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Original Article

Ramsey numbers of connected 5-cycle matchings

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Abstract

In this paper, we show that the exact values of the Ramsey numbers of connected 5-cycle matchings, denoted by $R_2(c(nC_5))$, are $11n-2$ for $n \geq 2$.

Keywords: graph coloring, connected 5-cycle matching, edge-coloring, multiple copies of graphs, Ramsey number

1. Introduction

In this paper, all graphs discussed are finite, undirected and simple, meaning they contain no loops or multiple edges. We denote by K_n the complete graph and by C_n the cycle graph of n vertices.

Let G and H be graphs with disjoint vertex sets. The vertex set and the edge set of the graph G are denoted by $V(G)$ and $E(G)$, respectively. The *disjoint union* or *sum* of graphs G and H, denoted by $G+H$, is the graph obtained by taking the disjoint union of the vertex sets and edge sets from both G and H. The disjoint union of n copies of G is denoted by nG. Let S be a subset of the vertex set of G, $G-S$ is the graph obtained by removing the vertices in S. An *induced subgraph*, $G[S]$, is $G-\bar{S}$ where $\bar{S}=V(G)-S$. We also call $G[S]$ the subgraph of G induced by S. A *k-edge-coloring* of a graph G is a labeling $f: E(G) \to S$ where $|S|=k$. The labels are called *colors*. The edges of one color form a *color class*. If all edges of G are assigned with the same color, then G is called *monochromatic*.

Next, we will introduce an important definition on the Ramsey number, which will be used throughout this paper. Note that most of our definitions and notations are primarily sourced from West (2001).

Definition 1. (West, 2001). Let $G_1, G_2, \dots, G_k$ be graphs. The (*graph*) *Ramsey number* is the smallest integer n such that every k-edge-coloring of K_n contains a copy of G_i in color i for some i, denoted by $R(G_1, G_2, \dots, G_k)$. When $G_i = G$ for all i, we write $R_k(G)$ instead of $R(G_1, G_2, \dots, G_k)$.

Definition 2. (Roberts, 2017). Let G be a graph and $c(nG)$ is the set of all connected graphs containing nG. The *k-color Ramsey number of connected G-matchings nG*, denoted by $R_k(c(nG))$, is the smallest integer N such that every k-edge-coloring on K_N contains a monochromatic copy of a graph in $c(nG)$.

In this paper, we focus solely on 2-color Ramsey numbers using red and blue. Nowadays, there are only results on the Ramsey numbers of connected graph matchings on complete graphs. Burr (1981) proved an important theorem that helps finding lower bounds for this type of Ramsey numbers. Cockayne and Lorimer (1975) first gave a result for a 2-color connected matching. Gyárfás and Sárközy (2016) also proved the exact value for a connected triangle matching. Later Roberts (2017) gave a general result for a connected clique matching. The next three theorems are the results of 2-color connected clique matchings.

Theorem 3. (Cockayne & Lorimer, 1975). For $n \geq 2$, $R_2(c(nK_2)) = R_2(nK_2) = 3n-1$.

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Theorem 4. (Gyárfás & Sárközy, 2016). For $n \geq 2$, $R_2\big(c(nK_3)\big) = 7n - 2$.

Theorem 5. (Roberts, 2017). For $r \geq 4$ and $n \geq R_2(K_r)$, we have $R_2\big(c(nK_r)\big) = (r^2 - r + 1)n - r + 1$.

The question on determining the Ramsey number of connected graphs matching besides the clique still remains wide open. In this paper, we will provide the Ramsey number of the connected 5-cycle matchings, which is $R_2\big(c(nC_5)\big)$.

2. Main Result

In this section, we will prove the Ramsey number of the connected 5-cycle matching. Before we proceed to the proof, we first need to introduce some necessary theorems. The bounds on the Ramsey number of multiple copies of C_5 were given by Denley (1996). Moreover, the Ramsey number of matchings versus regular graphs was provided by Faudree, Schelp and Sheehan (1980).

Theorem 6. (Denley, 1996). Let $r \geq 5$ and $m \geq n \geq 1$. Then

$$rm + 3n - 1 \leq R(mC_r, nC_5) \leq rm + 3n + r - 4.$$

In particular, setting $r = 5$ gives the general Ramsey number for disjoint 5-cycles,

$$5m + 3n - 1 \leq R(mC_5, nC_5) \leq 5m + 3n + 1.$$

Theorem 7. (Faudree, Schelp, & Sheehan, 1980). Let G be an r-regular graph $(1 \leq r \leq 6)$ of n vertices. Then

$$R(mK_2, G) = \max\{n + 2m - \alpha(G) - 1, n + m - 1\},$$

where $\alpha(G)$ is the independence number of G.

Again, if we set $G = (k+1)C_5$, then we have the following corollary.

Corollary 8. Let $m, k \in \mathbb{N}$. Then

$$R(mK_2, (k+1)C_5) = \begin{cases} 3k + 2m + 2, & m \geq 2k + 2, \\ 5k + m + 4, & m \leq 2k + 2, \end{cases}$$

where both cases are equal when $m = 2k + 2$.

Proof. Since $\alpha(C_5) = 2$, we have $\alpha\big((k+1)C_5\big) = 2(k+1)$. We know that $(k+1)C_5$ has $5(k+1)$ vertices. By Theorem 7, we obtain

$$R(mK_2, (k+1)C_5) = \max\{3k + 2m + 2, 5k + m + 4\}.$$

We see that $3k + 2m + 2 \geq 5k + m + 4$ if and only if $m \geq 2k + 2$. In addition, $3k + 2m + 2 = 5k + m + 4$ when $m = 2k + 2$. So, the result holds.

Theorem 9. For $n \geq 2$, $R_2\big(c(nC_5)\big) = 11n - 2$.

Proof. First, we need to prove the lower bound, i.e., $R_2\big(c(nC_5)\big) \geq 11n - 2$, by constructing a 2-edge-coloring on K_{11n-3} without monochromatic red or blue subgraphs in $c(nC_5)$. Consider two blue copies of K_{5n-1} and a red copy of K_{n-1}. Then we join these three graphs together only by red edges, and thus we obtain a 2-edge-coloring on K_{11n-3}. Clearly, there is no blue graph from $c(nC_5)$ in this coloring. For a red graph in $c(nC_5)$, since C_5 is not bipartite, in order to form a red C_5, we need to use at least one vertex in the red K_{n-1}. But, there are not enough vertices in the red K_{n-1}, which means this coloring contains no red graphs in $c(nC_5)$. This finishes the proof of the lower bound.

Next, for the upper bound, let G be the graph K_{11n-2} together with a 2-edge-coloring with red and blue. We will prove that G contains a monochromatic graph from $c(nC_5)$. For every simple graph G, either G or $\bar{G}$ is connected. This implies that at least one color class of G is connected. Thus, we can assume that the red is connected. By Theorem 6, the G contains a monochromatic nC_5. If that nC_5 is red, then we are done. If it is blue and the blue color class is also connected, then we are done as well. Therefore, we can assume that G contains a blue nC_5 and contains no blue graph in $c(nC_5)$. Let k denote the maximum number of disjoint blue copies of C_5 in G. From the discussion above, we know $k \geq n$. Since the blue graph is disconnected, G can be separated into two blue (not necessarily connected) subgraphs, where all edges between the two subgraphs are red. Let V_1 and V_2 be the vertex sets of these subgraphs. Since $k \geq n$ and there is no blue graph from $c(nC_5)$ in G, we can ensure that $G[V_1]$ contains a maximum of m disjoint blue copies of C_5 where $\frac{n}{2} \leq m \leq n - 1$. (If the subgraph initially contains more than $n - 1$ disjoint copies, it can be divided into two smaller subgraphs. At least one of these subgraphs will contain at least $\frac{n}{2}$ copies. Repeat the process until obtaining a subgraph containing m copies such that $\frac{n}{2} \leq m \leq n - 1$).

We show a contradiction by constructing a red nC_5 in G, where each one is formed by joining a red $K_2 + K_1$ in one subgraph and two vertices in another subgraph. We separate the proof into four cases depending on the size of V_1.

Case 1: $|V_1| \geq 9n - 1$.

By Theorem 6, $|V_1| \geq 9n - 1 \geq 8n + 1 \geq R_2(nC_5)$ for $n \geq 2$. Since $G[V_1]$ contains no blue nC_5, it must contain a red nC_5.

Case 2: $5m + n + 4 \leq |V_1| \leq 9n - 2$.

We know that $G[V_1]$ contains no blue $(m+1)C_5$. Since $m \geq \frac{n}{2}$, we have $2m + 2 > n$. By Corollary 8, $R(nK_2, (m+1)C_5) = 5m + n + 4 \leq |V_1|$. Thus $G[V_1]$ contains a red nK_2. Since $|V_1| \geq 3n$, $G[V_1]$ contains a red $n(K_2 + K_1)$. From $|V_2| = 11n - 2 - |V_1| \geq 2n$, we know $G[V_2]$ contains an $n(2K_1)$. Therefore G contains a red nC_5.

Case 3: $5m + 5 \leq |V_1| \leq 5m + n + 3$.

We will construct a red nC_5 using $p(K_2 + K_1) + (n - p)(2K_1)$ in $G[V_1]$ and $(n - p)(K_2 + K_1) + p(2K_1)$ in $G[V_2]$, where $1 \leq p \leq n - 1$.

Let $|V_1| = 5m + p + 4$, where $1 \leq p \leq n - 1$. Since $m \geq \frac{n}{2}$, we have $p < 2m + 2$. By Corollary 8, $R(pK_2, (m+1)C_5) = 5m + p + 4$. Thus, $G[V_1]$ contains a red pK_2. To show that $G[V_1]$ contains a red $p(K_2 + K_1) + (n -$

$p)(2K_1)$, we need to show that $|V_1| \geq 3p + 2(n-p) = 2n + p$. Since $m \geq \frac{n}{2}$, we obtain $|V_1| = 5m + p + 4 \geq \frac{5n}{2} + p \geq 2n + p$. Hence, $G[V_1]$ contains a red $p(K_2 + K_1) + (n - p)(2K_1)$.

Next, we need to show that $G[V_2]$ contains a red $(n-p)(K_2 + K_1) + p(2K_1)$.

We first show that there is a red $(n-p)K_2$ in $G[V_2]$. Since $|V_1| = 5m + p + 4$, we have $|V_2| = 11n - 2 - |V_1| = 11n - 5m - p - 6$. Suppose that $G[V_2]$ contains a maximum of m' disjoint blue copies of C_5.

First, we consider the case when $m' \leq 2n - m - 2$. By Corollary 8, we have

$$R\left((n-p)K_2, (m'+1)C_5\right) = \begin{cases} 3m' + 2n - 2p + 2, & n - p \geq 2m' + 2 \\ 5m' + n - p + 4, & n - p \leq 2m' + 2. \end{cases}$$

If $n - p \geq 2m' + 2$, then with $m' \leq 2n - m - 2$, we have

$$R\left((n-p)K_2, (m'+1)C_5\right) = 3m' + 2n - 2p + 2 \leq 8n - 3m - 2p - 4.$$

Since $n \geq m + 1 > 0$ and $p > 0$, we have

$$8n - 3m - 2p - 4 \leq 10n - 5m - 2p - 6 < 11n - 5m - p - 6 = |V_2|.$$

If $n - p \leq 2m' + 2$, again with $m' \leq 2n - m - 2$, we obtain $5m' \leq 10n - 5m - 10$.

Therefore,

$$R\left((n-p)K_2, (m'+1)C_5\right) = 5m' + n - p + 4 \leq 11n - 5m - p - 6 = |V_2|.$$

From both subcases, $G[V_2]$ contains a red $(n-p)K_2$.

Next, suppose that $m' \geq 2n - m - 1$. Since $m \leq n - 1$, we have $m' \geq n$. Since $G[V_2]$ contains no blue $c(nC_5)$, V_2 can be partitioned into U_1 and U_2, where $G[U_1]$ contains at least $\frac{n}{2}$ but at most $n-1$ disjoint copies of blue C_5. Note that $G[U_2]$ will contain at least $m' - n + 1$ disjoint copies of blue C_5.

If $m' \geq n - 1 + \frac{n-p}{5}$, then both $G[U_1]$ and $G[U_2]$ contain at least $\frac{n-p}{5}$ disjoint blue copies of C_5. This means $|U_1| \geq n - p$ and $|U_2| \geq n - p$. In addition, all edges between these two subgraphs are red. So, there is a red $(n-p)K_2$ in $G[V_2]$.

Suppose that $m' < n - 1 + \frac{n-p}{5}$. Since $m' \geq n$, let $m' = n - 1 + q$, for some $1 \leq q < \frac{n-p}{5}$. Then both $G[U_1]$ and $G[U_2]$ contain at least q disjoint blue copies of C_5. Therefore, we obtain a red $(5q)K_2$ from $5q$ vertices in each subgraph.

Next, we let V_2' be the set of vertices in V_2 after deleting all vertices contained in the red $(5q)K_2$ from the previous step. Then $G[V_2']$ contains at most $m' - 2q$ disjoint copies of blue C_5 and $|V_2'| = 11n - 5m - p - 10q - 6$. Now, we need a red $(n - p - 5q)K_2$ in $G[V_2']$. Since $m' \geq n$, we obtain $2m' - 4q + 2 > n - 5q - p$. By Corollary 8, we have

$$R\left((n-p-5q)K_2, (m' - 2q + 1)C_5\right) = 5m' + n - p - 15q + 4.$$

Since $m' = n - 1 + q$, we obtain

$$5m' + n - p - 15q + 4 = 5(n - 1 + q) + n - p - 15q + 4 = 6n - p - 10q - 1.$$

From $m \leq n - 1$, we can conclude that

$$6n - p - 10q - 1 = 11n - 5(n-1) - p - 10q - 6 \leq 11n - 5m - p - 10q - 6 = |V_2'|.$$

Therefore, $G[V_2']$ contains a red $(n - p - 5q)K_2$ as desired. Hence, we have a red $(n-p)K_2$ in $G[V_2]$.

In both cases, $G[V_2]$ contains a red $(n-p)K_2$. Clearly, $|V_2| = 11n - 5m - p - 6 \geq 5n > 3n - p$. This implies that V_2 has enough vertices to form $(n-p)(K_2 + K_1) + p(2K_1)$. So, $G[V_2]$ contains a red $(n-p)(K_2 + K_1) + p(2K_1)$. Thus, we have a red nC_5 in G.

Case 4: $|V_1| \leq 5m + 4$.

We will construct a red nC_5 in G using a red $n(2K_1)$ in $G[V_1]$ and a red $n(K_2 + K_1)$ in $G[V_2]$. Since V_1 contains at least $\frac{n}{2}$ disjoint blue C_5, we have $|V_1| \geq 2n$. Then $G[V_1]$ contains an $n(2K_1)$.

From $|V_1| \leq 5m + 4 \leq 5n - 1$, we have $|V_2| \geq 6n - 1$. Again, we suppose that $G[V_2]$ contains a maximum of m' disjoint blue copies of C_5. We consider four subcases.

Subcase 4.1: $m' \leq \frac{n-2}{2}$.

This means $n \geq 2m' + 2$. By Corollary 8, we have

$$R\left(nK_2, (m'+1)C_5\right) = 3m' + 2n + 2 \leq \frac{7n-2}{2} < 6n - 1 \leq |V_2|.$$

Subcase 4.2: $\frac{n-2}{2} < m' \leq n - 1$.

This means $n < 2m' + 2$. Again, by Corollary 8, we have

$$R\left(nK_2, (m'+1)C_5\right) = 5m' + n + 4 \leq 6n - 1 \leq |V_2|.$$

Subcase 4.3: $n \leq m' < n - 1 + \frac{n}{5}$.

Note that when n is less than 6, there is no such m'. In this subcase, we can assume that $n \geq 6$. Let $m' = n - 1 + q$, where $1 \leq q < \frac{n}{5}$. Then $G[V_2]$ can be separated into two blue subgraphs, where U_1 and U_2 are vertex sets of these subgraphs, in such a way that each of $G[U_1]$ and $G[U_2]$ contains at least q disjoint blue copies of C_5. (Similar to the construction of m at the beginning of the proof, it can be done so that $G[U_1]$ contains at least $\frac{n}{2}$ but at most $n-1$ copies.) Then there is a red $(5q)K_2$

in $G[V_2]$. Next, let V_2' be a set of vertices in V_2 apart from the vertices in the red $(5q)K_2$. We have

$$|V_2'| = |V_2| - 10q = 11n - 5m - 10q - 6.$$

So, we need a red $(n-5q)K_2$ in $G[V_2']$. Since $G[V_2']$ has at most $m' - 2q = n - q - 1$ disjoint blue copies of C_5, and $n - 5q < 2(n-q) + 2$, by Corollary 8, we have

$$R((n-5q)K_2, (n-q)C_5) = 5(n-q-1) + n - 5q + 4 = 6n - 10q - 1.$$

Since $n \geq m + 1$, we obtain

$$R((n-5q)K_2, (n-q)C_5) = 6n - 10q - 1 \leq 11n - 5m - 10q - 6 = |V_2'|.$$

Thus, we have a red $(n-5q)K_2$ together with the red $(5q)K_2$ that we have constructed prior. Hence, we have a red nK_2 in $G[V_2]$.

Subcase 4.4: $m' \geq n - 1 + \frac{n}{5}$.

Then V_2 can be partitioned into U_1 and U_2, such that both $G[U_1]$ and $G[U_2]$ contain at least $\frac{n}{5}$ disjoint blue copies of C_5. Thus, $|U_1|, |U_2| \geq n$. Pairing one vertex from U_1 with another vertex from U_2, we get a red nK_2 in $G[V_2]$.

In all four subcases, we can conclude that there is a red nK_2 in $G[V_2]$. In addition, since $|V_2| \geq 6n - 1 \geq 3n$, $G[V_2]$ contains a red $n(K_2 + K_1)$. Therefore, we obtain a red nC_5 in G. This completes the proof.

3. Conclusions

In this paper, we proved that $R_2(c(nC_5)) = 11n - 2$, for $n \geq 2$. In order to prove this result, the Ramsey number of multiple copies of 5-cycles is a very essential tool. But Ramsey numbers of multiple copies of k-cycles when $k \geq 6$ remain unknown. This makes it difficult and very interesting to prove Ramsey numbers for connected k-cycle matchings when $k \geq 6$.

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