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Original Article

Stochastic autoregressive volatility model for exchange rates

Nittaya McNeil[1], Don McNeil[2] and Nino Kordzakhia[2]

[1] *Faculty of Science and Technology,*
Prince of Songkla University, Pattani Campus, Muang, Pattani, 94000

[2] *Department of Statistics, School of Economic and Financial Studies,*
Macquarie Unversity, Sydney, Australia, 2109

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Abstract

A discrete time model for asset price changes is considered. The volatility process underlying these changes is modeled as a first-order Gaussian autoregressive series. Inversion of the marginal characteristic function of the return process simplifies the assessment of the tail behaviour of the probability density function of returns. The Generalized Method of Moments (GMM) is used to calibrate the model and implement an overidentification test. Daily Euro/USD, Pound/USD, AUD/USD, and Yen/USD exchange rates over the period January 1999 to October 2006 are used to illustrate the methods.

Keywords: Ornstein-Uhlenbeck process, stochastic volatility, characteristic function, fourier transform, Generalised Method of Moments

1. Introduction

Many studies have confirmed the presence of heavy tails in the distributions of daily returns of stock market indices and currency exchange rates, linked to statistically significant autocorrelations among squared returns. Taylor (1994) presented the different statistical models used to address these issues. In general, these models can be divided into two broad groups: (a) models dealing with a conditional variance or so called heteroscedastic variance, and (b) models with stochastic variance, where the variance is a function of a new noise term. Cox (1981) classified these respective models as data-driven and parameter driven, and Shephard (1996) gives a more recent review. The stochastic volatility model generated by an autoregressive time series with lognormal white noise is widely used (Taylor, 1986), and has a continuous time limit as the frequency of observations increases from daily to an infinitesimal interval. The limiting process is the geometric Brownian motion, which is widely used in the option pricing literature. Andersen (1992, 1994) has studied a more general class of stochastic volatility models (called polynomial stochastic volatility models). Chysels *et al.* (1996) surveyed the various approaches of volatility modeling and the associated parameter identification problem. The discrete time model under consideration is called the Gaussian autoregressive stochastic volatility (GASV) model arising from discretisation of the Ornstein-Uhlenbeck (OU) process. Stein and Stein (1991) used this model as a basis for option pricing, while Heston (1993) considered a variation incorporating correlation between the innovations driving the processes of returns and volatility. However, Heston's model may be of limited practical use, because it does not include the special case of constant volatility.

In exchange rate research it is now well known that daily returns on currency exchange rates are best described by distributions with fatter tails than the normal distribution. The study by Boothe and Glassman (1987) confirmed this fact by examining pairs of currencies GBP/USD, CAD/USD, DMK/USD and JPY/USD over the period January 1973 to

*Corresponding author.
Email address: nittaya@bunga.pn.psu.ac.th

August 1984. They found that the daily and weekly exchange rates display kurtosis far in excess of that for the normal distribution. Using the Pearson goodness-of-fit test they concluded that Student's distribution and a mixture of two normal distributions fitted the data better than stable Pareto and normal distribution models. In describing the daily exchange data of four pairs of currencies, Student's distribution ranked first most often. This study also indicated that the parameter estimates varied over the time periods. These issues were handled by introducing autoregressive conditional heteroscedasticity (ARCH) models (Engle, 1982). The paper by Baillie and Bollerslev (1989) gives a survey of previous analysis of exchange rate data and extends it further by considering a generalized ARCH (GARCH) parameterization. These models can generate fat tailed distributions showing excess kurtosis. On the other hand, Mellino and Turnbull (1990) found that daily returns simulated based on historical volatility from daily CAD/USD exchange rates from 1973 to 1984 do not allow for excess kurtosis and correlation among squared returns at the same time. However, a stochastic volatility model resolved the problem.

Since its introduction in 1999, the Euro has played an important role in the global currency market (Greene and Mole, 2003) and there have been several studies based on specific Euro/USD exchange rates (Corsetti 2004, Heaney and Pattenden 2005, Nautz and Scheithauer 2005).

The objectives of the present paper are to assess the unconditional marginal distribution of the return process based on the GASV model, to estimate the parameters and fit the model to daily exchange rate data. The Generalized Method of Moments (GMM) is found to be feasible for estimating the parameters. The model is fitted to four pairs of currencies namely Euro/USD, Pound/USD, AUD/USD and Yen/USD over the period from January 1999 to October 2006.

2. Stochastic autoregressive volatility model

We consider the following model

$$y_t = \mu + \sigma(1 + \delta u_t) z_t, \qquad (1)$$

$$u_t = \gamma u_{t-1} + \eta w_t, \qquad (2)$$

where y_t is the compounded return, given by $y_t = 100\ln(P_t / P_{t-1})$, P_t is the spot exchange rate at day t and μ, σ, δ, γ and η are unknown parameters associated with the return process, while w_t and z_t are independent standardised Gaussian white noise processes. The background for consideration of stochastic volatility u_t modelled by (2) is inspired by its continuous time counterpart, an Ornstein-Uhlenbeck process (Stein and Stein, 1991).

According to Chirtkiatsakul (2002), without loss of generality (2) is parameterised by specifying $\eta = \sqrt{1-\gamma^2}$. Based on the moment generating function, if $u_0 \sim N(0,1)$ and u_t is a stationary process, i.e. $|\gamma| < 1$, then the characteristic function of the stationary process y_t is

$$\varphi(\theta) = E[\exp(i\theta y_t)] = \frac{1}{\sqrt{1+\delta^2\sigma^2\theta^2}} exp\left(i\mu\theta - \frac{\sigma^2\theta^2}{2(1+\delta^2\sigma^2\theta^2)}\right). \qquad (3)$$

Due to the parameterisation of u_t the marginal distribution of y_t is invariant with respect to γ. For simplicity assume $\mu = 0$ since in this case the corresponding probability density function (pdf) is a symmetric function.

If $\mu = 0$, then the standard deviation of y_t is

$$sd[y_t] = \sigma\sqrt{1+\delta^2},$$

the kurtosis of y_t using Fisher's (1958) definition is

$$kurt[y_t] = \frac{6\delta^2(2+\delta^2)}{(1+\delta^2)^2},$$

and the autocorrelation function of y_t^2 at lag s is

$$corr[y_t^2, y_{t-s}^2] = \frac{\delta^2\gamma^s(2+\delta^2\gamma^s)}{1+8\delta^2+4\delta^4}, \quad s > 0.$$

3. Parameter estimates

The reparametrized version of the stochastic volatility model (1, 2) involves the three unknown parameters δ, σ and γ (assuming $\mu = 0$). The standard deviation, kurtosis and correlation of squared returns at lag 1, $E[y_t^2 y_{t-1}^2]$ may be used to derive the estimates for δ, σ and γ through the Method of Moments (MM). However, the estimate of γ can lead to a nonstationary solution of (2). The problem might be resolved by using the mixed moment of squares at some other lag, but the question of which moment to choose to estimate γ remains unanswered. A more natural way of deriving estimates is the Generalized Method of Moments (GMM). This approach has been used for lognormal SV models, e.g, Mellino and Turnbull (1990) and Andersen (1992, 1994).

In the implementation of GMM the mixed moments $m_s^{p,q} = E[y_t^p y_{t-s}^q]$ are used with combinations of p=2 and q=0, 2, experimenting with different lags $s \geq 0$. The moment discrepancies form a l×1 vector $\overline{m}(\theta)$, with $\theta = (\delta, \sigma, \gamma)$, of components of differences between sample and analytical moments, of the form $e_s^{p,q}$ and $m_s^{p,q}$ respectively, where

$$e_s^{p,q} = \frac{1}{n}\sum_{k=s+1}^{n} y_k^p y_{k-s}^q.$$

In the applications considered, $l = s+2$, where s is a number of mixed moments of squared returns.

The GMM estimate of q minimizes the quadratic form

$$q(\theta) = \overline{m}(\theta)' W_n^{-1} \overline{m}(\theta),$$

by

$$\hat{\theta}_n = \arg\min q(\theta),$$

where W_n^{-1} is a (possibly random) positive definite weighting matrix.

Under some regularity conditions $\hat{\theta}_n$ is consistent and asymptotically normal (Hansen, 1982):

$$\sqrt{n}(\hat{\theta}_n - \theta) \sim N(0,V), \text{ as } n \to \infty.$$

The asymptotic variance covariance matrix V can be consistently estimated by

$$V_n = (D_n' W_n^{-1} D_n^T)^{-1} D_n' W_n^{-1} S_n W_n^{-1} D_n^T (D_n' W_n^{-1} D_n^T)^{-1},$$

where D_n' is the $3\times l$ Jacobian matrix

$$D_n' = \frac{\partial \overline{m}(\theta)}{\partial \theta}\Big|_{\theta=\theta_n}$$

evaluated at θ_n, a consistent estimator of θ, and S_n is the sample variance-covariance matrix of the moment discrepancy vector $\overline{m}(\theta)$.

Let $S_n = (S_{n,ij})$, $i, j = 0, 1, \ldots, l$ be the appropriately standardised sample variance covariance matrix of moments with entries

$$s_{n,ij} = \frac{1}{n}\sum_{k=i+1}^{n}(y_k^p y_{k-i}^q - e_i^{p,q})(y_k^p y_{k-j}^q - e_j^{p,q}), \text{ if } i \geq j. \quad (4)$$

Care should be taken to ensure that the matrix S_n is of full rank, although this is a cumbersome task due to the involvement of a high number of similar moments. The trade-off between singularity of the weighting matrix and a number of restrictive moment conditions has been thoroughly studied for the lognormal stochastic volatility model by Andersen and Sørensen (1996) and by Jacquier *et al.* (1994). By setting $W_n = S_n$ the expression for V_n simplifies to

$$V_n = (D_n' S_n^{-1} D_n^T)^{-1}.$$

Different ways of estimating the weighting matrix have been proposed, and it is a common practice to use a nonparametric kernel estimate of the spectral density of the moment vector (Andersen and Sørensen, 1996).

In the underlying GASV model the method advocated by Newey and West (1987) has been implemented in order to provide an alternative way of handling the problem of the deterioration of the variance covariance matrix of the sample moments S_n. The weighting matrix S_n can be approximated by

$$\hat{S}_n = \hat{S}_0 + \sum_{r=1}^{m-1} w(r,m)(\hat{S}_r + \hat{S}_r')$$

with Bartlett weights $w(r,m) = 1 - r/(m+1)$, bandwidth m and $\hat{S}_r = (\hat{s}_{r,ij})$, a variance covariance matrix with entries $\hat{s}_{r,ij} = s_{n,(i+r)j}$ referring to (4). In particular, $\hat{S}_0 = S_n$. Based on empirical studies the recommended bandwidth is $m = 7$. Smoothing the weighting matrix reduces the variability of the autocovariances of the sample moments, though the parameter estimates do not change significantly for data sets considered in this paper.

Hansen and Singleton (1982) and Ferson and Foerster (1994) considered the iterative GMM procedure that consists of using an identity matrix to obtain the initial parameter estimates, and the procedure continues until convergence of the estimates. Jacquier *et al.* (1994) found that the estimates converged after only few iterations of the weighting matrix.

Under regularity conditions Hansen (1982) showed that the matrix V minimizes the limiting variance covariance matrix in the sense of matrix norm. Therefore, this limiting matrix appears to be an analogue of the Cramér-Rao bound arising in maximum likelihood theory. For a discussion of the issue of the efficiency of GMM estimates in consumption and dividend growth Markov chain and lognormal SV models, see Tauchen (1986) and Andersen *et al.* (1999).

The standard errors of GMM estimates of θ are derived as the vector of square roots of the diagonal elements of the matrix $(1/n)V_n$.

A by-product of the estimation procedure is a chi-squared goodness of fit test, (Hansen 1982) based on the overidentifying restrictions of the model. By construction a Wald statistic

$$q_n = \overline{m}(\hat{\theta}_n)' S_n^{-1} \overline{m}(\hat{\theta}_n),$$

asymptotically has a χ^2_α distribution with degrees of freedom α equal to the dimension of $\overline{m}(\theta)$ less the number of estimated parameters. Therefore, $\alpha = l - 3$ and q_n converges in distribution to χ^2_α, as $n \to \infty$.

4. Empirical results

The compounded daily returns of the currency exchange rates Euro/USD, Pound/USD, AUD/USD, and Yen/USD over the period January 1999 to October 2006 are analyzed. The summary statistics for the mean-corrected samples are given in Table 1.

The results show common characteristics of financial returns across the pairs of currencies over the same period of time and all have excess kurtosis relative to a normal distribution. The kurtosis uses Fisher's notation where the normal distribution has a kurtosis equal to zero. Figure 1 shows that the sample autocorrelations of the Euro/USD return are scattered around zero. The limits are based on 0.95 percentiles of the normal distribution $N(0,1/n)$. As observed by Shephard (1996) for data based on JPY/GBP and DMK/GBP over the period from January 1986 to April 1994, the correlograms of compounded returns show little activity compared to their squares. Batten and Ellis (2001) also presented the detailed statistical analysis for returns of four currency pairs (DMK/USD, SWF/USD, JPY/USD, and GBP/

Table 1. Numerical summary of daily returns over Jan 1999-Oct 2006

Compounded return	Size	St Dev	Kurt
Euro/USD	1953	0.619	0.696
Pound/USD	1953	0.514	0.573
AUD/USD	1953	0.488	1.920
Yen/USD	1953	0.233	2.016

USD) over the same period from January 1985 to December 1998 and Liu's study (2007) was based on seven pairs of currencies relatives to the US dollar from 1982 to 1997. These studies confirmed that the data fail to exhibit significant autocorrelation at extended lags.

The estimation of parameters is based on the second, fourth and mixed moments as follows.

$$m_0^{2,0} = \sigma^2(1+\delta^2),$$

$$m_0^{4,0} = 3\sigma^4(1+6\delta^2+3\delta^4) \text{ and}$$

$$m_s^{2,2} = \sigma^4[1+2\delta^2(1+2\gamma^s)+\delta^4(1+2\gamma^{2s})].$$

Table 2 shows the estimated parameters for each pair of currencies and the results based on the chi-squared goodness-of-fit test. The standard error for each parameter and the degrees of freedom α of the χ^2 distribution are given in parentheses. The GMM procedure achieved convergence with respect to an increasing number of moment restrictions.

The goodness-of-fit test is used to investigate the tail behaviour of the marginal distribution function of y_t. The probability density function (pdf) of y_t can be obtained by inverting its characteristic function, and evaluating the integral numerically, using a method of integration of oscillatory functions over an infinite interval (Davis and Rabinovitz, 1967). As an alternative to numerical integration, the modified Fourier Transform (FT) is suggested (Chirtkiatsakul, 2002). For $y \neq 0$ the stationary pdf of y_t is approximated by

$$f(y) \approx \frac{\sin(hy/2)}{\pi y}(1+2\sum_{j=1}^{N}\varphi(jh)\cos(jhy)),$$

where $\varphi(\theta)$ is the characteristic function of y_t. The choice of h and N may require some experimentation, which is a common problem in practice with FT implementations (Seal 1977; Waller *et al.*, 1995).

Figure 2 shows the fitted curves of both the probability density function of y (solid line) and the normal distribution (light line) superimposed on the histogram of the returns. The probability density function of y fits the data

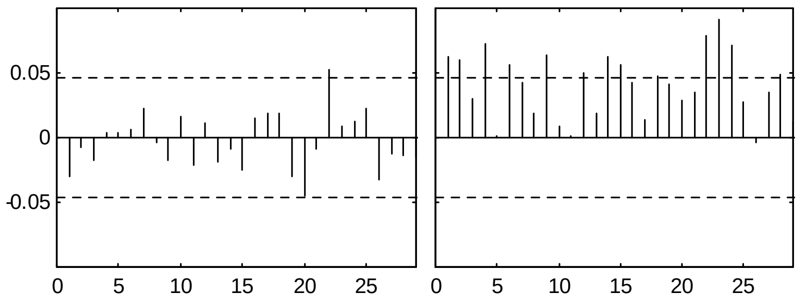

Figure 1. Sample autocorrelation plots of daily return (left panel) and squared returns (right panel) at lag 1 to 20 of Euro/USD Jan 1999 – Oct 2006

Table 2. Parameter estimates, standard errors and p-values corresponding to values of Wald statistics

Exchange rates	Estimated parameters			Wald χ^2	
	γ	δ	σ	p-value	Euro/USD
Euro/USD	0.9692 (0.0047)	0.2126 (0.0003)	0.5900 (0.0025)	10.38 (9)	0.32
Pound/USD	0.7751 (0.1257)	0.2158 (0.0002)	0.4977 (0.0033)	1.06 (3)	0.78
AUD/USD	0.9478 (0.0073)	0.2783 (0.0067)	0.6392 (0.0006)	8.52 (8)	0.39
Yen/USD	0.7862 (0.0509)	0.3618 (0.0003)	0.5766 (0.0100)	9.60 (8)	0.29

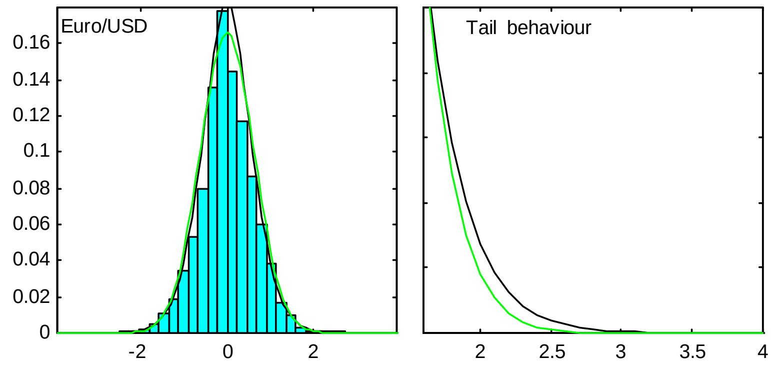

Figure 2. Histogram, f(y) and normal approximation, with light and solid lines corresponding to pdfs of normal and y_t distributions, respectively

Table 3. Pearson χ^2 goodness-of-fit test with 59 degree of freedom

Compounded return	χ^2_{59}	p-value
Euro/USD	75.60	0.108
Pound/USD	79.86	0.110
AUD/USD	66.65	0.231
Yen/USD	86.04	0.012

better than the normal distribution. The magnified tail behaviour is shown on the right hand panel of the graphs. After fitting this stochastic volatility model to the rest of the data, the tail behaviour showed the same pattern.

The χ^2 goodness-of-fit statistic is used to evaluate how well this model fits the data. The choices of classes for the test, based on Kendall and Stuart's (1967) recommendation, is $3(n - 1)^{2/5}$, where n is the sample size. The sample is 1953 and the number of classes is approximately 62. Table 3 shows that the stochastic volatility fits to all series except Yen/USD.

5. Conclusions and Discussion

A discrete time stochastic volatility model is considered to fit the asset price changes. The volatility underlying these changes is modeled as a first-order Gaussian autoregressive series. This model is fitted to the compounded daily returns of the foreign exchange rates Euro/USD, Pound/USD, AUD/USD, and Yen/USD assuming that there is no correlation between the change in the volatility and the daily return. The GMM is used to estimate the parameters and assessing the tail behaviour of the marginal distribution function of the compounded returns of this model by inverting its probability density function using a numerical inversion of Fourier transform. The fit of the model was evaluated using Pearson's chi-squared goodness-of-fit test. The model shows a reasonable fits to the all series that have quite a small kurtosis except Yen/USD.

However, in this result the dependence of observations in the goodness-of-fit test was not taken into account. The estimation using the GMM is generally not efficient. Knight *et al.* (2002) pointed out that this method can miss important information in the data when there are only a finite number of moment conditions.

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