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Original Article

Fixed point and periodic point theorems in fuzzy metric space

Binod Chandra Tripathy[1], Sudipta Paul[2]*, and Nanda Ram Das[3]

[1] *Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Paschim Boragoan, Garchuk, Guwahati, Assam, 781035 India.*

[2] *Department of Mathematics of Mathematics, National Institute of Technology, Silchar, Assam, 788010 India.*

[3] *Department of Mathematics, Gauhati University, Guwahati, Assam, 781014 India.*

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Abstract

In this paper we prove a periodic point theorem in fuzzy metric space and present some common fixed point theorems for occasionally weakly compatible mappings in fuzzy 3- metric spaces under various conditions. We prove the fuzzy analogue of the Edelsten theorem in fuzzy 3-metric space. Also we establish some more results in fuzzy 3-metric space.

Keywords: ε-chainable, ε-contractive, Fuzzy 3- metric space, weakly commuting, occasionally weakly compatible

1. Introduction

Fuzzy Set was introduced by L.A. Zadeh in the year 1965 and thereafter the potential of the notion was realized by the whole scientific community and many researchers were motivated for further investigation on its application. Besides its theoretical development, it has been successfully implemented in various branches, viz. cybernetics, artificial intelligence, expert system, control theory, pattern recognition, operation research, decision making, image analysis, projectiles, probability theory, agriculture, weather forecasting etc.. It has been applied and investigated in the recent past in different branches of mathematics by Tripathy *et al.* (2012); Tripathy and Borgohain (2011, 2013); Tripathy and Das (2012); Triapthy and Dutta (2012, 2013); Tripathy and Ray (2012); Tripathy and Sarma (2012) and many others.

Kramosil and Michalek (1975) introduced fuzzy metric space. George and Veermani (1994) modified the notion of fuzzy metric spaces with the help of continuous t-norm. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki (1999) proved fixed point theorems for R-weakly commutating mappings. Pant (1998a, 1998b) and Pant and Jha (2004) introduced the concept of reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam *et al.* (2002) have shown that the open problem on the existence of contractive definition, which generates a fixed point but does not force the mapping to be continuous at the fixed point, by Rhoades (1988) possesses an affirmative answer. Pant and Jha (2004) obtained some analogous results proved by Balasubramaniam *et al.* (2002). Further works on fixed point are due to Jungck (1998) and Jungck and Rhoades (1998). We are motivated by the works due to these researchers on fixed point theory in fuzzy metric spaces and present some common fixed point theorems for more general commutative condition *i.e.* occasionally weakly compatible mappings in fuzzy 3- metric space in this paper.

* Corresponding author.
Email address: sudiptapaul_math@rediffmail.com

2. Definitions and Preliminaries

Definition 2.1 A fuzzy set A on X is a function with domain X and values in [0, 1] *i.e.* $A:X\to[0,1]$.

Definition 2.2 A binary operation *: [0,1]x[0,1]→[0,1] is called a continuous t-norm if ([0,1], *) is an abelian topological monoid with unit 1 such that $a_1*b_1 \le a_2*b_2$ whenever $a_1 \le a_2$, $b_1 \le b_2$ for all $a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 2.3 The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X^2 x$[0,\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $t_1, t_2, t > 0$

(1) $M(x,y,0) = 0$;

(2) $M(x,y,t) = 1$ if and only if $x = y$;

(3) $M(x,y,t) = M(y,x,t)$;

(4) $M(x,y, t_1+t_2) \ge M(x,z, t_1) * M(z,y,t_2)$;

(5) $M(x,y, \times) : [0,\infty) \to [0,1]$ is left continuous.

then M is called Fuzzy Metric on X, and$(X,M,*)$ is called fuzzy metric space and $M(x,y,t)$ denotes the degree of nearness between x and y.

Definition 2.4 Let $(X,M,*)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to converge to x in X if for each ε > 0 and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n,x,t) > 1\text{-}\varepsilon$ for all $n \ge n_0$.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each ε > 0 and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n,x_m,t) > 1\text{-}\varepsilon$, for all $n,m \ge n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5 Let $(X,M,*)$ be fuzzy metric space.

A finite sequence

$x = x_0,x_1, \ldots, x_n = y$ is called ε-chain from x to y if ∃ a positive number $\varepsilon > 0$ such that

$M(x_i,x_{i-1},t) > 1\text{-}\varepsilon$ for every $t > 0$ and $i = 1,2,\ldots, n$.

A fuzzy metric space $(X,M, *)$ is called ε-chainable if for any x,y, X there exits ε-chain from x to y.

Definition 2.6 A map $T:X\to X$ is said to be fuzzy contractive if

$M(Tx, Ty,t) \ge p\, M(x,y,t)$ for all $x,y \in X$, and is said to be fuzzy ε-contractive if $M(Tx, Ty,t) \ge p\, M(x,y,t)$ whenever

$1\text{-}\varepsilon < M(x,y,t) < 1$ and $0 < \varepsilon < 1$, where $1\text{-}\varepsilon < p < \varepsilon$.

Definition 2.7 A pair of self mappings (f, g) of a fuzzy metric space $(X,M,*)$ is said to be

(1) Weakly commuting if

$M(fgx,gfx,t) \ge M(fx,gx,t)$ for all $x \in X$ and $t > 0$

(2) R-weakly commuting if there exists some $R > 0$ such that

$M(fgx,gfx,t) \ge M(fx,gx,t/R)$ for all $x \in X$ and $t > 0$.

Definition 2.8 Two self mappings f and g of a fuzzy metric space $(X,M,*)$ are called compatible if $\lim_{n\to\infty} M(fgx_n,gfx_n,t) = 1$

where $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

Definition 2.9 Two self maps f and g of a fuzzy metric space $(X,M,*)$ are called reciprocally continuous on X if

$\lim_{n\to\infty} fgx_n = fx$ *and* $\lim_{n\to\infty} gfx_n = gx$,

whenever there exist a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

Definition 2.10 Let X be a set, f, g self maps of X. A point $x \in X$ is called a coincidence point of f and g if and only if $fx = gx$, we shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.11 A pair of maps f and g is called a weakly compatible pair if they commute at coincidence points.

Definition 2.12 Two self-maps f and g of a set X are occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of f and g at which f and g commute.

Remark If self-maps f and g of a fuzzy metric space $(X,M,*)$ are compatible then they are weakly compatible.

Definition 2.13 A binary operation *: [0,1] x [0,1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1], *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \le a_2 * b_2 * c_2 * d_2$ whenever $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, $d_1 \le d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in [0,1].

Definition 2.14 The 3-tuple $(X,M,*)$ is called a fuzzy 3-metric space if X is an arbitrary set, * is its t-norm and M is a fuzzy set in X^4 x $[0, \infty)$ satisfying the following condition:

for all $x,y,z,w,u \in X$ and $t_1, t_2, t_3, t_4 > 0$

(1) $M(x,y,z,w,0) = 0$

(2) $M(x,y,z,w,t) = 1$ for all $t > 0$

(Only when the three simplex $\langle x,y,z,w \rangle$ degenerate)

(3) $M(x,y,z,w,t) = M(x,w,z,y,t) = M(y,z,w,x,t) = M(z,w,x,y,t) = \ldots\ldots$

(4) $M(x,y,z,w,t_1+t_2+t_3+t_4) \ge M(x,y,z,u,t_1)*M(x,y,u,w,t_2) * M(x,u,z,w,t_3) * M(u,y,z,w,t_4)$

(5) $M(x,y,z,w, \times) : [0, \infty) \to [0,1]$ is left continuous.

Definition 2.15 A sequence $\{x_n\}$ in a fuzzy 3-metric space $(X,M,*)$ is said to converge to x in X if $\lim_{n\to\infty} M(x_n,x,a,b,t) = 1$ for all $a,b \in X$ and $t > 0$.

Definition 2.16 Let $(X,M,*)$ be a fuzzy 3-metric space. A sequence $\{x_n\}$ in X called Cauchy sequence, if $\lim_{n\to\infty} M(x_{n+p},x_n,a,b,t) = 1$ for all $a,b \in X$, $p > 0$ and $t > 0$.

Definition 2.17 A fuzzy 3-metric space $(X,M,*)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 2.18 Two maps f and g from a fuzzy 3-metric space $(X,M,*)$ into itself are said to be compatible if

$\lim_{n\to\infty} M(fgx_n,gfx_n,a,b,t) = 1$ for all $a,b \in X$ and $t > 0$

whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X$.

Lemma 2.19 Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

3. Results

We prove the fuzzy analogue of the Edelsten theorem in fuzzy 3-metric space.

Theorem 3.1 Let T be an fuzzy -contractive mapping of a fuzzy metric space X into itself and let x_0 be a point of X such that the sequence $\{T^n x_0\}$ has a subsequence convergent to a point u of X. Then u is a periodic point of T, i.e. there is a positive integer k such that $T^k u = u$.

Proof: Let $\{n_i\}$ be a strictly increasing sequence of positive integer such that

$\lim_{i\to\infty} T^{n_i} x_0 = 0$ and let $x_i = T^{n_i} x_0$.

There exists m_0 such that

$M(x_i,u,t) > 1-\varepsilon/4$ for $j^3 m_0$.

Choose any $j^3 \geq m_0$ and let $k = n_{i+1} - n_i$. Then,

$M(x_{i+1}, T^k u,t) = M(T^k x_i, T^k u,t) \geq M(x_i,u,t) > 1 - /4$

and

$M(T^k u,u,t) \geq M(T^k u,x_{i+1},t/2) * M(x_{i+1},u,t/2) > (1-\varepsilon/4) * (1-\varepsilon/4)\, ^3\, 1-\varepsilon/4 > 1\varepsilon/2$

Suppose that $v = T^k u \neq u$. Then T being ε-contractive,

$M(Tu,Tv,t) \geq s\ M(u,v,t)$, where $1-\varepsilon < s < \varepsilon$.

Or $\dfrac{M(T_u, T_{v,t})}{M(u,v,t)} > 1-\varepsilon$

The function $(x,y,t) \to \dfrac{(T_x, T_{y,t})}{M(u,v,t)}$ is continuous at (u,v,t).

So, there exist $\delta > 0$ and $K > 0$ with $1-\varepsilon < K < \varepsilon$ such that,

$M(x,u,t) > 1-\delta$, $M(y,v,t) > 1-\delta$

implies that,

$M(Tx,Ty,t) > K\,M(x,y,t)$.

Since $lim_{r\to\infty} T^K x_r = T^K u = v$, there exists $N' \geq N$ such that,

$M(x_r,u,t) > 1-\delta$, $M(Tx_r,v,t) > 1-\delta$ for $r\, ^3\, N'$ and so

$$M(Tx_r, TT^K x_r,t) > KM(x_r, T^K x_r,t) \quad (1)$$

$$M(x_r, T^K x_r,t) \geq M(x_r,u,t/3) * M(u,T^K u,t/3) * M(T^K u,T^K x_r,t/3) \geq (1-\varepsilon/4) * (1-\varepsilon/4) * (1-\varepsilon/4) \geq 1-\varepsilon/4 > 1 - \varepsilon \text{ for } r \geq N' > N \quad (2)$$

From (1) & (2)

$M(Tx_r, TT^k x_r,t) > KM(x_r, T^K x_r,t) > 1-\varepsilon$ *for* $r \geq N'$

and so T being ε-contractive,

$$M(T^P x_r, T^P T^k x_r,t) > KM(x_r, T^k x_r,t) \text{ for } n \geq N', p > 0 \quad (3)$$

setting $p = n_{r+1} - n_{r-}$ in (3)

$M(x_{r+1}, T^k x_{r+1},t) > KM(x_r, T^k x_r,t)$ for any $r \geq N'$

Hence

$M(x_s, T^k x_s,t) > K^{s-r} M(x_r, T^k x_r,t) > K^{s-r}(1-\varepsilon)$

and

$M(u,v,t) > M(u,x_s,t/3) * M(x_s,T^k x_s,t/3) * M(T^k x_s,v,t/3) \to 1$ as $s \to \infty$

This contradicts the assumptions that,

$M(u,v,t) < 1$.

Thus $T^k u = v = u$.

Theorem 3.2 Let $(X,M,*)$ be a complete fuzzy 3-metric space and let A,B,S and T be self mappings of X. Let pairs $\{A,S\}$ and $\{B,T\}$ be owc. If there exists $q \in (0,1)$ such that

$$M(Ax,By,a,b,qt) \geq min\{M(Sx,Ty,a,b,t), M(Sx,Ax,a,b,t), M(By,Ty,a,b,t), [M(Ax,Ty,a,b,t) + M(By,Sx,a,b,t)]/2\} \quad (4)$$

for all $x,y \in X$ and for all $t > 0$, then there exists an unique point $w \in X$ such that

$Aw = Sw = w$ and an unique point $z \in X$ such that $Bz = Tz = z$, moreover $z = w$, so that there is an unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc,
so there are points $x, y \in X$
such that $Ax = Sx$ and $By = Ty$.
We claim that $Ax = By$.
If not, by inequality (1),

$M(Ax,By,a,b,qt) \geq min\{M(Sx,Ty,a,b,t), M(Sx,Ax,a,b,t), M(By,Ty,a,b,t), [M(Ax,Ty,a,b,t) + M(By,Sx,a,b,t)]/2\}$

$= min\,\{M(Ax,By,a,b,t), M(Ax,Ax,a,b,t), M(By,By,a,b,t), [M(Ax,By,a,b,t) + M(By,Ax,a,b,t)]/2\}$

$= M(Ax,By,a,b,t)$.

$\therefore Ax=By$ i.e. $Ax = Sx = By = Ty$.

Suppose that there is another point z such that $Az = Sz$ then by (4) we have

$Az = Sz = By = Ty$, so, $Ax = Az$ and $w = Ax = Sx$

is the unique point of coincidence of A and S.

By lemma (2.19) w is the only common fixed point of A and S. Similarly there is an unique point $z \in X$ such that

$z = Bz = Tz$

Assume that, $w \neq z$, we have,

$M(w,z,a,b,qt) = M(Aw,Bz,a,b,qt) \geq min\{M(Sw,Tz,a,b,t), M(Sw,Az,a,b,t), M(Bz,Tz,a,b,t), [M(Aw,Tz,a,b,t) + M(Bz,Sw,a,b,t)]/2\}$

$= min\,\{M(w,z,a,b,t), M(w,z,a,b,t), M(z,z,a,b,t), [M(w,z,a,b,t) + M(z,w,a,b,t)]/2\}$

$= M(w,z,a,b,t)$

Therefore we have $z = w$ by lemma (2.19) and z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (4) .

Theorem 3.3 Let $(X,M,*)$ be a complete fuzzy 3-metric space and let A, B, S and T be self mapping of X. Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc. If there exists $q \in (0,1)$ such that,

$$M(Ax,By,a,b,qt) \geq \psi\{M(Sx,Ty,a,b,t), M(Sx,Ax,a,b,t), M(By,Ty,a,b,t), [M(Ax,Ty,a,b,t) + M(By,Sx,a,b,t)]/2, M(By,Sx,a,b,t)\} \quad (5)$$

for all $a,b,x,y \in X$ and $\psi : [0,1]^5 \to [0,1]$ such that $\psi(t, 1, 1, t, t) > t \ \forall \ 0<t<1$, then there exists an unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ are owc, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.

We claim that $Ax = By$. By inequality (5) We have,

$$M(Ax,By,a,b,qt) \geq y\{M(Sx,Ty,a,b,t), M(Sx,Ax,a,b,t), M(By,Ty,a,b,t), [M(Ax,Ty,a,b,t)+M(By,Sx,a,b,t)]/2, M(By,Sx,a,b,t)\}$$
$$= \psi\{M(Ax,By,a,b,t), M(Ax,Ax,a,b,t), M(By,By,a,b,t), [M(Ax,By,a,b,t) + M(By,Ax,a,b,t)]/2, M(By,Ax,a,b,t)\}$$
$$= \psi\{M(Ax,y,a,b,t), 1, 1, M(Ax,By,a,b,t)\} > M(Ax,By,a,b,t)$$

a contradiction, therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$.

Suppose that there is another point z such that $Az = Sz$ then by (5) we have, $Az = Sz = By = Ty$, So $Ax = Az$ and $w = Ax = Tx$ is the unique point of coincidence of A and T.

By lemma (2.19) w is a unique common fixed point of A and S. Similarly, there is an unique point $z \in X$ such that $z = Bz = Tz$.

Thus z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (5).

4. Conclusions

We have established periodic point theorem in fuzzy metric space and established some results on fixed point theory for occasionally weakly compatible mappings in fuzzy 3-metric spaces under different conditions. Many workers can apply these results for further investigations and applications.

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