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Original Article

Maximal 3-local subgroups of symmetric groups

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Abstract

The subgroups in the set $\mathcal{N}_{\max}(G,B)$ consisting of all maximal 3-local subgroups of $G = Sym(n)$ with respect to *B*, the normalizer of a Sylow 3-subgroup of *G* in *G*, is investigated. Additionally, the structure of the subgroups in $\mathcal{N}_{\max}(G,B)$ was determined.

Keywords: symmetric group, Sylow 3-subgroup, normalizer, maximal 3-local subgroup

1. Introduction

Maximal 2-local geometries of the sporadic simple groups were first introduced by Ronan and Smith (1980). These geometries were inspired by the theory of buildings for the groups of the Lie type, which was developed by Tits (1956, 1974) in the 1950s. For *G*, a group of the Lie type with the characteristic *p*, its building is a geometrical structure whose vertex stabilizers are the maximal parabolic subgroups, which are also *p*-local subgroups of *G* containing a Sylow *p*-subgroup. It is well known that each building has a Coxeter diagram associated with it. Buekenhout (1979) generalized these concepts to obtain diagrams for many geometries related to sporadic simple groups. Ronan and Smith (1980) pursued these ideas further and introduced the maximal 2-local geometries. Other invariants on buildings for the sporadic simple groups have been defined, notably the minimal parabolic geometries as described by Ronan and Stroth (1984).

We now define what we mean, generally, by a minimal parabolic subgroup. Suppose that *H* is a finite group and *p* is a prime dividing the order of *H*. Let *S* be a Sylow *p*-subgroup of *H* and *B* the normalizer of *S* in *H*. A subgroup *P* of *H* properly containing *B* is said to be a minimal parabolic subgroup of *H* with respect to *B* if *B* lies in exactly one maximal subgroup of *P*.

The definition of minimal parabolic subgroups in terms of the normalizer of a Sylow *p*-subgroup is given in the works of Ronan and Smith (1980) and Ronan and Stroth (1984), in which they study minimal parabolic geometries for the 26 sporadic finite simple groups. The connection between minimal parabolic subgroups and group geometries is best illustrated in the case of groups of the Lie type in their defining characteristics. For a group of Lie type, its minimal parabolic system is always geometric. This is not always the case in general (see Ronan and Stroth, 1984). Many studies on the minimal parabolic system of special subgroups have been done over the years. For example, Lempken *et al.* (1998) determined all the minimal parabolic subgroups and system for the symmetric and alternating groups, with respect to the prime $p = 2$. Later, Covello (2000) has studied minimal parabolic subgroups and systems for the symmetric group with respect to an odd prime *p* dividing the order of the group. The main results are about the symmetric groups of degree p^r; she also established some more general results. More recently, Rowley and Saninta (2004) investigated the maximal

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2-local geometries for the symmetric groups. Furthermore, Saninta (2004) considered the relationship between the maximal 2-local subgroups and the minimal parabolic subgroups for the symmetric groups.

In this paper we shall investigate maximal 3-local subgroups for the symmetric groups. Throughout all groups considered, and in particular all our sets, will be finite. Let Ω be a set of cardinality $n > 1$. Set $G = Sym(\Omega)$, the symmetric group on the finite set Ω. We also use $Sym(m)$ or S_m to denote the symmetric group of degree m. Now let T be a fixed Sylow 3-subgroup of G and B be the normalizer of T in G.

Now we define

$$\mathcal{N}(G,B) = \left\{ H \,\middle|\, B \leq H \leq G \text{ and } O_3(H) \neq 1 \right\}$$

where $O_3(H)$ is the largest normal subgroup of H whose order is a power of 3. Notice that the subgroup $O_3(H)$ is well defined. Clearly, $O_3(H)$ is a characteristic subgroup of H. Furthermore, $O_3(H)$ can be characterized in terms of the Sylow 3-subgroups of H. By Proposition 1.2.2 of Covello (2000), is equal to the intersection of all the Sylow 3-subgroups of H. Since T is the unique Sylow 3-subgroup of $B = N_G(T)$, so $O_3(B) = T \neq 1$ and hence $B \in \mathcal{N}(G,B)$. A subgroup in $\mathcal{N}(G,B)$ is said to be a 3-local subgroup of G with respect to B and a subgroup in $\mathcal{N}(G,B)$, which is maximal under inclusion is said to be a maximal 3-local subgroup of G with respect to B. We denoted the collection of maximal 3-local subgroups of G with respect to B by $\mathcal{N}_{\max}(G,B)$. The aim is to study the subgroups in $\mathcal{N}_{\max}(G,B)$.

However, the general case looks already from the first approach more complicated. In fact, a Sylow 3-subgroup of the symmetric group is not self-normalized and significant work needs to be done in understanding the structure of the normalizer. For instance, in the case of $Sym(3^2)$, there is an isomorphism between the lattice of subgroups of a cyclic groups of order 2 and the lattice of certain over groups of the normalizer and a similar correspondence holds also for the case $Sym(3^2)$, with $m > 2$.

2. Preliminary Results

This section gathers together results that will be used.

Proposition 2.1: Let H be a group and suppose that $H = A \times B$. Let $S \in Syl_p(H)$, where p is a prime. Then $S = (S \cap A) \times (S \cap B)$ and

$$N_H(S) = (N_H(S) \cap A) \times (N_H(S) \cap B),$$

with $N_H(S) \cap A = N_A(S \cap A)$ and $N_H(S) \cap B = N_B(S \cap B)$.
Proof: See Covello (2000) (Proposition 1.1.10).

Lemma 2.2: Suppose that $H = X \times Y$ is a direct product of groups X and Y and suppose that $S \in Syl_p(H)$ where p is a prime which divides the order of both X and Y. Assume that L is a subgroup of H which contains $B = N_H(S)$. Then $L = (L \cap X) \times (L \cap Y)$, with $L \cap X = (B \cap X)^L$ and $L \cap Y = (B \cap Y)^L$.
Proof: See Lempken *et al.* (1998) (Lemma 2.5).

We denoted the wreath product of L by R, where L and R are groups, by $L \wr R$.

Lemma 2.3: Suppose that R is a transitive permutation group of degree n. Let $H = L \wr R$ and $P = K \wr R$, with L maximal subgroup of K, and let p be a prime dividing $|K|$. If L contains the normalizer of a Sylow p-subgroup of K, then H is a maximal subgroup of P.
Proof: See Covello (2000) (Lemma 2.6.8).

Lemma 2.4: (Jordan, Marggraf): Suppose that Σ is a finite set and L is a primitive subgroup of $Sym(\Sigma)$.

(i) If L contains a transposition, then $L = Sym(\Sigma)$.

(ii) Suppose L contains a fours group which is transitive on four points and fixes all the other points of Σ. If $|\Sigma| > 9$, then $L \geq Alt(\Sigma)$.
Proof: See Wielandt (1964) (Theorem 13.3 and 13.5).

Proposition 2.5: Let Ω be a set and $H = Sym(\Omega)$. Let $\mathcal{B} = \{\Omega_1, \ldots, \Omega_m\}$ be a partition of Ω into m subsets of the same cardinality. Then the stabilizer L of $\mathcal{B}$ in H is isomorphic to

$$Sym(\Omega_1) \wr Sym(\mathcal{B}).$$

In particular, L is imprimitive and $\mathcal{B}$ is a complete block system of L.
Proof: See Covello (2000) (Theorem 3.5.1).

Corollary 2.6: Let Ω be a set and $H = Sym(\Omega)$. Let $K \leq H$ be imprimitive and Γ be a block of K. Then the stabilizer in H of the complete block system $\mathcal{B}_\Gamma = \{\Gamma^k \mid k \in K\}$ is isomorphic to $Sym(\Gamma) \wr Sym(\mathcal{B}_\Gamma)$. In particular, K is isomorphic to a subgroup of $Sym(\Gamma) \wr Sym(\mathcal{B}_\Gamma)$.
Proof: See Covello (2000) (Corollary 3.5.2).

Lemma 2.7: Suppose p is a prime, n is a positive integer and $T_{p^n} \in Syl_p(Sym(p^n))$. Then $|Z(T_{p^n})| = p$.
Proof: See Saninta (2001) (Lemma 2.3.5).

Theorem 2.8: Let S be a Sylow p-subgroup of $Sym(p^n)$, where p is a prime and $n \in \mathbb{N}$. If $p > 2$, S has a unique abelian normal subgroup of order $p^{p^{n-1}}$ and this is an elementary abelian p-group.
Proof: See Covello (1998) (Theorem 4.4.6).

Theorem 2.9: Let S be a Sylow p-subgroup of $H = Sym(p^n)$, where p is a prime and $n \in \mathbb{N}$. Then the normalizer in H of S is contained in the normalizer in H of every abelian normal subgroup of S of order $p^{p^{n-1}}$.
Proof: See Covello (1998) (Theorem 4.4.11).

Theorem 2.10: Let $H = Sym(\Omega)$, with $|\Omega| = n$, and let $S \in Syl_p(H)$. Let

$$n = k_t p^t + k_{t-1} p^{t-1} + \cdots + k_1 p + k_0,$$

with $0 \le k_j < p$, for all $j = 0,\ldots,t$, be the p-adic decomposition of n. Then the normalizer B of S in H is given by

$$B = B_0 \times \cdots \times B_t,$$

where, for $j = 0,\ldots,t$, B_j is the normalizer of a Sylow p-subgroup of $Sym(\Omega_j)$ with $\Omega_j \subseteq \Omega$ and $|\Omega_j| = k_j p^j$. In particular,

$$|B| = |S| \prod_{j=0}^{t} k_j!(p\text{-}1)^{k_j j}$$

and the sets $\Omega_0, \Omega_1, \ldots, \Omega_t$ are the orbits of B on Ω.
Proof: See Covello (2000) (Theorem 5.4.1).

Theorem 2.11: Let $H = Sym(\Omega)$ with $|\Omega| = p^n$. Let $S \in Syl_p(H)$ and set $B = N_H(S)$. Then B is transitive on Ω and every block of B has a length of power of p. Furthermore, for $i = 1,\ldots,n-1$, B has blocks of length p^r, for all $r = 1,\ldots,n$.
Proof: See Covello (2000) (Theorem 5.2.9).

Theorem 2.12: Let $H = Sym(\Omega)$, with $|\Omega| = kp^n$ and $1 \le k < p$. Let $S \in Syl_p(H)$ and set $B = N_H(S)$. Then B is isomorphic to the wreath product of $\bar{B}$ by $Sym(k)$, where $\bar{B}$ is the normalizer in $Sym(p^n)$ of a Sylow p-subgroup of $Sym(p^n)$. In particular,

$$|B| = |S|\, k!\,(p\text{-}1)^{nk}$$

and B is transitive on Ω.
Proof: See Covello (2000) (Theorem 5.3.1).

Theorem 2.13: Let $H = Sym(\Omega)$, with $|\Omega| = n$ and $S \in Syl_p(H)$. Suppose that M is a primitive subgroup of G containing the normalizer in H of S. If $n \ge p+2$, then $M = G$.
Proof: See Covello (2000) (Theorem 5.5.2).

Corollary 2.14: Let $H = Sym(\Omega)$, with $|\Omega| = n$, $S \in Syl_p(H)$ and $B = N_H(S)$. Let $n = k_t p^t + k_{t\text{-}1} p^{t\text{-}1} + \cdots + k_1 p + k_0$ be the p-adic decomposition of n. Suppose that M is an imprimitive subgroup of H containing B. Then there exists $1 \le r \le t$ such that $p^r \mid n$ and M is isomorphic to a subgroup of $Sym(p^r) \wr Sym(n/p^r)$. In particular, $k_0 = k_1 = \cdots = k_{r\text{-}1} = 0$.
Proof: See Covello (2000) (Corollary 5.5.5).

Theorem 2.15: Let p be a prime, $p \ne 2, 3$, and $G = Sym(\Omega)$, with $|\Omega| = p$. Suppose that $T \in Syl_p(G)$ and $B = N_G(T)$. Then B is a maximal subgroup of G.
Proof: See Covello (2000) (Theorem 6.1.2).

Lemma 2.16: Let $G = Sym(\Omega)$, with $|\Omega| = n$, $T \in Syl_p(G)$ and $B = N_G(T)$. Suppose that $n = k_1 p^m + k_0$, with $a \ge 1$ and $1 \le k_0, k_1 < p$, is the p-adic decomposition of n. Then every transitive subgroup of G containing B is 2-transitive on Ω, such subgroups are primitive on Ω.
Proof: See Covello (2000) (Lemma 6.5.1).

3. Main Results

We maintain the notation introduced in Section 1. The aim of this section is to describe the structure of subgroups in $\mathcal{N}_{\max}(G,B)$. We start examining some specific cases. Recall that the normalizer of a Sylow p-subgroup of $Sym(p)$ is a maximal subgroup of.

Lemma 3.1: Let $G = Sym(\Omega)$ with $|\Omega| = p$, where p is a prime. Suppose that $T \in Syl_p(G)$ and $B = N_G(T)$. Then $\mathcal{N}_{\max}(G,B) = \{B\}$.
Proof: If $p = 2, 3$, then $B = G$, $O_p(G) \ne 1$ and there is nothing to prove. So assume that $p \ne 2, 3$. We know that $T \cong C_p$, where C_p is a cyclic group of order p. Since T is a normal p-subgroup of B, we have that $O_p(B) \ne 1$ and Theorem 2.15 implies that B is a maximal p-local subgroup of G. Let N be a maximal p-local subgroup of G with respect to B such that $N \ne B$. Then $B < N \le G$ and $O_p(N) \ne 1$. Using Theorem 2.15, $N = G$, which contradicts the fact that $O_p(G) = 1$. Thus B is a unique maximal p-local subgroup of G with respect to B, which completes the proof.

We now look at those subgroups in $\mathcal{N}_{\max}(G,B)$ which act transitively on Ω. Our next result concerns subgroup in $\mathcal{N}_{\max}(G,B)$, where $G = Sym(3^2)$.

Lemma 3.2: Let $G = Sym(\Omega)$ with $|\Omega| = 3^2$. Suppose that $T \in Syl_3(G)$ and $B = N_G(T)$. Then $Sym(3) \wr Sym(3) \in \mathcal{N}(G,B)$.
Proof: Let $L = Sym(3) \wr Sym(3)$. By Theorem 2.11, using Corollary 2.6, we know that $B \le L$ and so L is a maximal subgroup of G. Since, by Proposition 2.5, L is isomorphic to the stabilizer of $Sym(3)$ acting on

$$\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$$

in G. Therefore, $Sym(3) \wr Sym(3) \cong N_G(E)$, where

$$E = \langle (1,2,3),(4,5,6),(7,8,9) \rangle.$$

Using Theorem 2.8, E is a unique elementary abelian normal 3-subgroup of order 3^3 of T. As $E \trianglelefteq N_G(E)$, we have that $O_3(N_G(E)) \ne 1$. It follows that $Sym(3) \wr Sym(3) \cong N_G(E) \in \mathcal{N}(G,B)$.

Lemma 3.3: Let $G = Sym(\Omega)$ with $|\Omega| = 3^2$. Suppose that $T \in Syl_3(G)$ and $B = N_G(T)$. If $N \in \mathcal{N}_{\max}(G,B)$, then N is isomorphic to $Sym(3) \wr Sym(3)$.
Proof: Let $L = Sym(3) \wr Sym(3)$. Since N is a subgroup of G containing B, by Theorem 2.13, we may assume that N is imprimitive. By the transitivity of B, it follows that every subgroup containing B can only have blocks of length 1, 3 and 3^2. So every nontrivial block of N must have length 3 and by Corollary 2.6, N is isomorphic to a subgroup of L. By Lemma 3.2, we have that $Sym(3) \wr Sym(3) \in \mathcal{N}(G,B)$, hence $N \cong Sym(3) \wr Sym(3)$.

Theorem 3.4: Let $G=Sym(\Omega)$ with $|\Omega|=3^m$, where $m\in\mathbb{N}$ such that $m>1$. Suppose that $T\in Syl_3(G)$ and $B=N_G(T)$. If $N\in\mathcal{N}_{\max}(G,B)$, then N leaves invariant a block system with blocks of size 3. In particular, N is isomorphic to $Sym(3)\wr Sym(3^{m-1})$.

Proof: We have that N is transitive on Ω. We argue by induction on m starting with the case $m=2$. For $m=2$, the lemma clearly holds. Since N is a subgroup of G containing B, by Theorem 2.13, we may assume that N is imprimitive. Let $\mathcal{B}=\{\Delta_1,\Delta_2,\ldots,\Delta_k\}$ be a non-trivial block system invariant under N. Since N is transitive on Ω, it follows that N acts transitively on $\mathcal{B}$. Set $t=|\Omega|/k$. Then $t=|\Delta_i|$ for $i=1,\ldots,k$ and so t is a power of 3. Set $M=Stab_G(\mathcal{B})$. Then

$$T\le B\le N\le M\cong Sym(t)\wr Sym(k).$$

For $i=1,\ldots,k$, put $K_i=Sym(\Delta_i)$ and $K=K_1\times K_2\times\cdots\times K_k$. Then for $i=1,\ldots,k$, as $K_i\trianglelefteq K\trianglelefteq M$, $1\ne R_i=T\cap K_i\in Syl_3(K_i)$, $T\cap K=R_1\times R_2\times\cdots\times R_k\in Syl_3(K)$ and $B_i=B\cap K_i=N_{K_i}(R_i)$. Since t is a power of 3, R_i is transitive on Δ_i for all i. Suppose that $O_3(N)\cap K=1$. Since $[O_3(N),N\cap K]\le O_3(N)\cap K$, this gives $[O_3(N),N\cap K]=1$. As $R_i\le N\cap K$, for all i, $O_3(N)$ centralizes R_i and because of the structure of $Sym(t)\wr Sym(k)$ this forces $O_3(N)\le K$. But now $O_3(N)\cap K=O_3(N)\ne 1$, a contradiction. Therefore $O_3(N)\cap K\ne 1$.

Let $\varphi_i: K\to K_i$ be the projection map of K onto K_i and set $L_i=\varphi_i(N\cap K)$. We see that $R_i\le B_i\le L_i\le K_i$ and that L_i is transitive on Δ_i. If $O_3(L_i)=1$, then $O_3(N\cap K)\le\prod_{j\ne 1}K_j$. For all $n\in N$, as $O_3(N\cap K)\trianglelefteq N$, we then have $O_3(N\cap K)=O_3(N\cap K)^n\le(\prod_{j\ne 1}K_j)^n$. Let $l\in\{1,\ldots,k\}$. We may choose an $n\in N$ so as $\Delta_i=\Delta_l^n$. Therefore $(\prod_{j\ne 1}K_j)^n=\prod_{j\ne 1}K_j$, hence it follows that $O_3(N\cap K)\le\bigcap_{i=1}^{k}(\prod_{j\ne 1}K_j)=1$, a contradiction. Hence $O_3(L_i)\ne 1$. So $L_i\in\mathcal{N}(K_i,B_i)$ for all $i=1,\ldots,k$. Let $H_1\in\mathcal{N}_{\max}(K_1,B_1)$ be such that $H_1\ge L_1$. Since H_1 is transitive on Δ_1, by induction H_1 leaves invariant a block system with blocks of size 3. Then H_1 contains E_1, a normal elementary abelian 3-subgroup of order $3^{|\Delta_1|/3}=3^{t/3}$. Hence $E_1\trianglelefteq L_1$ and it follows that $E_1\trianglelefteq N\cap K$. Put $E=\langle E_1^N\rangle$. By the Frattini argument, $N=N_N(T\cap K)(N\cap K)$. So $E=\langle E_1^{N_N(T\cap K)}\rangle\le N\cap K$. Since N is transitive on $\mathcal{B}$, $N_N(T\cap K)$ is transitive on $\mathcal{B}$. Let $g\in N_N(T\cap K)$ be such that $R_1^g=R_j$ for some j. Since $E_1\trianglelefteq R_1$, E_1^g is an elementary abelian normal 3-subgroup of R_j of order $3^{t/3}$. Therefore, E is an elementary abelian normal 3-subgroup of T of order $3^{kt/3}=3^{3^{m-1}}$. Thus, using Theorem 2.8, up to conjugacy we see that

$$E=\langle(1,2,3),(4,5,6),\ldots,(3^m-2,3^m-1,3^m)\rangle.$$

By Theorem 2.9, we have that $B\le N_G(E)$, thus, as $N_G(E)\ge N$ and $N\in\mathcal{N}_{\max}(G,B)$, $N_G(E)=N$. Therefore N leaves invariant a block system with blocks of size 3. This completes the proof of Theorem.

Theorem 3.5: Let $G=Sym(\Omega)$ with $|\Omega|=k\cdot 3^m$, where $m,k\in\mathbb{N}$ such that $k<p$ and $m>1$. Suppose that $T\in Syl_3(G)$ and $B=N_G(T)$. If $N\in\mathcal{N}_{\max}(G,B)$, then N is isomorphic to $(Sym(3)\wr Sym(3^{m-1}))\wr Sym(k)$.

Proof: Since N is a subgroup of G containing B and, by Theorem 2.12, $B=N_{Sym(3^m)}(\bar{T})\wr Sym(k)$ where $\bar{T}\in Syl_3(Sym(3^m))$, so we have that, using Corollary 2.14, $N\le Sym(3^m)\wr Sym(k)$. Therefore, by Theorem 3.4,

$$\bar{N}=Sym(3)\wr Sym(3^{m-1})\in\mathcal{N}_{\max}(Sym(3^m),N_{Sym(3^m)}(\bar{T}))$$

and $\bar{N}$ is a maximal subgroup of $Sym(3^m)$. Thus, by Lemma 2.3, $\bar{N}\wr Sym(k)$ is a maximal subgroup of $Sym(3^m)\wr Sym(k)$. It follows that $N\le\bar{N}\wr Sym(k)$. As $O_3(\bar{N})\ne 1$, $O_3(\bar{N}\wr Sym(k))\ne 1$ and hence $\bar{N}\wr Sym(k)\in\mathcal{N}(G,B)$. Therefore, $N=\bar{N}\wr Sym(k)$.

Lemma 3.6: Let $G=Sym(\Omega)$ with $|\Omega|=3^m+1$, where $m\in\mathbb{N}$. Suppose that $T\in Syl_3(G)$ and $B=N_G(T)$. Then every proper subgroup of G containing B is contained in $Stab_G(\omega)\cong Sym(3^m)$, fixes $\omega\in\Omega$.

Proof: We know that T and B fix a unique point $\omega\in\Omega$ and operates transitively on $\Omega-\{\omega\}$. Suppose that $L\not\le Stab_G(\omega)$ and $G\ge L\ge B$. Then L is 2-transitive on Ω, and, as B contains a transpositions, Lemma 2.4 (i) implies that $L=G$. Thus, all proper subgroups of G which contain B are contained in $Stab_G(\omega)\cong Sym(3^m)$.

Lemma 3.7: Let $G=Sym(\Omega)$ with $|\Omega|=3^m+1$, where $m\in\mathbb{N}$. Suppose that $T\in Syl_3(G)$, $B=N_G(T)$ and put $H=Stab_G(\omega)$, fixed $\omega\in\Omega$. Then $\mathcal{N}_{\max}(G,B)=\mathcal{N}_{\max}(H,B)$.

Proof: Let $N\in\mathcal{N}_{\max}(G,B)$. Since B is transitive on $\Omega-\{\omega\}$, Lemma 3.6 implies that N is contained in $H\cong Sym(3^m)$. It follows that $\mathcal{N}_{\max}(G,B)\subseteq\mathcal{N}_{\max}(H,B)$. But $H\le G$, so that $\mathcal{N}_{\max}(H,B)\subseteq\mathcal{N}_{\max}(G,B)$ and the lemma is complete.

Lemma 3.8: Let $G=Sym(\Omega)$ with $|\Omega|=k_1\cdot 3^m+k_0$ is the 3-adic decomposition of n. Suppose that $T\in Syl_3(G)$ and $B=N_G(T)$. If $n\ge 5$ and $N\in\mathcal{N}_{\max}(G,B)$, then $N\le Sym(k_1\cdot 3^m)\times Sym(k_0)$.

Proof: Let $U=Sym(k_1\cdot 3^m)\times Sym(k_0)$. By Theorem 2.10, U contains B and we know that U is a maximal subgroup Assume that $N\not\le U$. Then N fuses the two orbits of U on Ω. Thus, by Lemma 2.16, N is primitive on Ω. Then Theorem 2.13 implies that $N=G$. Hence $N\le U$.

Theorem 3.9: Let $G=Sym(\Omega)$ with $|\Omega|=k_1\cdot 3^m+k_0$ is the 3-adic decomposition of n. Suppose that $T\in Syl_3(G)$ and $B=N_G(T)$. If $n\ge 5$ and $N\in\mathcal{N}_{\max}(G,B)$, then $N=\bar{N}\times Sym(k_0)$ where $\bar{N}$ is a maximal 3-local subgroup of $Sym(k_1\cdot 3^m)$ with respect to $B\cap Sym(k_1\cdot 3^m)$.

Proof: By Lemma 3.8, $N \le U \times V$ where $U = Sym(k_1 \cdot 3^m)$ and $V = Sym(k_0)$. Using Proposition 2.1, $T = (T \cap U) \times (T \cap V)$ with $T \cap U \in Syl_3(U)$, $T \cap V \in Syl_3(V)$ and $B = (B \cap U) \times (B \cap V)$ with $B \cap U = N_U(T \cap U)$, $B \cap V = N_V(T \cap V)$. As $T \cap V = 1$ and $1 \ne O_3(N) \le T$, we have $1 \ne O_3(N) \cap (T \cap U) \le O_3(N) \cap U$. Since $N \le U \times V$, $O_3(N) \cap U \trianglelefteq N$ and so $1 \ne O_3(N) \cap U \le O_3(NV)$. Therefore $NV \in \mathcal{N}(G,B)$ and, hence, as $N \in \mathcal{N}_{\max}(G,B)$, $N = NV$. So $V \le N$ which implies, using Dedekinds' Modular Law that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $\bar{N} \in \mathcal{N}_{\max}(U, B \cap U)$ with $N \cap U \le \bar{N}$. Since $1 \ne O_3(\bar{N}) \le O_3(\bar{N}V)$ and $B \le \bar{N}V$, $\bar{N}V \in \mathcal{N}(G,B)$ and so, as $N = (N \cap U)V \le \bar{N}V$, $N = \bar{N}V$.

Our next result concerns subgroups in $\mathcal{N}_{\max}(G,B)$ which do not act transitively on Ω. Recall that if $n = k_t \cdot 3^t + k_{t-1} \cdot 3^{t-1} + \cdots + k_1 \cdot 3 + k_0$, where k_j is an integer with $0 \le k_j < 3$, for all $j = 0,1,\ldots,t$, is the 3-adic decomposition of n, then T has $t+1$ orbits on Ω. Let $\Omega_0, \Omega_1, \ldots, \Omega_t$ denote these orbits where $|\Omega_i| = k_i \cdot 3^i$ for $i \in \{0,1,\ldots,t\}$. Note, that $T = T_0 \times T_1 \times \cdots \times T_t$, where $T_i \in Syl_3(Sym(\Omega_i))$, $i \in \{0,1,\ldots,t\}$ and, moreover, each T_i is the direct product of k_i factors each isomorphic to a Sylow 3-subgroup of $G = Sym(\Delta)$, with $|\Delta| = 3^i$ (see Findlay, 1904).

Theorem 3.10: Let $G = Sym(\Omega)$, with $|\Omega| = n$, $T \in Syl_3(G)$ and $B = N_G(T)$. Let $n = k_t \cdot 3^t + k_{t-1} \cdot 3^{t-1} + \cdots + k_1 \cdot 3 + k_0$, where $0 \le k_j < 3$, for all $j = 0,1,\ldots,t$, be the 3-adic decomposition of n and $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_t$, with $|\Omega_j| = k_j \cdot 3^j$, for all $j = 0,1,\ldots,t$, be the corresponding partition of Ω into B-orbits. Let J be a proper subset of $I = \{0,1,\ldots,t\}$. Set $\Delta = \bigcup_{i \in J} \Omega_i$, $U = Sym(\Delta)$ and $V = Sym(\Omega - \Delta)$. Suppose that $N \in \mathcal{N}_{\max}(G,B)$ and $N \le U \times V$.

(i): If $O_3(N) \cap U \ne 1$, then $N = N_U \times V$ where $N_U \in \mathcal{N}_{\max}(U, B \cap U)$.

(ii): If $O_3(N) \cap V \ne 1$, then $N = U \times N_V$ where $N_V \in \mathcal{N}_{\max}(V, B \cap V)$.

Proof: First we examine the case when $O_3(N) \cap U \ne 1$. Since $N \le U \times V$, $O_3(N) \cap U \trianglelefteq N$ and so $1 \ne O_3(N) \cap U \le O_3(NV)$. Therefore $NV \in \mathcal{N}(G,B)$ and hence, as $N \in \mathcal{N}_{\max}(G,B)$, $N = NV$. So $V \le N$ which implies, using Dedekinds' Modular Law that $N = (N \cap U)V$. Now, as $N \cap U \in \mathcal{N}(U, B \cap U)$, we may choose $N_U \in \mathcal{N}_{\max}(U, B \cap U)$ with $N \cap U \le N_U$. Since $1 \ne O_3(N_U) \le O_3(N_U V)$ and $B \le N_U V$, $N_U V \in \mathcal{N}(G,B)$, and so, as $N = (N \cap U)V \le N_U V$, $N = N_U V$. If we have $O_3(N) \cap V \ne 1$, the same argument yields $N = U \times N_V$ for some $N_V \in \mathcal{N}_{\max}(V, B \cap V)$.

Theorem 3.11: Let the hypothesis of Theorem 3.10 holds. Suppose that $0 \le k_j \le 1$ for all $j = 0,1,\ldots,t$. Then either $N = N_U \times V$, where $N_U \in \mathcal{N}_{\max}(U, B \cap U)$ and N_U is transitive on Δ, or $N = U \times N_V$, where $N_V \in \mathcal{N}_{\max}(V, B \cap V)$ and N_V is transitive on $\Omega - \Delta$.

Proof: Thanks to the study carried out in Theorem 3.10, we only need to eliminate the situation $O_3(N) \cap U = 1 = O_3(N) \cap V$. From

$$[O_3(N), T_U] \le O_3(N) \cap T_U \le O_3(N) \cap U = 1$$

and

$$[O_3(N), T_V] \le O_3(N) \cap T_V \le O_3(N) \cap V = 1,$$

where $T_U \in Syl_3(U)$, $T_V \in Syl_3(V)$, we deduce that $O_3(N) \le Z(T)$. Therefore, $C_G(Z(T)) \le C_G(O_3(N)) \le N_G(O_3(N)) = N$.

Let $1 \ne \sigma \in O_3(N)$, so $\sigma \in Z(T)$. For any $g \in N$, $\sigma^g \in O_3(N) \trianglelefteq N$ and hence $\sigma^g \in Z(T)$. Since $T = \prod_{i \in I} T_i$ where, for $i \in I$, $T_i \in Syl_3(Sym(\Omega_i))$, $Z(T) = \prod_{i \in I} Z(T_i)$. By Lemma 2.7, $Z(T_i) = \langle \sigma_i \rangle$ where σ_i has order 3 and cycle type $3^{3^{i-1}}$. Now let $1 \ne \mu \in Z(T)$ with $\mu \ne \sigma$. So $\sigma = \prod_{k \in K} \sigma_k$ and $\mu = \prod_{k \in K'} \sigma_k$, where $K, K' \subseteq I$ with $K \ne K'$ and consequently, as $t > t-1 > \cdots > 1$, σ and μ have different cycle types. Therefore $\sigma^g = \sigma$ and then $N \le C_G(\sigma)$. Since $\langle \sigma \rangle \subseteq Z(C_G(\sigma)) \le O_3(C_G(\sigma))$, $C_G(\sigma) \in \mathcal{N}(G,B)$. This implies that $N = C_G(\sigma)$ for all $1 \ne \sigma \in O_3(N)$, as $N \in \mathcal{N}_{\max}(G,B)$. We see that

$$C_G(\sigma) = \prod_{k \in K} C_{Sym(\Omega_k)}(\sigma_k) \times Sym\Big(\bigcup_{i \in I \setminus K} \Omega_i\Big)$$

and so $\langle \sigma_k \mid k \in K \rangle \le Z(C_G(\sigma))$. In particular, $\langle \sigma_k \mid k \in K \rangle \le O_3(C_G(\sigma)) = O_3(N)$. Now either $\langle \sigma_k \mid k \in K \rangle \cap T_U \ne 1$ or $\langle \sigma_k \mid k \in K \rangle \cap T_V \ne 1$ because $O_3(N) \trianglelefteq T = T_U \times T_V$, a contradiction.

Aiming for a contradiction we assume N_U is not transitive of Δ. Thus $N_U \le X \times Y \le U$ where $\theta = \bigcup_{i \in K} \Omega_i$, $X = Sym(\theta)$ and $Y = Sym(\Delta - \theta)$ for some $K \subset J$. Applying the previous part to $N_U \in \mathcal{N}_{\max}(U, B \cap U)$ we deduce that either $N_U \le N_X \times Y$ where $N_X \in \mathcal{N}_{\max}(X, B \cap X)$ or $N_U = X \times N_Y$ where $N_Y \in \mathcal{N}_{\max}(Y, B \cap Y)$. Without loss of generally we assume the former to hold. Since $O_3(N_X) \ne 1$ and $T \le N_X \times Sym(\Omega - \theta)$, clearly $N_G(O_3(N_X)) \in \mathcal{N}(G,B)$. However we have that

$$N = N_U \times V \le N_X \times Y \times V < N_X \times Sym(\Omega - \theta) \le N_G(O_3(N_X)),$$

a contradiction. Therefore we conclude that N_U is transitive on Δ and hence the proof of the theorem is complete.

Below we give some examples of the subgroups in $\mathcal{N}_{\max}(G,B)$ which illustrate some of the results proved earlier. Recall that $G=Sym(\Omega)$ with $|\Omega|=n$, $T\in Syl_3(G)$ and $B=N_G(T)$.

If $n=3$, then $T=\langle(1,2,3)\rangle$ with $|T|=3$ and $B=\langle(1,2,3),(2,3)\rangle=G$ with $|B|=6$ and $O_3(B)=T\neq 1$. Therefore $\mathcal{N}(G,B)=\{B\}=\mathcal{N}_{\max}(G,B)$.

If $n=4$, then $T=\langle(1,2,3)\rangle$ with $|T|=3$ and $B=\langle(1,2,3),(2,3)\rangle=Sym(3)$ with $|B|=6$ and $O_3(B)=T\neq 1$. Since $\langle(1,2,3)\rangle$ and $\langle(2,4,3)\rangle$ are Sylow 3-subgroups of G such that $\langle(1,2,3)\rangle\cap\langle(2,4,3)\rangle=1$, so $O_3(G)=1$ and we have that $G\notin\mathcal{N}(G,B)$. Therefore $\mathcal{N}(G,B)=\{Sym(3)\}=\mathcal{N}_{\max}(G,B)$.

If $n=5$, then $T=\langle(1,2,3)\rangle$ with $|T|=3$ and $B=\langle(1,2,3),(4,5),(2,3)\rangle$ with $|B|=12$ and $O_3(B)=T\neq 1$. Therefore the subgroup in $\mathcal{N}_{\max}(G,B)$ is $B=Sym(3)\times Sym(2)$.

If $n=6$, then $T=\langle(1,2,3),(4,5,6)\rangle$ with $|T|=9$ and $B=\langle(4,5,6),(1,2,3),(4,5),(2,3)(4,6),(1,4,3,6)(2,5)\rangle$ with $|B|=72$. We also have that the subgroup in $\mathcal{N}_{\max}(G,B)$ is $B\cong Sym(3)\wr Sym(2)$.

If $n=9$, then $T=\langle(1,2,3),(4,5,6),(7,8,9),(1,4,7)(2,5,8)(3,6,9)\rangle$ with $|T|=81$ and $\langle(1,5,9)(2,6,7)(3,4,8),(7,8,9),(4,5,6),(1,2,3),(4,9)(5,7)(6,8),(2,3)(4,8,5,7,6,9)\rangle$ with $|B|=324$. We have that G has a unique maximal 3-local subgroup with respect to B, which is $N=Sym(3)\wr Sym(3)\cong N_G(E)$, where $E=\langle(1,2,3),(4,5,6),(7,8,9)\rangle$. That is, $N=\langle(1,2,3),(1,2),(4,5,6),(4,5),(7,8,9),(7,8),(1,4,7)(2,5,8)(3,6,8),(1,4)(2,5)(3,6)\rangle$ with $|N|=1296$.

If $n=27$ and

$x_1\cong(1,2,3)$,

$x_2\cong(1,4,7)(2,5,8)(3,6,9)$,

$x_3\cong(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)$.

are its generators. The normalizer B of T in G can be described as $B=T\times\langle h_1,h_2,h_3\rangle$, with

$h_1\cong(2,3)(5,6)(8,9)(11,12)(14,15)(17,18)(20,21)(23,24)(26,27)$,

$h_2\cong(4,7)(5,8)(6,9)(13,16)(14,17)(15,18)(22,25)(23,26)(24,27)$,

$h_3\cong(10,19)(11,20)(12,21)(13,22)(14,23)(15,24)(16,25)(17,26)(18,27)$.

We have that the subgroup in $\mathcal{N}_{\max}(G,B)$ is $N=Sym(3)\wr Sym(9)$.

Studying maximal p-local subgroups of the symmetric groups where p is a prime number is an interesting task for future research.

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