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Original Article

Knowledge space theory and union-closed sets conjecture

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Abstract

The knowledge space theory provides a framework for knowledge management. One of major problems is to find core information for a body of knowledge. Union closed set conjecture, if true, guarantees that for a given knowledge space, there is an information that is linked to at least half of the knowledge units. This paper deals with a variant problem, where the knowledge space is also a topological space and possibly infinite. We prove that there is a point belonging to as many open sets as of the topological space itself.

Keywords: knowledge space, union-closed sets conjecture, topological space

1. Introduction

The knowledge space theory is based on the idea of visualizing knowledge as a directed graph of skill units (Doignon and Falmagne, 1985). To reach the terminal skill, students need to master certain number of prior skills. The theory proposes that there are many paths of the directed graph leading to the terminal skill. We can optimize the learning process by assessing student's skills and personalizing the learning path for each student. This will help us to assess students' skills and personalize their study plans. In this study, we consider the problem dealing with knowledge management. Specifically, we study the so called *union-closed sets conjecture*: "*For any finite union-closed family of finite sets, other than the family consisting only of the empty set, there exists an element that belongs to at least half of the sets in the family.*" (Frankl, 1979). The above is a conjecture made by Frankl in 1979; it is still an open problem (Poonen, 1992; Roberts and Simson, 2010). In this study we will consider the case of infinite knowledge space with a topological structure (Dugundji, 1966; Munkres, 2000). The continuum hypothesis plays a key role in our investigation. Specifically, we prove the following theorem:

Theorem: *Let (X, τ) be an infinite, 2^{nd} countable and Hausdorff topological space. Then there is x “ X such that $|\tau_x| = |\tau|$. where τ_x is the set of all open sets in τ containing x.*

2. Main Results

Definition 1. A tuple (X, K) consisting of a non-empty set X and a set K of subsets from X is called a **knowledge space** if K contains the empty set and X, and it is closed under union.

Definition 2. Let (X, S) be a knowledge space. An element $x \in X$ is called a **core unit** of S if x belongs to at least half of the elements in S when $|S| < \infty$ *or* $|S_x| = |S|$ if S is infinite, where $S_x = \{K \in S : x \in K\}$.

Definition 3. Let (X, τ) be a topological space and $x \in X$. The **subtopological space of τ containing x** is the set $\{\mathcal{O} \in \tau : x \in \mathcal{O}\} \cup \{\varnothing\}$. We denote it by τ_x. It is easy to see that, for each $x \in X$, τ_x is a subtopological space of τ. In this paper, we assume the continuum hypothesis. It states that there is no set T for which

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$$\aleph_0 < |T| < 2^{\aleph_0}, \text{ where } \aleph_0 = |\mathbb{N}|.$$

By assuming the axiom of choice, the continuum hypothesis is equivalent to

$$2^{\aleph_0} = \aleph_1, \text{ where } \aleph_1 = |\mathbb{R}|.$$

Lemma 2.1. *Let S be an infinite countable collection of nonempty sets. Then there is a choice function $\phi : S \to \cup S$ such that*

$$|\phi(S)| = \aleph_0 = |S|.$$

Proof. Assume that S is an infinite collection of nonempty sets. Let S_∞ be a set defined by $S_\infty = \{A \in S : |A| = \infty\}$. We consider two cases, depending on whether S_∞ is infinite or finite.

Case 1 S_∞ is infinite.

We can enumerate elements in S_∞ as $A_1, A_2, A_3, \ldots$. Let $a_1 \in A_1$. Then for $k \geq 1$, choose $a_k \in A_k \setminus \{a_1, a_2, \ldots, a_{k-1}\}$. We can see that $S_\infty \{a_k\}_{k=1}^{\infty}$ is an infinite sequence and a_k's are all different. Fix $a \in A$, define $\phi : S \to \cup S$ by

$$\phi(A) = \begin{cases} a_k & \text{if } A = A_k, \text{for some } k \\ a & \text{if } A \text{ is not in } S_\infty \end{cases}$$

Thus, $|\phi(S)| \geq |\{a_1, a_2, a_3, \ldots\}| = \aleph_0$. Since $|S| = \aleph_0$ and ϕ is a function, $|\phi(S)| \leq \aleph_0$. Thus, $|\phi(S)| = |S|$.

Case 2 S_∞ is finite.

Let $S - S_\infty = \{A_i : i \in \mathbb{N}\}$. Note that $S - S_\infty$ is an infinite set. We claim that there is a sequence $\{a_{k_i}\}_{i=1}^{\infty}$ such that for each $i \in \mathbb{N}$, there is $a_{k_i} \in A_{k_i}$ with a_{ki} $\{a_{k1}, a_{k2}, a_{k3}, \ldots, a_{ki-1}\}$. Let $a_1 \in A_{k_i} := A_1$. Assume that the above statement is true for $i = 1, 2, \ldots, j$ but false for all choices when $i = j+1$. Thus, for each choice of A_{kj+1} and $a_{kj+1} \in A_{kj+1}$, we have that $a_{kj+1} \in \{a_{k1}, a_{k2}, a_3, \ldots, a_{kj}\} \in P\left(\cup_{i=1}^{j} A_{k_i}\right) \subseteq P\left(\cup_{i=1}^{j+1} A_i\right)$. Thus, $A_r \in P\left(\cup_{i=1}^{j} A_{k_i}\right)$ for each $r \geq k_j$.

It implies that $A_r \in P\left(\cup_{i=1}^{k_j} A_i\right)$, for all $r \in \mathbb{N}$. Thus, $S - S_\infty \subseteq P\left(\cup_{i=1}^{k_j} A_i\right)$. However, A_i is finite for each i. Thus, $P\left(\cup_{i=1}^{k_j} A_i\right)$, is finite. Thus, S is finite, a contradiction. We can conclude that there is a one-to-one sequence $\{a_{k_i}\}_{i=1}^{\infty}$ in $S - S_\infty$ such that $a_{k_i} \in A_{k_i}$ for each $i \in \mathbb{N}$. Fix some $a \in A$. Define $\phi : S \to \cup S$ by

$$\phi(A) = \begin{cases} a_{k_i} & \text{if } A = A_{k_i}, \text{ for some } i \\ a \in A & \text{if } A \neq A_{k_i}, \text{ for all } i \end{cases}$$

Thus, we can have that $|\phi(S)| = |S|$. □

If a knowledge space (X, τ) has a finite topology (X is not necessarily finite), then we can prove that, for each $x \in X$, x is a core unit of K.

Theorem 2.2. *Let (X, τ) be a topological space where τ is finite. Then for every $x \in X$,*

$$|\tau_x| = |\tau|/2.$$

Proof. Let $x \in X$. Let $G_x = \{\mathcal{U} \in \tau : x \notin \mathcal{U}\}$. It follows that $\tau_x \cup G_x = \tau$. We can see that $G_x = \{\mathcal{U} \in \tau : \mathcal{OV} = \setminus\{x\}$ for some $V \in \tau_x$. Thus, $|\tau_x| = |G_x|$. Since $\hat{o}$ is a disjoint union of τ_x and G_x, $\tau_x = |\tau|/2$. □

In the following theorem, we will prove that for a given infinite topological space with certain properties, there is a core unit.

Theorem 2.3. *Let (X, τ) be an infinite, 2^{nd} countable and Hausdorff topological space. Then there is $x \in X$ such that $|\tau_x| = |\tau|$.*

Proof. Let $\{B_1, B_2, \ldots\}$ be a countable topological basis of τ. Since (X, τ) is 2^{nd} countable, we also have that $|\tau| \leq \aleph_1$. Let $x_i \in B_i$ for each i. Claim that there is i_0 such that $|\tau_{xi0}| = |\tau|$. Assume to the contrary that for each i, $|\tau_{xi}| < |\tau| \leq \aleph_1$.

Case 1 $|\tau| = \aleph_1$

By the assumption, $|\tau_{xi}| < |\tau| = \aleph_1$ for all $i \in \mathbb{N}$. By continuum hypothesis, $|\tau_{xi}| \leq \aleph_0$ for all $i \in \mathbb{N}$. Therefore, $\aleph_1 = |\tau| = |\cup_{i \in \aleph} \tau_{x_i}| \leq \aleph_0 < \aleph_1$, a contradiction.

Case 2 $|\tau| < \aleph_1$

Then, by assuming continuum hypothesis, $|\tau| = \aleph_0$. By the assumption that $|\tau_x| < |\tau|$ for all $x \in X$, we have that τ_x is finite for all $x \in X$. It follows that $\tau \setminus \tau_x$ is infinite for all $x \in X$. Let $x_0 \in X$ and $\tau \setminus \tau_{x0} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots\}$. Let $y_r \in \mathcal{O}_r$ for each $r \in \mathbb{N}$. Without loss of generality and by Lemma 2.1, we can assume that there are infinitely many such y_r's and they are all different. Since (X, τ) is Hausdorff, there are $\mathcal{U}_r, \mathcal{V}_r \in \tau$ such that $x_0 \in \mathcal{U}_r$, $y_r \in \mathcal{V}_r$ and $\mathcal{U}_r \cap \mathcal{V}_r = \varnothing$ for each $r \in \mathbb{N}$. Let $|\tau_{x_0}| = n_o \in \mathbb{N}$. For each $s \in \mathbb{N}$, define $\mathcal{U}'_s = \cap_{r=1}^{s} \mathcal{U}_r$. Thus, $\mathcal{U}_s \cap \mathcal{V}_r = \varnothing$ for all $s \geq r$ and $y_r \in \mathcal{V}_r$. Since (X, τ) is Hausdorff, we can redefine $\mathcal{V}_s$ (if necessary) for $1 \leq s \leq n_0 + 1$ so that $\mathcal{V}_{s'} \cap \mathcal{V}_s = \varnothing$ for $s' \neq s$. This can be done since y_r's are all different and we repeat the process finitely many times. Therefore, $\{\mathcal{U}'_r \cup \mathcal{V}_r \,|\, r = 1, \ldots, n_0 + 1|\}$ is a set of $n_0 + 1$ different open sets containing x_0, a contradiction.

From the above 2 cases, we can conclude that there is $x \in X$ such that $|\tau_x| = |\tau|$. □

Note that there exists a non-Hausdorff space without any core unit. For example, Let $X = \mathbb{N}$ and a topology τ on defined by $\tau = \{\{n, n+1, n+2, \dots\} : n \in \mathbb{N}\} \cup \{\varnothing\}$. We can see that for any $n_1, n_2 \in \mathbb{N}$, there are no $\mathcal{O}_1$, $\mathcal{O}$ $\mathcal{O}_2 \in \tau$ with the properties $n_1 \in \mathcal{O}_1$, $n_2 \in \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$. Thus, $(\mathbb{N}, \tau)$ is not Hausdorff. Moreover, any $n \in \mathbb{N}$, there are only n elements in containing . However, τ is an infinite set. Thus, there is no core unit in $(\mathbb{N}, \tau)$.

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