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*Original Article*

# Bending behaviors of simply supported rectangular plates with an internal line sagged and unsagged supports

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**Abstract**

This research work deals with the application of dual series equations to the problems of simply supported rectangular plate with an internal line support under uniformly distributed load. Two different types of problem are considered depending on the nature of singularities that allowed in the fields, The first is the plate having an internal line unsagged support that the moment singularity is assumed at the tip of internal line support. The second involves the advancing contact problem between the plate and the internal line sagged support in which the shear singularity exhibits at the tip of contact. However, both types of singularity are in the order of an inverse square root. By choosing the proper finite Hankel transform, the dual series can be converted to the inhomogeneous Fredholm integral equation. This equation, with a numerical technique, is then reduced to a set of simultaneous equations suitable for numerical solution. The physical quan-tities of the plates and the extent of contact related to the level of loading in the case of free contact are provided in the present work.

**Keywords:** plate bending, advancing contact, singularity, dual series equations, Fredholm integral equation

## 1. Introduction

Focusing on the problems of plate with mixed boundary conditions, there are numerous analytical and numerical methods used to analyze the problem. The numerical methods are generally found to be unsatisfactory (Leissa *et al.*, 1969), especially at the transition point of discontinuous boundary due to the problem singularity (Williams, 1952), and then, the use of integral transform (Sneddon, 1972) is one of the appropriately analytical methods to solve the problem which leads to determine the solution of an integral equation. Much attention has been said by many researchers to investigate the static bending problems (Yang, 1968; Keer and Sve, 1970; Kiattikomol *et al.*, 1974; Kiattikomol and Porn-anupapkul,1985; Kiattikomol and Sriswasdi, 1988), vibration characteristics, and stability and buckling behaviors (Leissa, 1969; Keer and Stahl, 1972; Stahl and Keer, 1972). However, the mentioned works are the problems of plate where the supports have the same level. Dundurs *et al.* (1974) first investigated the contact between the plates and the sagged supports in which the sagged supports are located at the plate edges. For the cases of sagged support placed in domain of the plate, recently, Sompornjaroensuk and Kiattikomol (2006) treated the two problems of rectangular plate simply supported on the two opposite edges and either free or clamped on the two other edges with an internal line sagged support. The singularity at the tips of contact between the plate and an internal line sagged support is introduced in the order of an inverse square root in the shear. Therefore, this research considers the two problems of simply supported rectangular plate with an internal line

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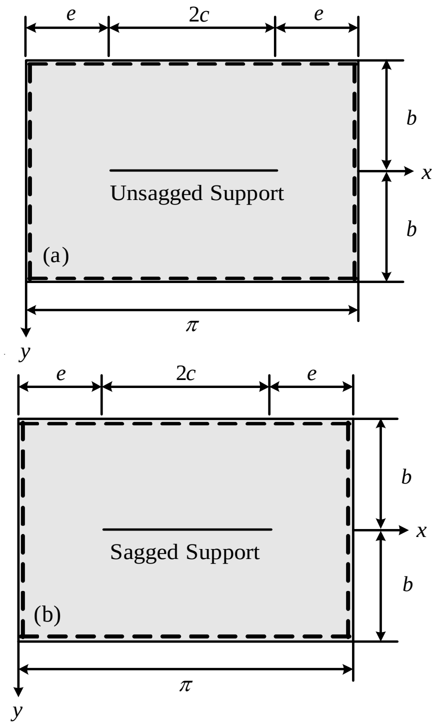


**Figure 1. Rectangular plates with internal line: (a) unsagged support and (b) sagged support**

sagged and unsagged supports. The solution technique similar to the previous work is applied to the present work while both types of problem are treated here within the same formulation except that, for the problem of plate with unsagged support, the order of singularity is an inverse square root in the moment instead of an inverse square root in the shear as in the case of an internal line sagged support for which the moment is bounded at the tips of contact.

## 2. Governing Equation and Boundary Conditions

Considering the geometry of plate as illustrated in Figure 1, the actual dimensions of the plate are $a$ and $2\overline{b}$ and scaled by the factor $\pi/a$. The partial differential equation governing the deflection function $w(x,y)$ of the plate under the uniformly distributed load $q$ is expressed by

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{qa^4}{\pi^4 D}, \tag{1}$$

where $D=Eh^3/12(1-\nu^2)$ = flexural rigidity, $E$ = Young's modulus, $\nu$ = Poisson's ratio, and $h$ = plate thickness.

Due to the two-fold symmetry in deflection function, the boundary conditions need only to be written on one quadrant $0\le x\le\pi/2$ and $0\le y\le b$ as follows:

$$w=\frac{\partial^2 w}{\partial y^2}=0\ :\ 0\le x\le\frac{\pi}{2},\ y=b, \tag{2a,b}$$

$$\frac{\partial w}{\partial y}=0\ \quad :\ 0\le x\le\frac{\pi}{2},\ y=0, \tag{3}$$

$$w=\delta,\ \frac{\partial w}{\partial x}=\frac{\partial^2 w}{\partial x^2}=0\ :\ e<x\le\frac{\pi}{2},\ y=0, \tag{4a,b,c}$$

$$\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}=0\ :\ 0\le x<e,\ y=0, \tag{5}$$

whilst $\delta=$ the initial gap between the plate and the internal line sagged support. Since $\delta=0$, the problem considered becomes the plate with an internal line unsagged support. It is also remarkable that Eqs (4) and (5) are the mixed boundary conditions and the others are the regular boundary conditions.

## 3. Formulation

Utilizing the Levy-Nadai approach (Timoshenko and Woinowsky-Krieger, 1959), the deflection function satisfying the support conditions of the plate edges at $x=0,\pi$ is taken as the sum of the particular ($w_p$) and complementary ($w_c$) solutions of Eq.(1),

$$w=w_p+w_c, \tag{6}$$

where

$$w_p=\frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5,...}^{\infty} m^{-5}\sin(mx);\ w_c=\sum_{m=1,3,5,...}^{\infty} Y_m\sin(mx), \tag{7a,b}$$

and

$$Y_m=\frac{qa^4}{D}\left[A_m\cosh(my)+B_m my\sinh(my)+C_m\sinh(my)+D_m my\cosh(my)\right] \tag{8}$$

in which the constants $A_m$, $B_m$, $C_m$, and $D_m$ are determined from the boundary conditions of Eqs.(2) and (3) that yield the following relations, with letting $\beta=mb$,

$$A_m=-\frac{2(2+\beta\tanh\beta)}{\pi^5 m^5\cosh\beta}+D_m\frac{(\sinh\beta\cosh\beta-\beta)}{\cosh^2\beta}, \tag{9}$$

$$B_m=\frac{2}{\pi^5 m^5\cosh\beta}-D_m\tanh\beta, \tag{10}$$

$$C_m=-D_m. \tag{11}$$

It can be noted that the problems are now reduced to the determination of a single constant $D_m$. The application of the mixed boundary conditions given in Eqs.(4c) and (5) leads to the dual series equations as follows:

$$\sum_{m=1,3,5,...}^{\infty} m^2 P_m\sin(mx)=0,\ \ e<x\le\frac{\pi}{2}, \tag{12}$$

$$\sum_{m=1,3,5,...}^{\infty} m^3 P_m(1+F_m)\sin(mx)=\sum_{m=1,3,5,...}^{\infty} G_m\sin(mx),\ \ 0\le x<e, \tag{13}$$

where

$$P_m=\frac{2}{\pi^5 m^5}\left[2-\left(\frac{2+\beta\tanh\beta}{\cosh\beta}\right)\right]+D_m\left(\frac{\sinh\beta\cosh\beta-\beta}{\cosh^2\beta}\right), \tag{14}$$

$$1+F_m=\frac{\cosh^2\beta}{\sinh\beta\cosh\beta-\beta}, \tag{15}$$

$$G_m = \frac{2}{\pi^5 m^5}\left[2-\left(\frac{2+\beta\tanh\beta}{\cosh\beta}\right)\right]\left(\frac{\cosh^2\beta}{\sinh\beta\cosh\beta-\beta}\right). \quad (16)$$

For the case of an internal line unsagged support, the dual series equations can be reduced to a single integral equation by representing the unknown function $P_m$ by a finite Hankel transform (Kiattikomol *et al.*, 1974; Kiattikomol and Porn-anupapkul, 1985)

$$m^2 P_m = \int_0^e t\phi(t)\left[J_1(mt)-\frac{t}{e}J_1(me)\right]dt; \quad m=1,3,5,\ldots, \quad (17)$$

and for an internal line sagged support, the function $P_m$ is replaced by (Dundurs *et al.*, 1974; Sompornjaroensuk and Kiattikomol, 2006)

$$m^2 P_m = \int_0^e t\phi(t)J_1(mt)\,dt; \quad m=1,3,5,\ldots, \quad (18)$$

while $\phi(t)$ is the unknown auxiliary function and $J_1(\ )$ is the Bessel function of the first kind and order 1.

By the same procedure as used in the previous works (Kiattikomol *et al.*, 1974; Sompornjaroensuk and Kiattikomol, 2006), substituting $P_m$ from Eqs.(17) and (18) that satisfy to each case of the plates into Eq.(13) and using the identity given in Stahl and Keer (1972),

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt)\cos(mx) = \frac{1}{2t} - \frac{xH(x-t)}{2t\left(x^2-t^2\right)^{1/2}} + \int_0^\infty \frac{I_1(ts)\cosh(xs)}{\exp(\pi s)+1}ds, \quad x+t<\pi, \quad (19)$$

where $H(\ )$, $I_1(\ )$ are the Heaviside unit step function and the modified Bessel function of the first kind and order 1, respectively, and then, after some manipulations and with the help of certain identities found in Gradshteyn and Ryzhic (1956), the second dual series equations, Eq.(13) can be reduced to the final form of an inhomogeneous Fredholm integral equation of the second kind

$$\Phi(\rho)+\int_0^1 K^{(i)}(\rho,r)\Phi(r)\,dr = f(\rho); \quad 0\le\rho\le 1, \quad (20)$$

in which

$$\Phi(\rho)=\phi(e\rho); \quad \Phi(r)=\phi(er), \quad (21)$$

$$i=\begin{cases}1 & \text{for an internal line unsagged support}\\ 2 & \text{for an internal line sagged support}\end{cases}, \quad (22)$$

$$K^{(1)}(\rho,r)=2e^2 r\left\{\begin{aligned}&\sum_{m=1,3,5,\ldots}^{\infty} mF_m\left[J_1(mer)-rJ_1(me)\right]J_1(me\rho)\\ &-\int_0^\infty \frac{s\left[I_1(ser)-rI_1(se)\right]I_1(se\rho)}{\exp(\pi s)+1}ds\end{aligned}\right\}, \quad (23a)$$

$$K^{(2)}(\rho,r)=2e^2 r\left\{\sum_{m=1,3,5,\ldots}^{\infty} mF_m J_1(mer)J_1(me\rho)-\int_0^\infty \frac{sI_1(ser)I_1(se\rho)}{\exp(\pi s)+1}ds\right\}, \quad (23b)$$

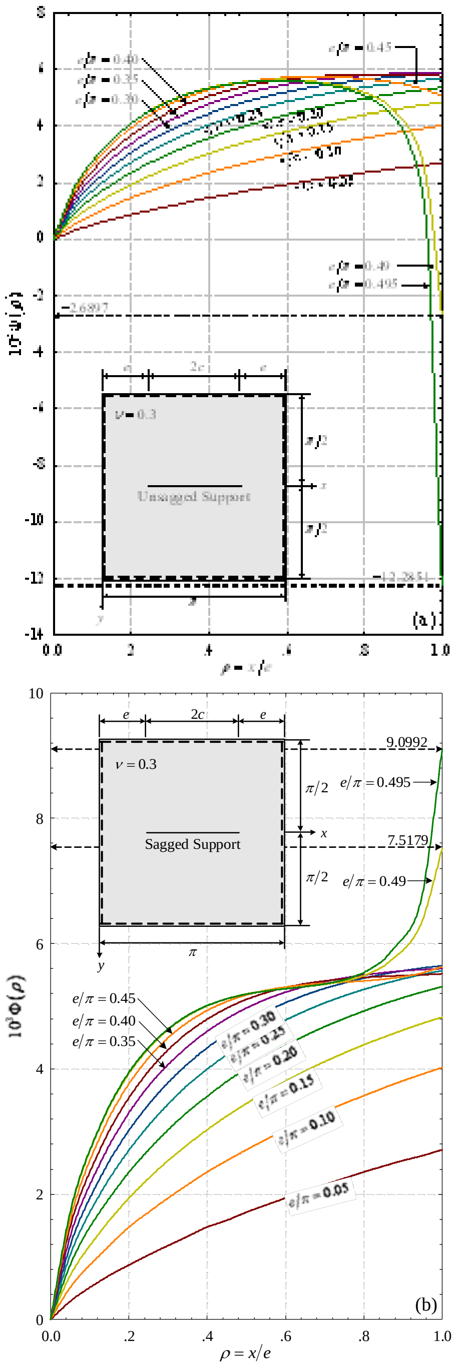


**Figure 2. Auxiliary functions of square plate with internal line: (a) unsagged support and (b) sagged support**

$$f(\rho)=2\sum_{m=1,3,5,\ldots}^{\infty} G_m J_1(me\rho). \quad (24)$$

An integral equation presented in Eq.(20) can be solved numerically to obtain the unknown auxiliary function $\Phi(\rho)$ by using standard methods.

## 4. Numerical Analysis and Results

Two different solutions can be determined from the same formulation by solving Eq.(20). In order to evaluate the

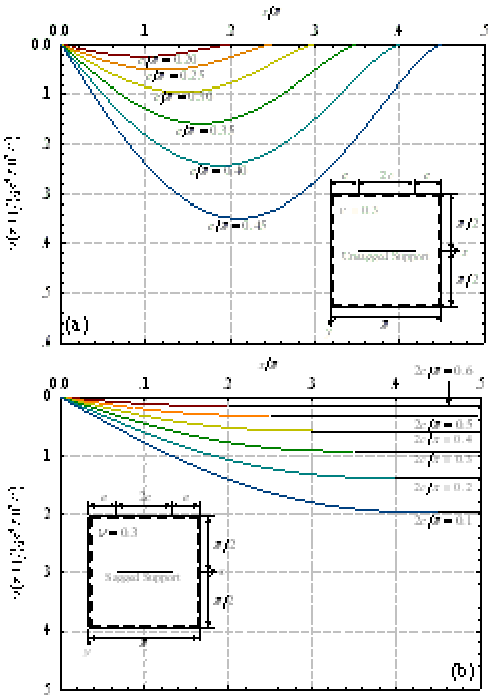


**Figure 3. Deflections at $0 \le x \le e, y = 0$ for square plate with internal line: (a) unsagged support and (b) sagged support**

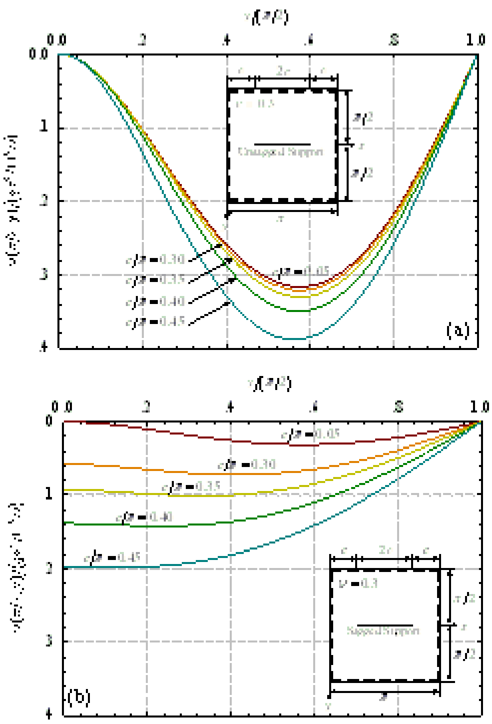


**Figure 4. Deflections at $x = \pi/2, 0 \le y \le \pi/2$ for square plate with internal line: (a) unsagged support and (b) sagged support**

physical quantities, the unknown auxiliary function $\Phi(\rho)$ must be obtained. Simpson's rule was chosen for this purpose because it is a simple method, leading to a system of linear

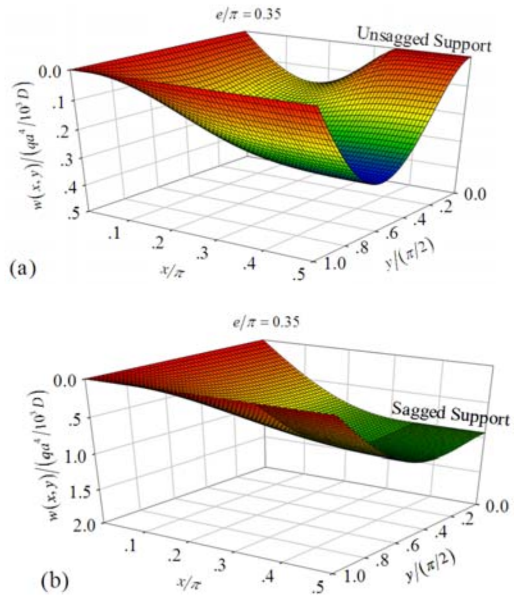


**Figure 5. Deflections surfaces bounded by the region $0 \le x, y \le \pi/2$ for square plate with internal line: (a) unsagged support and (b) sagged support**

algebraic equations and using the Gaussian elimination with partial pivoting to solve for the discretized values of $\Phi(\rho)$ as shown in Figure 2. It should be remarked that the closed form expressions for the improper infinite integral in Eqs. (23a,b) were not found. However, a numerical integration is easily performed with using a 16-point Gauss-Legendre quadrature formula. The infinite series in the kernel and in the function $f(\rho)$ were calculated to a relative error of 0.00001. The numerical evaluation was carried out only for the case of a square plate of scaled length $\pi$ with various cases of the scaled half-length of internal line support *c* and the Poisson's ratio was taken as 0.3.

Therefore, the deflection function of the plates that demonstrated in Figures 3 to 5 can be evaluated from Eq.(6). It is seen that the deflections of both cases are increased with the increasing of $e/\pi$-ratio. After the deflection is obtained, the stress resultants are calculated as follows:

$$M_x = -D\left(\frac{\pi}{a}\right)^2\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{25}$$

$$M_y = -D\left(\frac{\pi}{a}\right)^2\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{26}$$

$$V_x = -D\left(\frac{\pi}{a}\right)^3\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right], \tag{27}$$

$$V_y = -D\left(\frac{\pi}{a}\right)^3\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right], \tag{28}$$

where $M_x$, $M_y$ are the bending moments and $V_x$, $V_y$ are the support reactions.

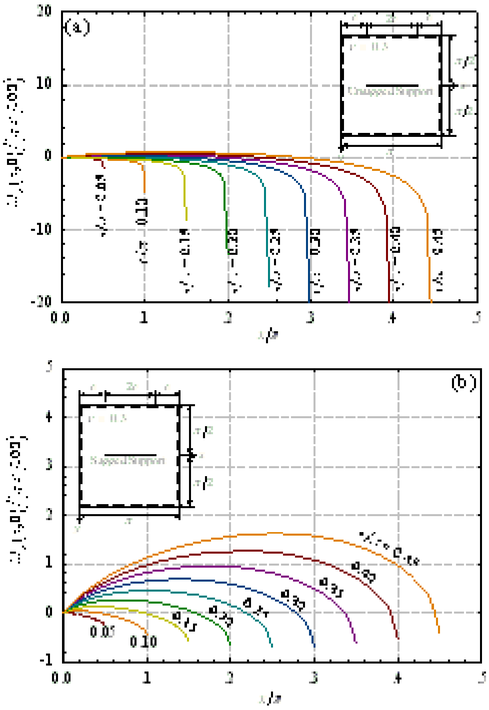

**Figure 6. Bending moments $M_x(x,0)$ and $0 \le x \le e$ for square plate with internal line: (a) unsagged support and (b) sagged support**

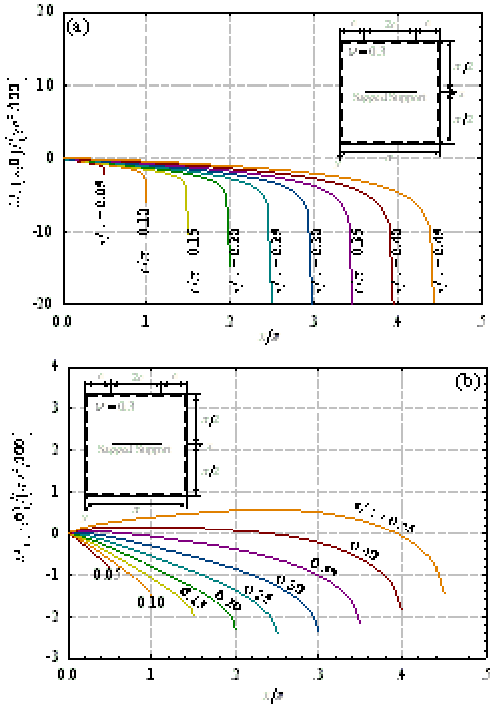

**Figure 7. Bending moments $M_y(x,0)$ and $0 \le x \le e$ for square plate with internal line: (a) unsagged support and (b) sagged support**

Figures 6 and 7 show the bending moments outside an internal line support $(0 \le x \le e, y=0)$ for both cases of the plate with internal line sagged and unsagged supports.

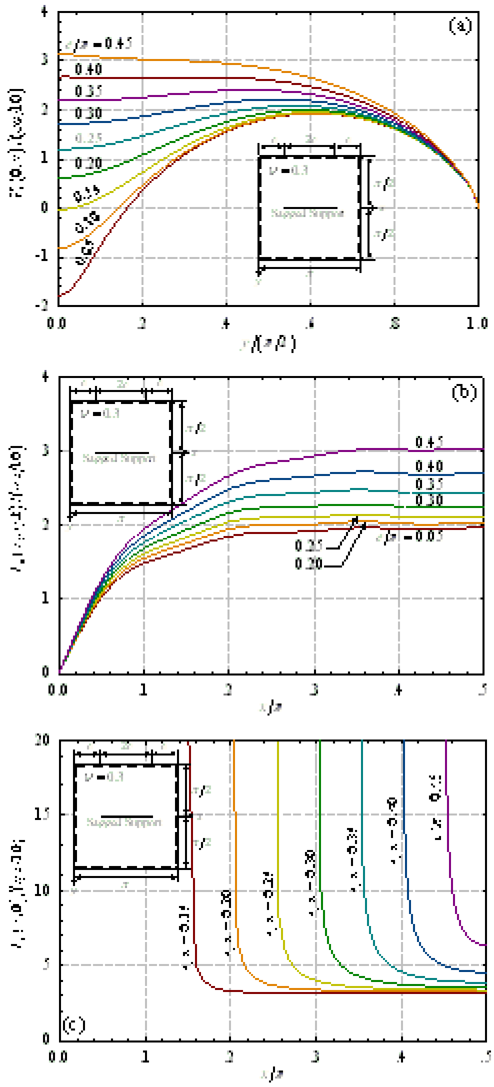

**Figure 8. Support reactions for square plate with internal line sagged support: (a) $V_x(0,y)$ and $0 \le y \le \pi/2$, (b) $V_y(x,\pi/2)$ and $0 \le x \le \pi/2$, (c) $V_y(x,0)$ and $0 \le x \le e$**

The moments at $x = e$ are singular for plate having an internal line unsagged support but bounded for the case of plate with an internal line sagged support. This is the difference of the two problems.

In case of free contact, since the distribution of support reaction is singular of order square root at the tips of contact, it is integrable and the results are illustrated as in Figure 8 for the plate with an internal line sagged support. It is seen that, at $x = e$, the support reaction $V_y(x,0)$ is singular and the reactions are not proportional to the applied load.

Only for the case of free contact between the plate and the internal line sagged support, the contact curve (Dundurs *et al.*, 1974) of plate can be determined from the deflection $w(x,0)$ by supposing the contact length $2c$. If the sag of internal line support is specified as $\delta$, the uniform load required to produce the deflection $w = \delta$ for each specified

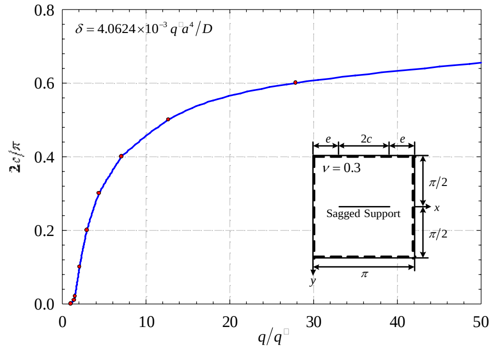


**Figure 9. Extent of contact with internal line sagged support versus level of loading for square plate**

contact length can be calculated by writing the sag in the form

$$\delta = \alpha \frac{q^{\circ} a^4}{D}, \tag{29}$$

where $q^{o}$ is the given uniform load for plate without partial internal line sagged support or the load at which the plate starts to touch the internal line sagged support, and $\alpha$ is the constant value.

When the contact length is specified as $2c$, the maximum deflection can be computed from Eq.(6) with setting $y = 0$ and has the form

$$w_{max} = \beta \frac{q a^4}{D}, \tag{30}$$

in which $q$ is the applied uniform load and $\beta$ is the constant value.

In view of Eqs.(29) and (30), thus, the uniform load $q$ required to produce the deflection $w_{max} = \delta$ with contact length $2c$ is given by the following relation

$$\frac{q}{q^{\circ}} = \frac{\alpha}{\beta}, \tag{31}$$

Hence, the relationship between the contact length $2c$ and the increased load $q/q^{\circ}$ required to produce this contact is obtained. Figure 9 shows the extent of contact versus the level of loading. It revealed that the lengths of contact depend on the level of loading.

## 5. Conclusions

The analytical investigations presented in this work are concerned with the bending problem of plate having an internal line unsagged support and the advancing contact problem between the plate and internal line sagged support. Both problems are treated within the same formulation by writing the mixed boundary conditions in the form of dual series equations. Utilizing the proper finite Hankel transform that satisfied the singularity order in each problem type, then the dual series equations can be reduced to the inhomogeneous Fredholm integral equation of the second kind, which is suitable for numerical evaluation. The deflection and stress resultants of the plate are determined numerically and presented graphically. From the obtained results, it found that the magnitude of deflection for both cases of plate is increased when the ratio of $e/\pi$ is also increased. The bending moment in both directions at $x = e$ is singular for plate with internal line unsagged support but finite value for plate with internal line sagged support. Although the support reaction at the tips of contact between the plate and the internal line sagged support is singular, however, it can be determined. It is seen that the reaction is not proportional to the applied load. Finally, the extent of contact with internal sagged support depends on the level of loading where the required load at which the plate starts to touch the sagged support is $246.16\, D\delta/a^4$ for the Poisson's ratio taken as 0.3.

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