=1.

4.1 Steady state condition

The PGF of the queue length has to satisfy P(1) = 1, for which we apply L' Hospital's rule and evaluating $\lim_{z\to 1} P(z)$ z1P(z), then equating to 1, we have, $N''=D''$. Since d_r, p_r, and q_r represents the probabilities of 'r' customers waiting in the queue, it follows that N'' must be positive. Thus P(1) = 1 is satisfied iff D'' > 0. The inequality

$\rho=\frac{\lambda X_1 E(H)}{b}<1$ (39)

is the stability condition for the proposed model.

4.2 Computational aspects

Equation (38) has $b+a$ unknowns $d_0, d_1, \ldots, d_{a-1}, q_0, \ldots, q_{a-1}$, and $p_a, p_{a+1}, \ldots, p_{b-1}$. We can express $q_i(i=0,1,\ldots,a-1)$ in terms of d_0 so that the numerator has only b constants. Now from equation (38) which is PGF of number of customers involving b unknowns. By Rouches's theorem $(z^b-\widetilde{H}(\phi(\text{z})))$ has one zero on the boundary and b-1 inside the unit circle. Due to P(z) being analytic, the numerator must vanish at that points and give b equations with b unknowns, which can be solved by a numerical technique.

4.3 Result

Let q_n can be expressed in terms of d_0 as

$$q_n = \frac{\beta_n}{(1-\beta_0)} d_0 + \frac{\sum_{r=1}^{n} \beta_r q_{n-r}}{(1-\beta_0)}, n = 1,2,\ldots,a-1,$$

where $q_0 = \dfrac{\beta_0}{1-\beta_0} d_0$

$\beta_i's$ are the probabilities of the 'i' customers arriving during multiple vacation times.

4.4 PGF of queue size at various completion epochs

Low-capacity batch service: Let $P_L(z)$ be the probability that the server is in the busy Low-capacity batch service at time t. From equation (34)

$$P_L(z) = \frac{(1-\tilde{L}(\phi(z)))[\sum_{r=1}^{a-1} d_r]}{\phi(z)}. \tag{40}$$

High-capacity batch service: Let $P_H(z)$ be the probability that the server is in the busy High-capacity batch service at time t. From equations (32) and (33)

$$P_H(z) = \frac{\begin{array}{l}(1-\tilde{H}(\phi(z)))[(z^b-\tilde{H}(\phi(z)))\sum_{r=a}^{b-1} p_r+\tilde{L}(\phi(z))\sum_{n=1}^{a-1} d_n \\ +\tilde{H}(\phi(z))\sum_{r=a}^{b-1} p_r+\tilde{V}(\phi(z))[d_0+\sum_{n=0}^{a-1} q_n z^n]-\sum_{n=0}^{b-1} p_n z^n]\end{array}}{\phi(z)(z^b-\tilde{H}(\phi(z)))}. \tag{41}$$

Service completion: Let $P_S(z)$ be the probability that the server's service completion at time t.
From equations (32), (33) and (34)

$$P_S(z) = \frac{\begin{array}{l}[(z^b-\tilde{H}(\phi(z)))[(1-\tilde{L}(\phi(z)))\sum_{r=1}^{a-1} d_r+((1-\tilde{H}(\phi(z))))\sum_{r=a}^{b-1} p_r] \\ +(1-\tilde{H}(\phi(z)))\{\tilde{L}(\phi(z))\sum_{n=1}^{a-1} d_n+\tilde{H}(\phi(z))\sum_{r=a}^{b-1} p_r-\sum_{n=0}^{b-1} p_n z^n \\ +\tilde{V}(\phi(z))[d_0+\sum_{n=0}^{a-1} q_n z^n]\}]\end{array}}{\phi(z)(z^b-\tilde{H}(\phi(z)))}. \tag{42}$$

Vacation completion: Let $P_V(z)$ be the probability that the server is on vacation at time t. From equations (35) and (36)

$$P_V(z) = \frac{(1-\tilde{V}(\phi(z)))}{\phi(z))}[d_0 + \sum_{n=0}^{a-1} q_n z^n]. \tag{43}$$

4.5 Probability of various server states

The server on vacation:

$$\lim_{z\to 1} \tilde{V}(z,0) = P(V) = E(V)[d_0 + \sum_{n=0}^{a-1} q_n z^n]. \tag{44}$$

The server is busy:

$$\lim_{z\to 1}\left[\sum_{r=1}^{a-1} \tilde{L}_r(z,0) \sum_{r=a}^{b-1} \tilde{H}_r(z,0)\right] = P(B) = \frac{b(L_1 \sum_{r=1}^{a-1} d_r + H_1 \sum_{r=a}^{b-1} p_r) + H_1\{V_1(d_0 + \sum_{n=0}^{a-1} q_n z^n) + \sum_{n=0}^{a-1} n q_n - \sum_{n=0}^{b-1} n p_n\}}{\lambda X_1 (b - C_1)} \tag{45}$$

4.6 Performance measures

The expected length of the idle period: Let I is the random variable denoting the 'Idle period due to multiple vacation process'. Let Y be the random variable defined by

$$Y = \begin{cases} 0 & if\ the\ server\ finds\ atleast\ 'a' customers\ after\ the\ first\ vacation \\ 1 & if\ the\ server\ finds\ less\ 'a' customers\ after\ the\ first\ vacation \end{cases}$$

Now

$$\begin{aligned} E(I) &= E(I\,/\,Y = 0)\,P(Y = 0) + E(I\,/\,Y = 1)\,P(Y = 1) \\ &= E(V)\,P(Y = 0) + (E(V) + E(I))\,P(Y = 1) \end{aligned}$$

Solving for $E(I)$, we get

$$E(I) = \frac{E(V)}{1-P(Y=1)} = \frac{E(V)}{1-\sum_{n=0}^{a-1} \beta_n d_0} \tag{46}$$

The mean queue length: The mean queue length: The expected length of the queue E(Q) at an arbitrary time epoch, is obtained by differentiating P(z) at z=1 and is given by $\lim_{z\to 1} P(z) = E(Q)$

$$E(Q) = \frac{Nr^{(III)}Dr^{(II)} - Nr^{(II)}Dr^{(III)}}{3(Dr^{(II)})^2}, \tag{47}$$

where

$$Nr^{(II)} = -(2bL_1 - 2H_1)\sum_{r=1}^{a-1} d_r - 2H_1 \sum_{r=a}^{b-1} (b-r)p_r - 2bV_1(d_0 + \sum_{n=0}^{a-1} q_n),$$

$$Nr^{(III)} = -\big(H_2 + 3(bL_2 + b(b-1)L_1)\big)\sum_{r=1}^{a-1} d_r + \Big(H_3 \sum_{r=1}^{a-1} d_r + 3(H_2 \sum_{r=1}^{a-1} rd_r + H_1 \sum_{r=1}^{a-1} r(r-1)d_r)\Big)$$
$$-3\big(H_2 \sum_{r=a}^{b-1} (-r)p_r + H_1 \sum_{r=a}^{b-1} \big(b(b-1) - r(r-1)\big)p_r\big) - 6bV_1 \sum_{r=0}^{a-1} rq_r$$
$$-3(bV_2 + b(b-1)V_1)(d_0 + \sum_{n=0}^{a-1} q_n),$$

$$Dr^{(II)} = -\lambda X_1(b - H_1),$$

$$Dr^{(III)} = -3\{\lambda X_2(b - H_1) + \lambda X_1[b(b-1) - H_2]\}.$$

5. Cost Model

The cost function incorporates the following cost components per unit of time, which are given in Table 3.

Table 3. Cost components per unit of time

C_0	The operating cost per unit of time
C_h	The holding cost per unit time for each unit of the system.
C_r	The reward per unit of time due to vacation
C_s	The start-up cost

$$Total\ Average\ Cost = \frac{C_s - C_r E(I)}{E(T_c)} + C_h E(Q) + C_0 \rho \tag{48}$$

where expected length of cycle $E(T_c) = \frac{E(V)}{1-\sum_{n=0}^{a-1} \beta_n d_0} + [E(L) - E(H)] + \frac{E(H)}{[d_0 + \sum_{n=0}^{a-1} q_n]}$.

6. Numerical Illustration

Numerical methods are employed to estimate the unknown chances of different queue sizes.The roots of the equation $(z^b - \widetilde{H}(\phi(z)))$ are obtained and solved in MATLAB. A numerical example is presented with these assumptions:

1. The arrivals come in batches with a geometric distribution and an average of 2.
2. The service time for both low and high-capacity batches follows a 2-Erlang distribution
3. The vacation time follows an exponential distribution with a parameter $\gamma = 5$.

The effects on performance characteristics from varying the system parameters, arrival rate (λ), server's service rate (μ_1) with $\mu_2 \neq 0$ and $\mu_2 = 0$ are analyzed. The service times for both low and high-capacity batches follow a 2-Erlang distribution and Server's vacation time follows an exponential distribution.

The results have been analyzed in tabular forms and two-dimensional graphs. The stability condition is satisfied for arbitrary chosen values with a minimum service capacity a=5, maximum service capacity b=10.

The effect of increasing arrival rate (λ) is shown in Table 4 and Figures 2 and 3. Thus, if the arrival rate increases the stability condition (ρ), mean queue length (E(Q)), mean waiting time (E(W)) are increasing and mean Idle time E(I) is decreasing.

Table 4. Arrival rate influences performance measures ($Let\ \mu_1 = 25, \mu_2 = 15\ and\ \gamma = 5$)

λ	ρ	$E(Q)$	$E(W)$	$E(I)$
10.0	0.160	57.8939	2.8947	0.6023
10.5	0.168	62.3707	2.9700	0.5650
11.0	0.176	67.0076	3.0458	0.5330
11.5	0.184	71.8137	3.1223	0.5051
12.0	0.192	76.8001	3.2000	0.4807
12.5	0.200	81.9785	3.2791	0.4591
13.0	0.208	87.3615	3.3601	0.4399
13.5	0.216	92.9628	3.4431	0.4227
14.0	0.224	98.7973	3.5285	0.4072
14.5	0.232	104.8810	3.6166	0.3931
15.0	0.240	111.2320	3.7077	0.3804
15.5	0.248	117.8690	3.8022	0.3687
16.0	0.256	124.8130	3.9004	0.3581
16.5	0.264	132.0880	4.0027	0.3482
17.0	0.272	139.7170	4.1093	0.3392

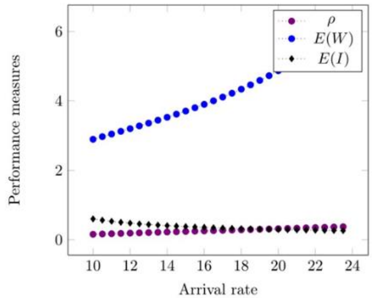

Figure 2. Performance measures vs. arrival rate

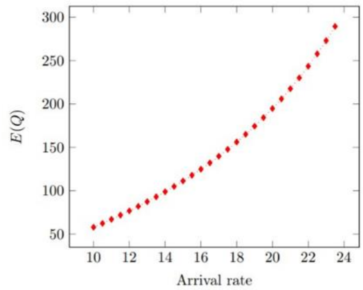

Figure 3. $E(Q)$ vs. arrival rate

The increase in the server's service rate (μ_1) with $\mu_2 \neq 0$ leads to decreases of ρ, E(Q), E(W) and increases E(I) as given in Table 5 and Figures 4 and 5. Similarly, Table 6 and Figures 6 and 7 depict that server's service rate (μ_1) with $\mu_2 = 0$ leads to decreases of ρ, E(Q), E(W) and increases E(I).

Table 5 assumes a low-capacity batch service, while Table 6 assumes no low-capacity batch service. The numerical results show that as the service rate increases, both queue length and waiting time decrease in Table 5, being lower than in Table 6.

Table 5. High-capacity batch service rate influences performance measures (Let $\lambda = 10, \mu_2 = 15$ and $\gamma = 5$)

μ_1	ρ	EQ	EW	EI
18.00	0.2222	47.1218	2.3561	0.4905
18.25	0.2192	46.8030	2.3402	0.4951
18.50	0.2162	46.4992	2.3250	0.4997
18.75	0.2133	46.2093	2.3105	0.5042
19.00	0.2105	45.9324	2.2966	0.5087
19.25	0.2078	45.6679	2.2834	0.5131
19.50	0.2051	45.4148	2.2707	0.5175
19.75	0.2025	45.1726	2.2586	0.5218
20.00	0.2000	44.9405	2.2470	0.5261
20.25	0.1975	44.7181	2.2359	0.5303
20.50	0.1951	44.5047	2.2252	0.5345
20.75	0.1928	44.2998	2.2150	0.5386
21.00	0.1905	44.1030	2.2052	0.5427
21.25	0.1882	43.9138	2.1957	0.5468
21.50	0.1860	43.7318	2.1866	0.5508
21.75	0.1839	43.5566	2.1778	0.5547
22.00	0.1818	43.3878	2.1694	0.5586
22.25	0.1798	43.2252	2.1613	0.5625
22.50	0.1778	43.0684	2.1534	0.5663
22.75	0.1758	42.9171	2.1459	0.5701
23.00	0.1739	42.7710	2.1386	0.5738
23.25	0.1720	42.6299	2.1315	0.5775
23.50	0.1702	42.4935	2.1247	0.5812

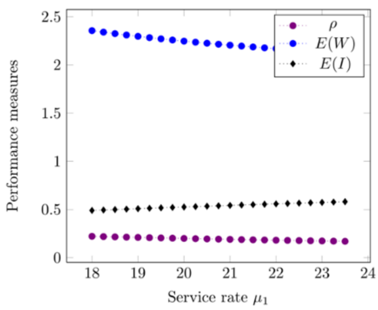

Figure 4. Performance measures vs. service rate (μ_1)

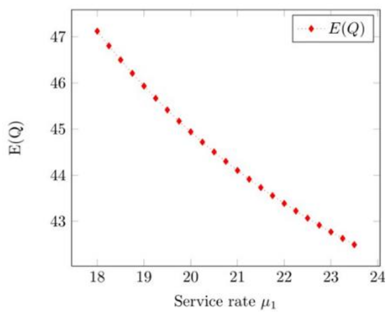

Figure 5. $E(Q)$ vs. service rate (μ_1)

Table 6. High-capacity batch service rate influences performance measures (Let $\lambda = 10, \mu_2 = 0$ and $\gamma = 5$)

μ_1	ρ	EQ	EW	EI
18.00	0.2222	69.9624	3.4981	11.4185
18.25	0.2192	69.3820	3.4691	12.4678
18.50	0.2162	68.8272	3.4414	13.6738
18.75	0.2133	68.2956	3.4148	15.0631
19.00	0.2105	67.7866	3.3893	16.6930
19.25	0.2078	67.2987	3.3649	18.6266
19.50	0.2051	66.8308	3.3415	20.9651
19.75	0.2025	66.3809	3.3191	23.8061
20.00	0.2000	65.9491	3.2975	27.3888
20.25	0.1975	65.5338	3.2767	32.0033
20.50	0.1951	65.1339	3.2567	38.1495
20.75	0.1928	64.7495	3.2375	46.8538
21.00	0.1905	64.3786	3.2189	59.8671
21.25	0.1882	64.0213	3.2011	81.7738
21.50	0.1860	63.6764	3.1838	125.7630
21.75	0.1839	63.3440	3.1672	262.7660

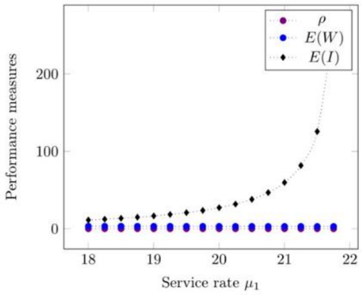

Figure 6. Performance measures vs. service rate (μ_1)

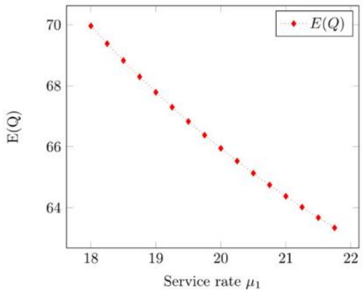

Figure 7. $E(Q)$ vs. service rate (μ_1)

7. Conclusions

The bulk service queuing system with queue length-dependent service and multiple vacations using the supplementary variable technique was investigated. A probability-generating function for the queue length, expected busy period, idle period, and waiting time was derived. MATLAB software was used for detailed numerical illustrations.

The analysis done assessed mean queue length, mean waiting time, and mean idle period. Table 5 assumes a low-capacity batch service, Table 6 assumes no low-capacity batch service. The numerical results show that as the service rate increases both queue length and waiting time decrease with the expected queue length and waiting time in Table 5 being lower than in Table 6.

In manufacturing industries, low quantities of raw materials delay the production, increasing the consumer waiting time. Providing low-capacity batch service can reduce queue length and waiting time, making this model applicable to real-world scenarios with similar complexities.

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