Songklanakarin J. Sci. Technol.
38 (5), 507-520, Sep. - Oct. 2016

http://www.sjst.psu.ac.th

Original Article

Performance analysis of preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption

Pakkirisami Rajadurai[1]*, Saravanan Yuvarani[2], and Mookkaiya Chandran Saravanarajan[2]

[1] *Department of Mathematics, Srinivasa Ramanujan Centre, Shanmugha Arts, Science, Technology and Research Academy (SASTRA University), Kumbakonam, Tamilnadu, India.*

[2] *Department of Mathematics, School of Advanced Sciences, Vellore Engineering College (VIT University), Vellore, Tamilnadu, India.*

Received: 1 October 2015; Accepted: 23 January 2016

Abstract

The present investigation deals with performance analysis of single server preemptive priority retrial queue with immediate Bernoulli feedback. There are two types of customers are considered, which are priority customers and ordinary customers. The priority customers do not form any queue and have an exclusive preemptive priority to receive their services over ordinary customers. After completion of regular service for ordinary customer, the customer is allowed to make an immediate feedback with probability *r*. When the orbit becomes empty at service completion instant for a priority customer or ordinary customer; the server goes for multiple working vacations. By using the supplementary variable technique, we obtained the steady state probability generating functions for the system/orbit. Some important system performance measures, the mean busy period and the mean busy cycle are discussed. Finally, some numerical examples are presented.

Keywords: retrial queue, preemptive priority queue, immediate Bernoulli feedback, working vacations, supplementary variable technique

1. Introduction

In queueing theory, retrial queues have been intensive research topics for quite some time; we can find general models in retrial queues from Gomez-Corral (2006), Artalejo and Gomez-Corral (2008) and Artalejo (2010). In retrial queueing system, queues with repeated attempts are characterized by an arriving customer who finds the server busy, leaves the service area and repeats its demand after some time. Between trials, the blocked customer joins a pool of unsatisfied customers called orbit. Such a retrial queues play a special role in telecommunication systems, communication protocols and retail shopping queues, etc.

In the past years, retrial queues with two types of customers have been widely studied by many researchers (Artalejo *et al*., 2001; Wang, 2008; Dimitriou, 2013; Wu *et al.*, 2013; Rajadurai *et al*., 2015d). The high priority customers are formed in queue or not queue and served according to discipline of preemptive or non-preemptive. Blocked pool of customers, low priority customers (called as ordinary customers), leave the system and join the retrial group to retry its service after some time when the server is free. Moreover, in some of the systems, an arriving higher priority customer may push out the lower priority customers whose service is ongoing to the queue or the orbit. For a comprehensive analysis of priority queueing models the reader may refer Liu *et al*. (2009), Liu and Gao (2011), Senthilkumar *et al*. (2013), Wu and Lian (2013), Gao (2015), and Peng (2015). Priority retrial queues are used in many applications like real-time systems, operating systems, manufacturing system, and

* Corresponding author.
Email address: psdurai17@gmail.com

simulations.

One additional feature which has been widely discussed in retrial queueing systems is feedback of customers. After completion of service the customers have to join the queue and wait for their service once again. In this aspect, the concept of optional re-service can be considered as immediate feedback. The customer completes his service as first step and if he finds any defect in his service or wants service one more time he will immediately get his service once again without joining the queue.

This model have many real life applications in situations like bank counters, working ATM machines, super markets, doctor clinics, etc. Some authors like Baruah *et al.* (2012), Kalidass and Kasturi (2014), and Rajadurai *et al.* (2014), have discussed the concept of immediate feedback (re-service). In the above mentioned papers, customer who wishes to obtain another round of service has to go to the server immediately one more time.

In a vacation queueing system, the server completely stops the service and unavailable for primary customers at short period of time. This period of time is referred as a vacation. But in working vacation period (WV), the server gives service to customer at lower service rate. This queueing system has major applications in providing network service, web service, file transfer service and mail service, etc.

In 2002, an M/M/1 queueing system with working vacations was first introduced by Servi and Finn (2002). Later, Wu and Takagi (2006) extended the M/M/1/WV queue to an M/G/1/WV queue. Very recently, Arivudainambi *et al.* (2014) introduced M/G/1 retrial queue with single working vacation. Furthermore, during the working vacation period, if there are customers at a lower service completion instant, the server can stop the vacation and come back to the normal busy state. This policy is called vacation interruption. Recently, authors like, Zhang and Hou (2012), Gao *et al.* (2014), Gao and Liu (2013), Rajadurai *et al.* (2015a,b,c, 2016) analyzed a single server retrial queue with working vacations and vacation interruptions.

To the authors best of knowledge, there are many works available in the concept of retrial queueing system with working vacation by using the method of matrix geometry analysis, but there is no work published in the queueing literature with the combination of preemptive priority retrial queueing system with general retrial times, immediate Bernoulli feedback, multiple working vacations and vacation interruption by using the method of supplementary variable technique.

In this paper, we consider a generalization of the well-known model discussed by Gao (2015) and Gao *et al.* (2014) with concepts of a single server preemptive priority retrial queue with general retrial times in two types customers, immediate Bernoulli feedback, multiple working vacations and vacation interruption. The rest of this paper is given as follows. The detailed mathematical model description and practical applications of this model are given is section 2. In section 3, the steady state joint distribution of the server state and the number of customers in the orbit/system are obtained. Some system performance measures, the mean busy period, the mean busy cycle are discussed in section 4. In section 5, important special cases are derived. In section 6, the effects of various parameters on the system performance are analyzed numerically. Conclusion and summary of the paper are presented in section 7.

2. Description of the Model

In this section, we consider a preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption. The detailed description of model is given as follows:

The arrival process: There are two types of customers arrive into the system: priority customers and ordinary customers. Priority customers have preemptive priorities over ordinary customers in service time of busy server. Assume that both priority customers and ordinary customers arrive according to two independent Poisson processes with rates λ and δ, respectively.

The retrial process: An arriving priority (or ordinary) customer finds the server is free, the customer begins its service immediately, otherwise the arrival time of a priority customer, the server gives service for a priority customer or lower speed serving in working vacation, the newly arriving priority customer will depart the system directly without service. While the regular busy server is working with an ordinary customer, the arriving priority customer will interrupt the service of the ordinary customer and the server begins its service immediately. We assume that when an ordinary customer is preempted by a priority customer, the ordinary customer who was just being served before starts the service of the priority customer and waits in the service area for the remaining service to complete.

If an arriving ordinary customer finds the server is being busy or on working vacation, the arrivals join pool of blocked customers called an orbit in accordance with FCFS discipline. That is, only one customer at the head of the orbit queue is allowed access to the server. Then measured from the instant the server becomes free, an external potential priority customer or ordinary customer and a retrial ordinary customer compete to entire the server. Inter-retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace Stieltijes Transform (LST) $R^*(\vartheta)$. The retrial ordinary customer is required to give up the attempt for service if an external priority customer or ordinary customer arrives first. In that case, the retrial ordinary customer goes back to its position in the retrial queue.

The multiple working vacation process: The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ. During a vacation period if any customer arrives, the server gives service at a lower speed service rate. If any customers in the orbit at a lower speed service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy period which means vacation interruption happens. Otherwise, it continues the vacation. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. Otherwise, the server begins another vacation. During the working vacation period, the service time follows a general random variable S_v with distribution function $S_v(t)$ and LST $S_v^*(\vartheta)$.

The regular service process: In the normal busy period, there is a single server which provides regular service and there is an option for re-service. That is called immediate Bernoulli feedback. As soon as the ordinary customer completes his service, he may repeat same service (without joining the orbit) with probability r or may leave the system with probability $(1-r)$. It is further assumed that the re-service may be repeated only once. The service time of priority customers follows a general distribution and denoted by the random variable S_p with distribution function $S_p(t)$, having LST $S_p^*(\vartheta)$ and the first and second moments are $\beta_p^{(1)}$ and $\beta_p^{(2)}$. The service time of ordinary customers follows a general distribution and denoted by the random variable S_b with distribution function $S_b(t)$, having LST $S_b^*(\vartheta)$ and the first and second moments are $\beta_b^{(1)}$ and $\beta_b^{(2)}$. Various stochastic processes involved in the system are assumed to be independent of each other. Throughout the rest of the paper, we denote by $\bar{F}(x) = 1 - F(x)$ the tail of distribution function $F(x)$. We also denote $F^*(s) = \int_0^\infty e^{-sx} dF(x)$, the LST of $F(x)$ and $\tilde{F}(s) = \int_0^\infty e^{-sx} F(x)dx$, to be the Laplace transform of $F(x)$ and we assume the notation $\bar{F}^*(s) = \frac{1 - F^*(s)}{s}$.

2.1 Practical application of the proposed model

Our model has a potential practical application in the area of computer processing, telecommunications, production and manufacturing system, inventory control system, operating systems and simulations. We consider a telecommunication system for example. In telecommunications, call centers play an important role in many industries and businesses. The customers are contacting to the call centers by talking to a customer service representative (CSR) or an agent over the telephone (the regular server). In addition to contacting over the phone (priority customer), the customers can contact the center over the internet either via e-mail, fax, or live chat sessions (the ordinary customers). An arriving voice call or e-mails handled by the idle CSR directly. Suppose, at the time of voice calling, if the CSR is not available, i.e. busy with other calls, the arriving voice call will be lost its service, but the CSR is busy with e-mails the voice call has a preemptive priority over an e-mail service and the preempted service will wait to complete its service. If an arriving message (the ordinary customers) found the CSR serving in voice call, the messages are temporarily stored in a retrial buffer (orbit) finite capacity should there be a space to be served some time later (retrial time) according to FCFS. After completion of message processing, the internet service may demand the same service to the CSR (the immediate feedback), if any failures in pervious process. When the CSR finds no voice call or mail services, it will perform a sequence of maintenance jobs, such as virus scan (multiple working vacations) in the system. During the maintenance period, the traditional center has different components such as an automatic call distributor (ACD) and an interactive voice response (IVR) unit (the working vacation server), these components can deal with the messages at the slower rate (working vacation period). This type of priority retrial queue with multiple working vacations discipline is a good approximation of such telecommunication processing system.

3. Steady State Analysis of the System

In this section, we develop the steady state difference-differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times and the elapsed working vacation times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for number of customers in the system and orbit.

3.1 The steady state equations

In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_p(0) = 0$, $S_p(\infty) = 1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $S_v(0) = 0$, $S_v(\infty) = 1$ are continuous at $x = 0$. So that the function $a(x)$, $\mu_p(x)$, $\mu_b(x)$ and $\mu_v(x)$ are the conditional completion rates (hazard rate) for retrial, service of a priority customer and ordinary customer, lower rate service respectively.

$$i.e.,\ a(x)dx = \frac{dR(x)}{1 - R(x)};\ \mu_p(x)dx = \frac{dS_p(x)}{1 - S_p(x)};$$

$$\mu_b(x)dx = \frac{dS_b(x)}{1 - S_b(x)};\ \mu_v(x)dx = \frac{dS_v(x)}{1 - S_v(x)}.$$

In addition, let $R^0(t)$, $S_p^0(t)$, $S_b^0(t)$ and $S_v^0(t)$ be the elapsed retrial time, elapsed service time of the priority customer, elapsed service time of the ordinary customer, elapsed service time of the ordinary feedback customer and elapsed working vacation time respectively at time t. Further, we introduce the random variable,

$$C(t)=\begin{cases}0, & \text{if the server is idle and in working vacation period,}\\ 1, & \text{if the server is free and in regular service period,}\\ 2, & \text{if the server is busy with a priority customer without preempting an ordinary customer and in regular service period at time } t,\\ 3, & \text{if the server is busy with a priority customer with preempting an ordinary customer and in regular service period at time } t,\\ 4, & \text{if the server is busy with an ordinary customer and in regular service period at time } t,\\ 5, & \text{if the server is busy with an ordinary feedback customer and in regular service period at time } t,\\ 6, & \text{if the server is busy and in working vacation period at time } t.\end{cases}$$

Let $\{t_n;\ n = 1,2,...\}$ be the sequence of epochs of the regular service completion times for priority customers, ordinary customers, ordinary feedback customers or working vacation period completion occurs ends. Then the state of the queueing system can be described by the bivariate Markov process $\{C(t), N(t);\ t \ge 0\}$, where $C(t)$ denotes the server state (0,1,2,3,4,5,6) depending on the server is free, busy for priority customers, ordinary customers, ordinary feedback customers and working vacation. $N(t)$ denotes the number of ordinary customers in the orbit. If $C(t)=1$ and $N(t)>0$ then $R^0(t)$ represent the elapsed retrial time. If $C(t)=2$ and $N(t)>0$ then $S_p^0(t)$ corresponding to the elapsed service time of the priority customer being served in regular busy period. If $C(t)=3$ and $N(t)>0$ then $S_p^0(t)$ corresponding to the elapsed service time of the interrupted ordinary customer being served in regular busy period. If $C(t)=4$ and $N(t)>0$ then $S_p^0(t)$ corresponding to the elapsed service time of the ordinary customer being served in regular busy period. If $C(t)=5$ and $N(t)>0$ then $S_p^0(t)$ corresponding to the elapsed service time of the feedback customer being served in regular busy period. If $C(t)=6$ and $N(t)>0$ then $S_p^0(t)$ corresponding to the elapsed time of the customer being served in lower rate service period. Then the sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 3.1: The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda+\delta)$, where $\rho = \left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)+\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}$

Proof: To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (Pakes, 1969), (Choudhury and Ke, 2012), which states that the chain $\{Z_n;\ n \in N\}$ is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{n+1}) - f(z_n)/z_n = j]$ is finite for all $j \in N$ and $\psi_j \le -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. Then we have

$$\psi_j=\begin{cases}\rho-1, & \text{if } j=0,\\ \rho-R^*(\lambda+\delta),, & \text{if } j=1,2...\end{cases}$$

Clearly, the inequality $\rho < R^*(\lambda+\delta)$, is sufficient condition for ergodicity. To prove the necessary condition, as noted in Sennott *et al.* (1983), if the Markov chain $\{Z_n;\ n \ge 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \ge 0$ and there exits $j_0 \in N$ such that $\psi_j \ge 0$ for $j \ge j_0$. Notice that in our case Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i-k$, and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n;\ n \in N\}$. Then $\rho \ge R^*(\lambda+\delta)$, implies the non-ergodicity of the Markov chain. For the process, we define the limiting probabilities $Q_0(t) = P\{C(t)=0,\ N(t)=0\}$ and the probability densities

$$P_n(x,t)dx = P\{C(t)=1, N(t)=n,\ x<R^0(t)\le x+dx\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 1.$$

$$\Pi_{1,n}(x,t)dx = P\{C(t)=2, N(t)=n,\ x<S_p^0(t)\le x+dx\},\ \text{for } t\ge 0,\ x\ge 0,\ n\ge 0.$$

$$\Pi_{2,n}(x,y,t)dx = P\{C(t)=3, N(t)=n,\ x<S_p^0(t)\le x+dx,\ y<S_b^0(t)\le y+dy\},\ \text{for } t\ge 0,\ x\ge 0,\ y\ge 0,\ n\ge 0.$$

$$\Pi_{b,n}(x,t)dx = P\{C(t)=4, N(t)=n,\ x<S_b^0(t)\le x+dx\},\ \text{for } t\ge 0,\ x\ge 0,\ n\ge 0.$$

$$\Omega_n(x,t)dx = P\{C(t)=5,\ N(t)=n,\ x<S_b^0(t)\le x+dx\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0.$$

$$Q_{v,n}(x,t)dx = P\{C(t)=6, N(t)=n,\ x<S_v^0(t)\le x+dx\},\ \text{for } t\ge 0,\ x\ge 0 \text{ and } n\ge 0.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $Q_0 = \lim_{t\to\infty} Q_0(t)$; and limiting densities for $t \ge 0$, $x \ge 0$ and $n \ge 1$.

$$P_n(x) = \lim_{t\to\infty} P_n(x,t);\ \Pi_{1,n}(x) = \lim_{t\to\infty}\Pi_{1,n}(x,t);\ \Pi_{2,n}(x,y) = \lim_{t\to\infty}\Pi_{2,n}(x,y,t);\ \Pi_{b,n}(x) = \lim_{t\to\infty}\Pi_{b,n}(x,t)$$

$$\text{and } Q_{v,n}(x) = \lim_{t\to\infty} Q_{v,n}(x,t).$$

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$(\lambda+\delta+\theta)Q_0 = \theta Q_0 + \int_0^\infty \Pi_{1,0}(x)\mu_p(x)dx + (1-r)\int_0^\infty \Pi_{b,0}(x)\mu_b(x)dx + \int_0^\infty \Omega_0(x)\mu_b(x)dx + \int_0^\infty Q_{v,0}(x)\mu_v(x)dx \quad (1)$$

$$\frac{dP_n(x)}{dx} + (\lambda+\delta+a(x))P_n(x) = 0,\ n\ge 1 \quad (2)$$

$$\frac{d\Pi_{1,0}(x)}{dx} + (\lambda+\mu_p(x))\Pi_{1,0}(x) = 0,\ n=0, \quad (3)$$

$$\frac{d\Pi_{1,n}(x)}{dx} + (\lambda+\mu_p(x))\Pi_{1,n}(x) = \lambda\Pi_{1,n-1}(x),\ n\ge 1 \quad (4)$$

$$\frac{\partial\Pi_{2,0}(x,y)}{\partial x} + (\lambda+\mu_p(x))\Pi_{2,0}(x,y) = 0,\ n=0, \quad (5)$$

$$\frac{\partial\Pi_{2,n}(x,y)}{\partial x} + (\lambda+\mu_p(x))\Pi_{2,n}(x,y) = \lambda\Pi_{2,n-1}(x,y),\ n\ge 1, \quad (6)$$

$$\frac{d\Pi_{b,0}(x)}{dx}+\left(\lambda+\delta+\mu_b(x)\right)\Pi_{b,0}(x)=\int_0^\infty \Pi_{2,0}(y,x)\mu_p(y)dy,$$

$$n=0 \quad (7)$$

$$\frac{d\Pi_{b,n}(x)}{dx}+\left(\lambda+\delta+\mu_b(x)\right)\Pi_{b,n}(x)=$$

$$\lambda\Pi_{b,n-1}(x)+\int_0^\infty \Pi_{2,n}(y,x)\mu_p(y)dy,\ n\geq 1 \quad (8)$$

$$\frac{d\Omega_0(x)}{dx}+\left(\lambda+\delta+\mu_b(x)\right)\Omega_0(x)=\int_0^\infty \Pi_{2,0}(y,x)\mu_p(y)dy,\ n=0 \quad (9)$$

$$\frac{d\Omega_n(x)}{dx}+\left(\lambda+\delta+\mu_b(x)\right)\Omega_n(x)=\lambda\Omega_{n-1}(x)+$$

$$\int_0^\infty \Pi_{2,n}(y,x)\mu_p(y)dy,\ n\geq 1 \quad (10)$$

$$\frac{dQ_{v,0}(x)}{dx}+\left(\lambda+\theta+\mu_v(x)\right)Q_{v,0}(x)=0,\ n=0 \quad (11)$$

$$\frac{dQ_{v,n}(x)}{dx}+\left(\lambda+\theta+\mu_v(x)\right)Q_{v,n}(x)=\lambda Q_{v,n-1}(x),\ n\geq 1 \quad (12)$$

To solve Equations 2 to 12, the steady state boundary conditions at $x = 0$ and $y = 0$ are followed,

$$P_n(0)=\int_0^\infty \Pi_{1,n}(x)\mu_p(x)dx+(1-r)\int_0^\infty \Pi_{b,n}(x)\mu_b(x)dx$$

$$+\int_0^\infty \Omega_n(x)\mu_b(x)dx+\int_0^\infty Q_{v,n}(x)\mu_v(x)dx,\ n\geq 1 \quad (13)$$

$$\Pi_{1,n}(0)=\delta\int_0^\infty P_n(x)dx,\ n\geq 1 \quad (14)$$

$$\Pi_{2,n}(0,x)=\delta\left(\Pi_{b,n}(x)+\Omega_n(x)\right),\ n\geq 0 \quad (15)$$

$$\Pi_{b,0}(0)=\left(\int_0^\infty P_1(x)a(x)dx+\theta\int_0^\infty Q_{v,0}(x)dx\right),\ n=0 \quad (16)$$

$$\Pi_{b,n}(0)=\left(\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\int_0^\infty P_n(x)dx+\theta\int_0^\infty Q_{v,n}(x)dx\right),\ n\geq 1, \quad (17)$$

$$\Omega_n(0)=r\int_0^\infty \Pi_{b,n}(x)\mu_b(x)dx,\ n\geq 0 \quad (18)$$

$$Q_{v,n}(0)=\begin{cases}\left(\lambda+\delta\right)Q_0, & n=0\\ 0, & n\geq 1\end{cases} \quad (19)$$

The normalizing condition is

$$Q_0+\sum_{n=1}^\infty\int_0^\infty P_n(x)dx+$$

$$\sum_{n=0}^\infty\left(\int_0^\infty \Pi_{1,n}(x)dx+\int_0^\infty\int_0^\infty \Pi_{2,n}(x,y)dxdy+\int_0^\infty \Pi_{b,n}(x)dx+\int_0^\infty \Omega_n(x)dx+\int_0^\infty Q_{v,n}(x)dx\right)=1 \quad (20)$$

3.2 The steady state solution

The steady state solution of the retrial queueing model is obtained by using the PGF function technique. To solve the above equations, the PGFs are defined for $|z| \leq 1$ as follows:

$$P(x,z)=\sum_{n=1}^\infty P_n(x)z^n;\ P(0,z)=\sum_{n=1}^\infty P_n(0)z^n;$$

$$\Pi_1(x,z)=\sum_{n=0}^\infty \Pi_{1,n}(x)z^n;\ \Pi_1(0,z)=\sum_{n=0}^\infty \Pi_{1,n}(0)z^n;$$

$$\Pi_2(x,y,z)=\sum_{n=0}^\infty \Pi_{2,n}(x,y)z^n;\ \Pi_2(x,0,z)=\sum_{n=0}^\infty \Pi_{2,n}(x,0)z^n;$$

$$\Pi_b(x,z)=\sum_{n=0}^\infty \Pi_{b,n}(x)z^n;\ \Pi_b(0,z)=\sum_{n=0}^\infty \Pi_{b,n}(0)z^n;$$

$$\Omega(x,z)=\sum_{n=0}^\infty \Omega_n(x)z^n;\ \Omega(0,z)=\sum_{n=0}^\infty \Omega_n(0)z^n \quad \text{and}$$

$$Q_v(x,z)=\sum_{n=0}^\infty Q_{v,n}(x)z^n;\ Q_v(0,z)=\sum_{n=0}^\infty Q_{v,n}(0)z^n;$$

On multiplying the Equations 2 to 12 by z^n and summing over n, (n = 0,1,2...) and solving the partial differential equations, we get

$$P(x,z)=P(0,z)[1-R(x)]e^{-(\lambda+\delta)x} \quad (21)$$

$$\Pi_1(x,z)=\Pi_1(0,z)[1-S_p(x)]e^{-A_p(z)x}, \quad (22)$$

$$\Pi_2(x,y,z)=\Pi_2(0,y,z)[1-S_p(x)]e^{-A_p(z)x}, \quad (23)$$

$$\Pi_b(x,z)=\Pi_b(0,z)[1-S_b(x)]e^{-A_b(z)x}, \quad (24)$$

$$\Omega(x,z)=\Omega(0,z)[1-S_b(x)]e^{-A_b(z)x}, \quad (25)$$

$$Q_v(x,z)=Q_v(0,z)[1-S_v(x)]e^{-A_v(z)x}, \quad (26)$$

where

$$A_p(z)=\left(\lambda(1-z)\right),\ A_b(z)=\left(\lambda(1-z)+\delta\left(1-S_p^*\left(A_p(z)\right)\right)\right)$$

and $A_v(z)=\left(\theta+\lambda(1-z)\right)$

From Equations 13 to 19 we can obtain

$$P(0,z)=\int_0^\infty \Pi_1(x,z)\mu_p(x)dx+(1-r)\int_0^\infty \Pi_b(x,z)\mu_b(x)dx+\int_0^\infty \Omega(x,z)\mu_b(x)dx+\int_0^\infty Q_v(x,z)\mu_v(x)dx-(\lambda+\delta)Q_0 \quad (27)$$

$$\Pi_1(0,z)=\delta\int_0^\infty P(x,z)dx, \quad (28)$$

$$\Pi_2(0,x,z)=\delta\left(\Pi_b(x,z)+\Omega(x,z)\right), \quad (29)$$

$$\Pi_b(0,z)=\left(\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\lambda\int_0^\infty P(x,z)dx+\theta\int_0^\infty Q_v(x,z)dx\right), \quad (30)$$

$$\Omega(0,z)=r\int_0^\infty \Pi_b(x,z)\mu_b(x)dx, \quad (31)$$

$$Q_v(0,z)=(\lambda+\delta)Q_0 \quad (32)$$

Inserting the Equations 21 in 28 we get

$$\Pi_1(0,z)=\delta P(0,z)\bar{R}^*(\lambda+\delta) \quad (33)$$

where $\bar{R}^*(\lambda+\delta)=\left(\frac{1-R^*(\lambda+\delta)}{\lambda+\delta}\right)$

Inserting Equation 21, 28, 33 in 30 and make some manipulation we finally get,

$$\Pi_b(0,z)=\frac{P(0,z)}{z}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+(\lambda+\delta)Q_0V(z) \quad (34)$$

where $V(z)=\frac{\theta}{\theta+\lambda(1-z)}\left[1-S_v^*\left(A_v(z)\right)\right]$

Inserting the Equation 24 in 31 we get

$$\Omega(0,z)=r\Pi_b(0,z)S_b^*\left(A_b(z)\right) \quad (35)$$

Using Equations 24-25 in 29 we get

$$\Pi_2(0,x,z)=r\left[\Pi_b(0,z)+\Omega(0,z)\right](1-S_b(x))e^{-A_b(x)} \quad (36)$$

Using Equation 22-26 in 27 and make some manipulation we get

$$P(0,z)=\Pi_1(0,z)S_p^*\left(A_p(z)\right)+(1-r)\Pi_b(0,z)S_b^*\left(A_b(z)\right)+\Omega(0,z)S_b^*\left(A_b(z)\right)+Q_v(0,z)S_v^*\left(A_v(z)\right)-(\lambda+\delta)Q_0 \quad (37)$$

Using Equations 32-36 in 37, we get

$$P(0,z)=\frac{Nr(z)}{Dr(z)} \quad (38)$$

$$Nr(z)=z(\lambda+\delta)Q_0\left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}$$

$$Dr(z)=z-\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)-z\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)$$

Using Equation 38 in 33 we get

$$\Pi_1(0,z)=\delta(\lambda+\delta)zQ_0\bar{R}^*(\lambda+\delta)\left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}\Big/Dr(z) \quad (39)$$

Using Equation 38 in 34 we get

$$\Pi_b(0,z) = (\lambda+\delta)Q_0\left\{\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\right\}\Big/Dr(z) \tag{40}$$

Using the Equation 40 in 35 we get

$$\Omega(0,z) = r(\lambda+\delta)\left(S_b^*(A_b(z))\right)Q_0\left\{\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\right\}\Big/Dr(z) \tag{41}$$

Using the Equation 40-41 in 36 we get

$$\Pi_2(0,x,z) = \left\{\begin{array}{l}\delta\left(1+rS_b^*(A_b(z))\right)(\lambda+\delta)zQ_0\bar{R}^*(\lambda+\delta)(1-S_b(x))e^{-A_b(x)} \\ \times\left\{\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\right\}\end{array}\right\}\Bigg/Dr(z) \tag{42}$$

Using Equation 32 and 38-42 in 21-26, then we get the results for the following PGFs $P(x,z)$, $\Pi_1(x,z)$, $\Pi_2(x,y,z)$, $\Pi_b(x,z)$, $\Omega(x,z)$ and $Q_v(x,z)$. Next, we are interested in investigating the marginal orbit size distributions due to system state of the server.

Corollary 1: The marginal probability distributions of the number of customers in the orbit when server being idle, busy serving a priority customers without preempting an ordinary customer, busy serving a priority customers with preempting an ordinary customer, busy serving an ordinary customers, busy serving an ordinary feedback customer and on working vacation is given by

$$P(z) = \frac{Nr(z)}{Dr(z)} = \int_0^\infty P(x,z)dx \tag{43}$$

$$Nr(z) = z(\lambda+\delta)\bar{R}^*(\lambda+\delta)Q_0\left\{\left(S_v^*(A_v(z))-1\right)+V(z)\left((1-r)S_b^*(A_b(z))+r\left(S_b^*(A_b(z))\right)^2\right)\right\}$$

$$Dr(z) = z-\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left((1-r)S_b^*(A_b(z))+r\left(S_b^*(A_b(z))\right)^2\right)-z\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)$$

$$\Pi_1(z) = \int_0^\infty \Pi_1(x,z)dx = \left\{\begin{array}{l}\delta(\lambda+\delta)zQ_0\bar{R}^*(\lambda+\delta)\left(1-S_p^*(A_p(z))\right) \\ \times\left\{\left(S_v^*(A_v(z))-1\right)+V(z)\left((1-r)S_b^*(A_b(z))+r\left(S_b^*(A_b(z))\right)^2\right)\right\}\end{array}\right\}\Bigg/\left(A_p(z)\times Dr(z)\right) \tag{44}$$

$$\Pi_2(z) = \int_0^\infty \Pi_2(x,z)dx = \left\{\begin{array}{l}\delta(\lambda+\delta)\left(1+rS_b^*(A_b(z))\right)\left(1-S_p^*(A_p(z))\right)zQ_0\bar{R}^*(\lambda+\delta) \\ \times\left(1-S_b^*(A_b(z))\right)\left\{\begin{array}{l}\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right) \\ +zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\end{array}\right\}\end{array}\right\}\Bigg/\left(A_p(z)\times A_b(z)\times Dr(z)\right) \tag{45}$$

$$\Pi_b(z) = \int_0^\infty \Pi_b(x,z)dx = \left\{(\lambda+\delta)\left(1-S_b^*(A_b(z))\right)Q_0\left\{\begin{array}{l}\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right) \\ +zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\end{array}\right\}\right\}\Bigg/\left(A_b(z)\times Dr(z)\right) \tag{46}$$

$$\Omega(z) = \int_0^\infty \Omega(x,z)dx = \left\{\begin{array}{l}r(\lambda+\delta)\left(1-S_b^*(A_b(z))\right)S_b^*(A_b(z))Q_0 \\ \times\left\{\begin{array}{l}\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right) \\ +zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right)\end{array}\right\}\end{array}\right\}\Bigg/\left(A_b(z)\times Dr(z)\right) \tag{47}$$

$$Q_v(z) = \int_0^\infty Q_v(x,z)dx = \{(\lambda+\delta)Q_0 V(z)/\theta\} \tag{48}$$

Applying the normalizing condition $Q_0 + P(1) + \Pi_1(1) + \Pi_2(1) + \Pi_b(1) + \Omega(1) + Q_v(1) = 1$ and using the equations by setting $z = 1$ in Equation 43-48 we get

$$Q_0 = \frac{R^*(\lambda+\delta)-\rho}{\left\{R^*(\lambda+\delta)-\rho+((\lambda+\delta)/\theta)\left(1-S_v^*(\theta)\right)\left\{\begin{bmatrix}(1+r)\left(1+\delta\beta_p^{(1)}\right)\beta_b^{(1)}\left(\lambda\left(1-\delta\bar{R}^*(\lambda+\delta)\right)\right.\\ \left.+(\theta-1)\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\right)+\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\end{bmatrix}\right\}\right\}} \tag{49}$$

where

$$\rho = \left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)+\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)};\ A_p(z) = \left(\lambda(1-z)\right);$$

$$A_b(z) = \left(\lambda(1-z)+\delta\left(1-S_p^*\left(A_p(z)\right)\right)\right) \text{ and } A_v(z) = \left(\theta+\lambda(1-z)\right)$$

Corollary 2. The probability generating function of number of customers in the system and orbit size distribution at stationary point of time is

$$K_s(z) = \frac{Nr_s(z)}{Dr_s(z)} = Q_0 + P(z) + \Pi_1(z) + z\left(\Pi_2(z)+\Pi_b(z)+\Omega(z)+Q_v(z)\right) \tag{50}$$

$$Nr_s(z) = Q_0\left\{\begin{array}{l}\lambda(1-z)\begin{bmatrix}\left(\left[1+((\lambda+\delta)zV(z)/\theta)\right]\times Dr(z)\right)+z\left(1-R^*(\lambda+\delta)\right)\\ \left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}\end{bmatrix}\\ +z\delta\left(1-R^*(\lambda+\delta)\right)\left(1-S_p^*\left(A_p(z)\right)\right)\left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}\\ +z(\lambda+\delta)\left(1+rS_b^*\left(A_b(z)\right)\right)\left(1-S_b^*\left(A_b(z)\right)\right)\\ \left\{\left(S_v^*\left(A_v(z)\right)-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)\right)\right\}\end{array}\right\}$$

$$Dr_s(z) = A_p(z)\times\left\{z-\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)-z\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)\right\}$$

$$K_o(z) = \frac{Nr_o(z)}{Dr_s(z)} = Q_0 + P(z) + \Pi_1(z) + \Pi_2(z) + \Pi_b(z) + \Omega(z) + Q_v(z) \tag{51}$$

$$Nr_o(z) = Q_0\left\{\begin{array}{l}\lambda(1-z)\begin{bmatrix}\left(\left[1+((\lambda+\delta)V(z)/\theta)\right]\times Dr(z)\right)+z\left(1-R^*(\lambda+\delta)\right)\\ \left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}\end{bmatrix}\\ +z\delta\left(1-R^*(\lambda+\delta)\right)\left(1-S_p^*\left(A_p(z)\right)\right)\left\{\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left((1-r)S_b^*\left(A_b(z)\right)+r\left(S_b^*\left(A_b(z)\right)\right)^2\right)\right\}\\ +(\lambda+\delta)\left(1+rS_b^*\left(A_b(z)\right)\right)\left(1-S_b^*\left(A_b(z)\right)\right)\\ \left\{\left(S_v^*\left(A_v(z)\right)-1\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)+zV(z)\left(1-\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)\right)\right\}\end{array}\right\}$$

where Q_0 is given in Equation 49.

4. System Performance Measures

In this section, we derive some system probabilities, mean number of customers in the orbit/system, mean busy period and busy cycle of the model.

4.1 System state probabilities

From Equations 43 to 48, by setting $z \to 1$ and applying L-Hospital's rule whenever necessary, then we get the following results,

(i) The probability that the server is idle during the retrial, is given by,

$$P = P(1) = Q_0\left(1 - R^*(\lambda+\delta)\right)\left\{\frac{\left(1 - S_v^*(\theta)\right)\left((\lambda/\theta) + (1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\right)}{R^*(\lambda+\delta) - \rho}\right\}$$

(ii) The probability that the server is busy serving a priority customers without preempting an ordinary customer, is given by,

$$\Pi_1 = \Pi_1(1) = \delta Q_0\left(1 - R^*(\lambda+\delta)\right)\left\{\frac{\beta_p^{(1)}\left(1 - S_v^*(\theta)\right)\left((\lambda/\theta) + (1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\right)}{R^*(\lambda+\delta) - \rho}\right\}$$

(iii) The probability that the server is busy serving a priority customers with preempting an ordinary customer, is given by,

$$\Pi_2 = \Pi_2(1) = \delta Q_0(\lambda+\delta)(1+r)\beta_b^{(1)}\beta_p^{(1)}\left(1 - S_v^*(\theta)\right)\left\{\frac{R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)} + (\lambda/\theta)\left(1 - \delta\bar{R}^*(\lambda+\delta)\right)}{R^*(\lambda+\delta) - \rho}\right\}$$

(iv) The probability that the server is busy serving an ordinary customers, is given by,

$$\Pi_b = \Pi_b(1) = (\lambda+\delta)Q_0\beta_b^{(1)}\left(1 - S_v^*(\theta)\right)\left\{\frac{R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)} + (\lambda/\theta)\left(1 - \delta\bar{R}^*(\lambda+\delta)\right)}{R^*(\lambda+\delta) - \rho}\right\}$$

(v) The probability that the server is busy serving an ordinary feedback customers, is given by,

$$\Omega = \Omega(1) = r(\lambda+\delta)Q_0\beta_b^{(1)}\left(1 - S_v^*(\theta)\right)\left\{\frac{R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)} + (\lambda/\theta)\left(1 - \delta\bar{R}^*(\lambda+\delta)\right)}{R^*(\lambda+\delta) - \rho}\right\}$$

(vi) The probability that the server is on working vacation, is given by,

$$Q_v = Q_v(1) = \left\{(\lambda+\delta)Q_0\left(1 - S_v^*(\theta)\right)\big/\theta\right\}$$

4.2 Men system size and orbit size

If the system is in steady state condition,

(i) The excepted number of customers in the orbit (L_q) is obtained by differentiating Equation 51 with respect to *z* and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z) = Q_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_q''(1) = -2\lambda\left(R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}\right) - 2\lambda\left(R^*(\lambda+\delta) + \lambda\bar{R}^*(\lambda+\delta)\right)$$
$$\left\{((\lambda+\delta)/\theta)\left(1 - S_v^*(\theta)\right) - (1+r)\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\left(\lambda - (\lambda+\delta)\left(1 - S_v^*(\theta)\right)\right)\right\}$$

$$Nr_q'''(1) = 3\tau\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\left(\lambda S_v^*(\theta)-\delta\left(1-S_v^*(\theta)\right)\right)-6\left(\lambda/\theta\right)(\lambda+\delta)V'(1)\left(R^*(\lambda+\delta)-\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}\right)$$
$$+3\left[\left((\lambda+\delta)/\theta\right)\left(1-S_v^*(\theta)\right)+1\right]\left[2\lambda^2\bar{R}^*(\lambda+\delta)(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)+\lambda\delta\bar{R}^*(\lambda+\delta)\left(\lambda^2\beta_p^{(2)}+2\lambda\beta_b^{(1)}\right)\right]$$
$$-3\left(1-R^*(\lambda+\delta)\right)\left(\left(\lambda/\theta\right)+(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\right)\left[2\lambda\left(1+V'(1)\left(1+\delta\beta_p^{(1)}\right)\right)+\delta\left(1-S_v^*(\theta)\right)\left(\lambda^2\beta_p^{(2)}+2\lambda\beta_b^{(1)}\right)\right]$$
$$+(1+r)(\lambda+\delta)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\begin{bmatrix}V'(1)+2\lambda S_v^{*'}(\theta)\left(1-R^*(\lambda+\delta)+\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}\right)+\\ \delta\bar{R}^*(\lambda+\delta)\left(1-S_v^*(\theta)\right)\left(\lambda^2\beta_p^{(2)}+2\lambda\left(1+\beta_b^{(1)}\left(\left(\lambda/\theta\right)+1\right)\right)\right)\end{bmatrix}$$

$$Dr_q''(1) = -2\lambda\left(R^*(\lambda+\delta)-\rho\right)$$
$$Dr_q'''(1) = 3\lambda\left(\tau\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)+\bar{R}^*(\lambda+\delta)\left(2\lambda^2\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)+\delta\left(\lambda^2\beta_p^{(2)}+2\lambda\beta_b^{(1)}\right)\right)\right)$$

where $V'(1)=\frac{\lambda}{\theta}\left(1-S_v^*(\theta)+\theta S_v^{*'}(\theta)\right)$; $V''(1)=\left(\frac{\lambda}{\theta}\right)^2\left[2\left(1-S_v^*(\theta)+\theta S_v^{*'}(\theta)\right)-\theta^2 S_v^{*''}(\theta)\right]$; $S_v^{*'}(\theta)=\int_0^\infty xe^{-\theta x}dS_v(x)$;

$$\tau=(1+r)\left(\lambda^2\beta_b^{(2)}\left(1+\delta\beta_p^{(1)}\right)^2+\delta\lambda^2\beta_p^{(2)}\beta_b^{(1)}\right)+2r\left(\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\right)^2$$

$$\rho=\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)+\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)};$$

(ii) The excepted number of customers in the system (L_s) is obtained by differentiating Equation 50 with respect to z and evaluating at $z = 1$

$$L_s = K_s'(1) = \lim_{z\to1}\frac{d}{dz}K_s(z) = Q_0\left[\frac{Nr_s'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

where $Nr_s'''(1) = Nr_q'''(1)-6(\lambda+\delta)\left(1-S_v^*(\theta)\right)\begin{Bmatrix}\left(\lambda/\theta\right)\left(R^*(\lambda+\delta)-\rho\right)+(1+r)\lambda\beta_b^{(1)}\left(1+\delta\beta_p^{(1)}\right)\\ \left(R^*(\lambda+\delta)-\delta\bar{R}^*(\lambda+\delta)\left(\left(\lambda/\theta\right)+\lambda\beta_b^{(1)}\right)+\left(\lambda/\theta\right)\right)\end{Bmatrix}$

(iii) The average time a customer spends in the system (W_s) and its orbit (W_q) are found by using the Little's formula

$$W_s=\frac{L_s}{\lambda} \text{ and } W_q=\frac{L_q}{\lambda}$$

4.3Mean busy period and busy cycle

Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions. By applying the argument of an alternating renewal process which leads to

$$Q_0=\frac{E(T_0)}{E(T_b)+E(T_0)};\ E(T_b)=\frac{1}{(\lambda+\delta)}\left(\frac{1}{Q_0}-1\right) \text{ and } E(T_c)=\frac{1}{(\lambda+\delta)Q_0}=E(T_0)+E(T_b) \qquad (52)$$

where T_0 is length of the system in empty state and $E(T_0)=\left(1/(\lambda+\delta)\right)$. Substituting the Equation 49 into Equation 52 and use the above results, then we can get

$$E(T_b)=\frac{\left((\lambda+\delta)/\theta\right)\left(1-S_v^*(\theta)\right)\left\{\begin{array}{l}(1+r)\left(1+\delta\beta_p^{(1)}\right)\beta_b^{(1)}\left(\lambda\left(1-\delta\overline{R}^*(\lambda+\delta)\right)\right.\\ \left.+(\theta-1)\left(R^*(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\right)\right)+\left(R^*(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\right)\end{array}\right\}}{(\lambda+\delta)\left(R^*(\lambda)-\rho\right)} \tag{53}$$

$$E(T_c)=\frac{\left(R^*(\lambda)-\rho\right)+\left((\lambda+\delta)/\theta\right)\left(1-S_v^*(\theta)\right)\left\{\begin{array}{l}(1+r)\left(1+\delta\beta_p^{(1)}\right)\beta_b^{(1)}\left(\lambda\left(1-\delta\overline{R}^*(\lambda+\delta)\right)\right.\\ \left.+(\theta-1)\left(R^*(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\right)\right)+\left(R^*(\lambda+\delta)+\lambda\overline{R}^*(\lambda+\delta)\right)\end{array}\right\}}{(\lambda+\delta)\left(R^*(\lambda)-\rho\right)} \tag{54}$$

5. Special Cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

***Case* (i):** *No priority arrival, No feedback, No vacation interruption and Single working vacation*

In this case, our model becomes an $M/G/1$ retrial queue with single working vacation. We assume that $(\delta, r, \theta)\to(0,0,0)$ in the main result is obtained as follows,

$$K_s(z)=\left\{\frac{S_v^*(\lambda)\left(R^*(\lambda)-\lambda\beta^{(1)}\right)}{\lambda E(S_v)-R^*(\lambda)S_v^*(\lambda)}\right\}\left\{\frac{\left\{\left[\left(S_v^*(\lambda-\lambda z)-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+(1-z)R^*(\lambda)S_v^*(\lambda)\right]\right\}S_b^*(\lambda-\lambda z)}{S_v^*(\lambda)\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(\lambda-\lambda z)\right)}\right\}$$

This coincides with the result of Arivudainambi *et al*. (2014).

***Case* (ii):** *No priority arrival and No feedback*

In this case, our model becomes a single server retrial queueing system with working vacations. We assume that $\delta=r=0$ in the main result $K_s(z)$ can be yielded as follows,,

$$K_s(z)=P_0\frac{\left\{\begin{array}{l}(1-z)\left[\begin{array}{l}\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(A_b(z))\right)\left((\lambda V(z)/\theta)+1\right)\\ +z\left(1-R^*(\lambda)\right)\left[\left(S_v^*(A_v(z))+V(z)S_b^*(A_b(z))\right)-1\right]\end{array}\right]\\ +\left(1-S_b^*(A_b(z))\right)\left[\left(S_v^*(A_v(z))-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+zV(z)\right]\end{array}\right\}}{(1-z)\left(z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_b^*(A_b(z))\right)}$$

This coincides with the result of Gao *et al*. (2014).

***Case* (iii):** *No feedback, No working vacation and No vacation interruption*

In this case, we put $r=\theta=0$, our model can be reduced to a single server retrial queueing system with working vacations and $K_s(z)$ can be obtained as follows,

$$K_s(z)=\frac{Q_0R^*(\lambda+\delta)S_b^*(A_b(z))(1-z)\left(1+\delta S_p^*(A_p(z))\right)}{\left\{z-\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)S_b^*(A_b(z))-z\delta\overline{R}^*(\lambda+\delta)\left(S_p^*(A_p(z))\right)\right\}}$$

This coincides with the result of Gao (2015).

6. Numerical Examples

In this section, we present some numerical examples to study the effect of various parameters. For the purpose of a numerical illustration, we assume that all distribution function like retrial, regular service for priority and ordinary customers, working vacation are exponentially, Erlang and hyper-exponentially distributed. All the parameters values are selected with the aim of satisfying the steady state condition $\rho < R^*(\lambda+\delta)$. MATLAB software has been used to illustrate the results numerically. Where the exponential distribution is $f(x)=\upsilon e^{-\upsilon x}, x>0$, Erlang–2 stage distribution is $f(x)=\upsilon^2 x e^{-\upsilon x}, x>0$ and hyper-exponential distribution is $f(x)=c\upsilon e^{-\upsilon x}+(1-c)\upsilon^2 e^{-\upsilon^2 x}, x>0$. The interpretation of the results based on numerical illustration carried out for the different performance measures is as follows in Table 1 to 3:

Table 1 shows that when immediate feedback probability (r) increases, the probability that server is idle (Q_0) decreases, then the mean orbit size (L_q) increases and probability that server is busy with feedback customer (Ω) also increase for the values of $\lambda = 0.5$; $\theta = 3$; $\mu_p = 8$; $\mu_b = 5$; $a = 3$; $\delta = 1$; $\mu_v = 3$; $c = 0.7$. Table 2 shows that when priority arrival rate (δ) increases, the probability that server is idle (Q_0) decreases, the mean orbit size (L_q) increases and probability that server is busy with priority customer over non preemptive ordinary customer (Π_1) also increases for the values of $\lambda = 0.5$; $\theta = 3$; $\mu_p = 8$; $\mu_b = 5$; $a = 3$; $r = 0.5$; $\mu_v = 3$; $c = 0.7$. Table 3 shows that when lower speed service rate (μ_v) increases,

Table 1. Effect of immediate feedback probability (r) on Q_0, L_q and Ω.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
r	Q_0	L_q	Ω	Q_0	L_q	Ω	Q_0	L_q	Ω
Immediate feedback probability									
0.10	0.7341	0.0894	0.0095	0.4735	0.2475	0.0244	0.7058	0.1366	0.0081
0.20	0.7249	0.0933	0.0189	0.4520	0.2745	0.0482	0.6978	0.1401	0.0161
0.30	0.7156	0.0974	0.0284	0.4310	0.3030	0.0715	0.6898	0.1436	0.0241
0.40	0.7064	0.1015	0.0378	0.4104	0.3334	0.0942	0.6818	0.1472	0.0321
0.50	0.6971	0.1057	0.0473	0.3903	0.3658	0.1165	0.6738	0.1508	0.0401

Table 2. Effect of priority arrival rate (δ) on Q_0, L_q and Π_1.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
δ	Q_0	L_q	Π_1	Q_0	L_q	Π_1	Q_0	L_q	Π_1
Priority arrival rate									
0.50	0.7752	0.0942	0.0018	0.4935	0.3439	0.0129	0.7567	0.1245	0.0025
1.00	0.6971	0.1057	0.0029	0.3903	0.3658	0.0186	0.6738	0.1508	0.0059
1.50	0.6297	0.1173	0.0036	0.3133	0.3990	0.0210	0.6050	0.1684	0.0099
2.00	0.5713	0.1287	0.0040	0.2549	0.4392	0.0217	0.5471	0.1799	0.0140
2.50	0.5203	0.1399	0.0043	0.2099	0.4844	0.0215	0.4977	0.1871	0.0183

Table 3. Effect of lower speed service rate (μ_v) on Q_0, L_q and Q_v.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
μ_v	Q_0	L_q	Q_v	Q_0	L_q	Q_v	Q_0	L_q	Q_v
Lower speed service rate									
2.00	0.5899	0.1186	0.1770	0.3172	0.3772	0.1332	0.5779	0.1525	0.1942
3.00	0.6332	0.1142	0.1583	0.3422	0.3742	0.1283	0.6154	0.1525	0.1769
4.00	0.6682	0.1098	0.1432	0.3668	0.3702	0.1235	0.6469	0.1517	0.1624
5.00	0.6971	0.1057	0.1307	0.3903	0.3658	0.1189	0.6738	0.1508	0.1500
6.00	0.7214	0.1020	0.1202	0.4125	0.3611	0.1146	0.6970	0.1500	0.1394

the probability that server is idle (Q_0) increases, then the mean orbit size (L_q) decreases and probability that server is on working vacation (Q_v) also decrease for the values of λ = 0.5; θ = 1.5; μ_p = 8; μ_b = 5; a = 3; r = 0.5; δ= 3; c = 0.7. For the effect of the parameters λ, a, r, δ, θ and μ_v on the system per¢formance measures, three dimensional graphs are illustrated in Figure 1 to 3. In Figure 1, the surface displays an upward trend as expected for increasing the value of arrival rate (λ) and priority arrival rate (δ) against the mean orbit size (L_q). Figure 2 shows that the probability that server is idle (Q_0) decreases for increasing the value of feedback probability (r) and priority arrival rate (δ). In Figure 3, we demonstrate the effect of variation of the probability that

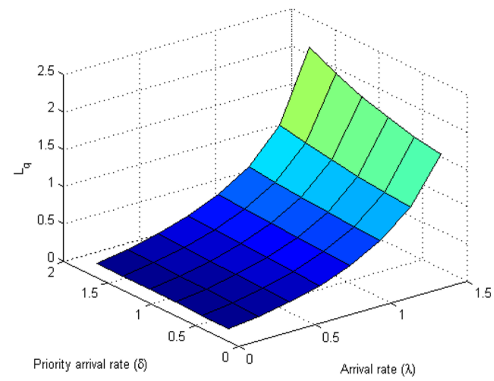

Figure 1. L_q versus δ and δ

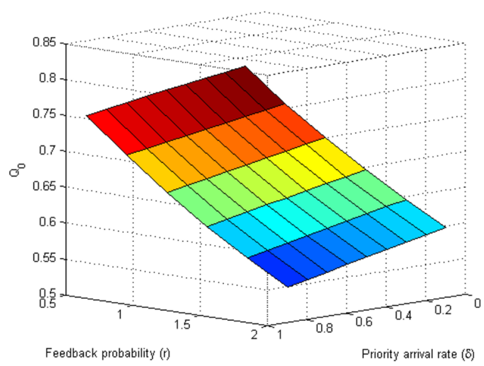

Figure 2. Q_0 versus r and δ

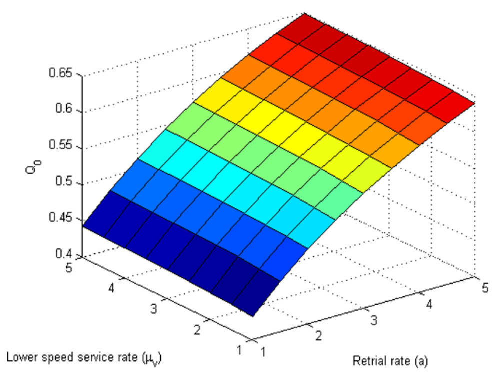

Figure 3. Q_0 versus μ_v and a

server is idle (Q_0) increases for increasing the value of lower speed service rate (μ_v) and retrial rate (a). From the above numerical examples, we can find the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

7. Conclusions

In this paper, we studied performance analysis of M/G/1 preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption. By using the probability generating function approach and the method of supplementary variable technique, the probability generating functions for the numbers of customers in the system and its orbit when it is free, busy, on working vacation, under repair are derived. Some varieties of performance measures of the system are calculated. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples are presented to study the impact of the system parameters. The novelty of this investigation is the introduction of multiple working vacations and immediate feedback in presence of priority retrial queues. This proposed model has potential practical real life application in telecommunication system and telephone consultation of medical service systems.

Acknowledgements

The authors are thankful to the reviewers for their valuable comments and suggestions for the improvement of the paper.

References

Arivudainambi, D., Godhandaraman, P., and Rajadurai, P. 2014. Performance analysis of a single server retrial queue with working vacation. OPSEARCH. 51(3), 434-462.

Artalejo, J.R. 2010. Accessible bibliography on retrial queues: progress in 2000-2009. Mathematical and Computer Modelling. 51(9-10), 1071-1081.

Artalejo, J.R., Dudin, A.N., and Klimenok, V.I. 2001. Stationary analysis of a retrial queue with preemptive repeated attempts. Operations Research Letters. 28(4), 173-180.

Artalejo, J.R. and Gomez-Corral A. 2008. Retrial queueing systems: a computational approach. Springer, Berlin, Germany.

Baruah, M., Madan, K.C., and Eldabi, T. 2012. Balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service. Journal of Mathematics Research. 4(4), 114-124.

Choudhury, G. and Ke, J.C. 2012. A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. Applied Mathematical Modelling. 36(1), 255-269.

Dimitriou, I. 2013. A preemptive resume priority retrial queue with state dependent arrivals, unreliable server and negative customers. TOP. 21(3), 542-571.

Gao, S. 2015. A preemptive priority retrial queue with two classes of customers and general retrial times. Operational Research an International Journal. 15(2), 233-251.

Gao, S. and Liu, Z. 2013. An M/G/1 queue with single working vacation and vacation interruption under Bernoulli schedule. Applied Mathematical Modelling. 37(3), 1564-1579.

Gao, S., Wang, J., and Li, W. 2014. An M/G/1 retrial queue with general retrial times working vacations and vacation interruption. Asia-Pacific Journal of Operational Research. 31(2), 6-31.

Gomez-Corral, A. 2006. A bibliographical guide to the analysis of retrial queues through matrix analytic techniques. Annals of Operations Research. 141(1), 163–191.

Kalidass, K. and Kasturi, R. 2014. A two phase service M/G/1 queue with a finite number of immediate Bernoulli feedbacks. OPSEARCH. 51(2), 201-218.

Liu, Z. and Gao, S. 2011. Discrete-time retrial queue with two classes of customers and feedback. Mathematical and Computer Modelling. 53(5-6), 1208-1220.

Liu, Z., Wu, J., and Yang, G. 2009. An M/G/1 retrial G-queue with preemptive resume and feedback under N-policy subject to the server breakdowns and repairs. Computers and Mathematics with Applications. 58 (9), 1792–1807.

Pakes, A.G. 1969. Some conditions for Ergodicity and recurrence of Markov chains. Operations Research. 17(6), 1058-1061.

Peng, Y. 2015. On the discrete-time Geo/G/1 retrial queueing system with preemptive resume and Bernoulli feedback. OPSEARCH. DOI 10.1007/s12597-015-0218-5

Rajadurai, P., Saravanarajan, M.C., and Chandrasekaran, V.M. 2014. Analysis of an $M^{[X]}/(G_1,G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption. Ain Shams Engineering Journal. 5(3), 935-950.

Rajadurai, P., Chandrasekaran, V.M., and Saravanarajan, M.C. 2015a. Steady state analysis of batch arrival feedback retrial queue with two phases of service, negative customers, Bernoulli vacation and server breakdown. International Journal of Mathematics in Operational Research. 7(5), 519-546.

Rajadurai, P., Varalakshmi, M., Saravanarajan, M.C., and Chandrasekaran, V.M. 2015b. Analysis of $M^{[X]}/G/1$ retrial queue with two phase service under Bernoulli vacation schedule and random breakdown. International Journal of Mathematics in Operational Research. 7(1), 19-41.

Rajadurai, P., Saravanarajan, M.C., and Chandrasekaran, V.M. 2015c. Analysis of M/G/1 retrial queue with balking, negative customers, working vacation and server breakdown. International Journal of Applied Engineering Research. 10(55), 4130-4136.

Rajadurai, P., Chandrasekaran, V.M., and Saravanarajan, M.C. 2015d. Analysis of an $M^{[X]}/G/1$ unreliable retrial G-queue with orbital search and feedback under Bernoulli vacation schedule, OPSEARCH. 53(1), 197-223.

Rajadurai, P., Saravanarajan, M.C., and Chandrasekaran, V.M. 2016. Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule. Ain Shams Engineering Journal. DOI: http://dx.doi.org/10.1016/j.asej.2016.03.008

Sennott, L.I., Humblet, P.A., and Tweedi, R.L. 1983. Mean drifts and the non ergodicity of Markov chains. Operation Research, 31(4), 783-789.

Senthilkumar, M., Chakravarthy, S.R., and Arumuganathan, R. 2013. Preemptive resume priority retrial queue with two classes of MAP arrivals. Applied Mathematical Sciences. 7(52), 2569-2589.

Servi, L.D. and Finn, S.G. 2002. M/M/1 queues with working vacations. Performance Evaluation. 50(1), 41 -52.

Wang, J. 2008. On the single server retrial queue with priority subscribers and server breakdowns. Journal of Systems Science and Complexity. 21(2), 304-315.

Wu, J. and Lian, Z. 2013. A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. Computers and Industrial Engineering. 64(1), 84-93

Wu, D. and Takagi, H. 2006. M/G/1 queue with multiple working vacations. Performance Evaluation. 63(7), 654-681.

Wu, J., Wang, J., and Liu, Z. 2013. A discrete-time Geo/G/1 retrial queue with preferred and impatient customers. Applied Mathematical Modelling. 37(4), 2552–2561.

Zhang, M. and Hou, Z. 2012. M/G/1 queue with single working vacation. Journal of Applied Mathematics and Computing. 39(1), 221-234.